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Institutions, Equilibria and Efficiency

Essays in Honor of Birgit Grodal

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Essays in Honor of Birgit Grodal

With 21 Figures
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Birgit Grodal

Preface

This symposium is published in honor of Birgit Grodal (1943–2004). Birgit Grodal was diagnosed with cancer for the second time in the spring of 2002. Many of her friends, colleagues and former students wanted to show their support. John Roemer expressed the idea that we should have a conference in honor of her. The Institute of Economics, University of Copenhagen organized a conference, The Birgit Grodal Symposium: Topics in Mathematical Economics, in September 2002, where 50 of Birgit Grodal's friends participated. They heard co-authors of Birgit Grodal's unpublished papers present the papers, as well as an impressive list of fellow economists express their appreciation of Birgit Grodal's friendship, advise, and cooperation. The participants decided to write papers for a volume in her honor. Economic Theory agreed to publish an issue dedicated to Birgit Grodal. On Birgit Grodal's 60'th birthday June 2003 20 papers in her honor appeared as working papers from Institute of Economics. For different reasons some of the papers will be published in other journals, and a couple of papers were added later. The authors were asked to write very short papers to ET, and it was agreed that complete versions would appear in this symposium.

Birgit Grodal started as a mathematician. Werner Fenchel was her teacher. Her first publications were on atomless economies and representations of preferences. Her latest publications have mostly been jointly with Hildegard and Egbert Dierker and with Bryan Ellickson, Suzanne Scotchmer, and Bill Zame. The papers in this issue are in many different areas of economics, they are almost all directly or indirectly influenced by Birgit Grodal. This shows that Birgit Grodal's importance for economics has not been just her own publications. Her interest in economics has been very broad, and her advice to students and colleagues has had an importance also outside the areas of her own research.

Birgit Grodal has been very active in other ways as well. She has been department chairman and head of the institute. She has been on the council of Econometric Society, and she has been very active in European Economic Association. She was elected President for EEA for 2005. Birgit Grodal died May 4, 2004.

Copenhagen, *Christian Schultz*
April 2004 and June 2005 *Karl Vind*

This symposium as well as the preceding volume of ET has been edited by Karl Vind, Birgit's old advisor and life long friend. Karl was instrumental in setting up the conference in honor of Birgit and it was his idea that the contributions in honor of her should appear in this volume.

While working on the volume, Karl was diagnosed with cancer in June 2004 and died after a short period of illness July 14th 2004. It became my task to finish Karl's work with the volume.

Copenhagen, *Christian Schultz*
June 2005

Karl Vind was born in 1933 and studied economics in Copenhagen from 1951. There he found an environment of economists who generally did not appreciate the use of mathematics in economics - with Professor Frederik Zeuthen as an important exception. From the very beginning of his studies Karl Vind saw the potential in using mathematical language and tools in the study of economic problems. After finishing his master degree in economics and military service he started teaching mathematics and statistics for economic students at University of Copenhagen. In 1962 he got a Rockefeller fellowship to visit University of California, Berkeley. The inspiration from Gerard Debreu became very important for the research of Karl Vind.

Karl Vind initiated his research in the beginning of the 1960's and published his first contribution to economic theory in 1964: an influential paper on the core (or the Edgeworth allocations) of atomless economies. At the age of 33 in 1966 he became a full professor of economics at the University of Copenhagen, a position he held until retirement in 2003. He continued as an active researcher even after retirement. The editorship of this volume witnesses to this fact.

During his carrier he has made profound contributions to many parts of mathematical economics including among others core theory, general equilibrium under different institutional structures, representation of preferences under risk and uncertainty as well as of time preferences. His results have been widely published, and in 2003 Springer Verlag published his monograph Independence, Additivity, Uncertainty, which summarizes and develops his research on preference representations - and contains contributions by Birgit Grodal.

At the Institute of Economics, University of Copenhagen, he created a strong research group within the field of mathematical economics, attracting and inspiring talented students and researchers from economics and mathematics. They founded the international reputation of the Institute, and kept a strong international network. Karl Vind had close and long lasting collaborations with many scholars and he repeatedly returned to Berkeley at sabbaticals.

The group also had a large influence on the development of the economics programme at the University of Copenhagen in turning it into a modern programme taking the students to the research frontier. Karl Vind pushed for many years for a joint programme in mathematics and economics, and in 1986 these efforts were rewarded with the introduction of the math-econ programme.

Karl Vind was an inspiring teacher, a good friend, an excellent scholar, and a wonderful storyteller. The Institute of Economics and his colleagues and friends world wide will miss him.

Copenhagen, *Jørgen Birk Mortensen, Christian Schultz, and Birgitte Sloth*
June 2005

Contents

1

Birgit Grodal: A Friend to Her Friends*†

Andreu Mas-Colell

Universitat Pompeu Fabra

It was much too soon. We knew it could happen. She participated in the proceedings of the conference in her honor held in Copenhagen in September 2002 with a display of her usual acumen and vigor, but we left worried. I do not know if the time ever comes for anyone, but it had not come for her. It is hard to accept. With the passing of Birgit Grodal the European Economic Association loses its 2005 president, the community of European economists one of its shapers and leading figures, the worldwide community of economic theorists and mathematical economists an outstanding general equilibrium theorist, and her many friends, a friend.

Birgit started her career as a mathematician. She did her Ph.D. under the great Danish convexity specialist William Fenchel. For a whole generation of economists the legendary Fenchel notes on convexity were a source of knowledge and of inspiration. It was most natural to move from Fenchel and convexity analysis to mathematical economics and economic theory, especially under the gentle push of economic and social concerns. This was the road that Grodal took.

On it she got connected early to the two cauldrons of mathematical economics research of the late 1960s and early 1970s: Berkeley, where Gerard Debreu, Dan McFadden, Roy Radner, David Gale, Steve Smale, John Harsanyi, William Shephard, were mentoring; and CORE, with Jacques Drèze, Jean François Mertens, Jean Gabszewicz, Werner Hildenbrand, and many others. Both institutions teemed with visitors, including many Americans at CORE and many Europeans at Berkeley. Birgit Grodal became a thriving member of this network, and from these origins a successful research career was launched.

* Acknowledgment: I want to acknowledge the help of Karl Vind in the preparation of this text. Note added in proofs: it is with much regret that I learned at the beginning of last July of the passing of Karl Vind, eminent economist, lifelong friend and dear colleague of Birgit Grodal.

† This obituary is reprinted with the kind permission of the Journal of European Economic Association

I will attempt to describe and summarize Grodal's contributions to economic theory, all of them falling into the realm of general equilibrium theory broadly understood, under six general headings.

1.1 Preference and Demand Theory

Her background in convexity theory and the intellectual environment of the late 1960s made preference and demand theory, on the one hand, and the theory of the core on the other, the points of entry of Grodal into general equilibrium analysis.

In preference and demand theory she carried out a very useful task early on, and she did it at the moment that it was needed. The measure theoretic approach to atomless economies, viewed as a model of perfect competition, which was initiated by Robert Aumann, Karl Vind, and, emphasizing a distributional approach, by Werner Hildenbrand, rested, technically speaking, on the possibility to define topological and metric structures on spaces of agent's characteristics—in particular, on spaces of preferences. In one of her first published papers (modestly called "A Note on…", see Grodal 1974) she carried out the most thorough and general analysis of this matter. Without doubt, it constitutes the definitive contribution.

Samuelson once said that preference and demand theory is for many economists a partial but lifelong devotion. Grodal did not fail to go back to these theories on a number of occasions. For a particularly nice instance we refer to the paper she wrote with Hildenbrand (Grodal and Hildenbrand 1989), where a most simple example of an aggregate excess demand function not satisfying the weak axiom of revealed preference is offered: a type of result with significant implications for the uniqueness and the stability analysis of equilibrium. In the example there are four commodities: two factors of production that do not enter into the utility functions and are initially owned by consumers, and two consumption goods. Suppose that there are at least two consumers that are different (in a precise and natural sense). Then the aggregate excess demand will not satisfy the weak axiom!

1.2 Core Theory

As a tool for the analysis of equilibria, the core is present in Grodal's research throughout her career. But, more specifically, in her first period Grodal made two important contributions to the then emerging core equivalence theorem. In Grodal (1972) she inaugurated, with two companion papers (by David Schmeidler and Karl Vind), an important theme in core analysis. Namely, the idea that if an allocation can be improved upon (that is, it is not in the core), then there are many improving coalitions; and, in fact, improving allocations can be subjected to a number of additional restrictions without losing the validity of the core equivalence theorem. In the quoted contribution Grodal showed that the Theorem would still obtain if the improving coalitions were required to be formed as the union of at most L small clusters of almost identical consumers (here L is the number of commodities).

In the second contribution (Grodal 1975), she extended, in a far-reaching way, a theorem of Debreu on the rate of convergence of the core to the set of Walrasian equilibriua. The result of Debreu was for a type economy (that is, the number of consumers tends to infinity, but their characteristics—preferences and endowments—belong to an a priori given finite set). In contrast, Grodal's result was completely general. I cannot resist recording the key, and beautiful, mathematical result that she established in order to unlock the door to a generalization. (Maybe I amnot seeing this in an entirely objective way. At the time we shared an office at Berkeley and I was witness to the weeks of impasse and to the feeling of exhilaration when at last the nut was cracked). It goes as follows: Suppose that for every Nyou have a set of $2N$vectors $z_i(N)$in $\mathbf{R}^\mathbf{L}$.These vectors add up to zero and, independently of N,they are all norm-bounded by a constant K.Then for every Nyou can select a group of Nvectors from the collection corresponding to Nsuch that their sum adds up to a vector norm-bounded by K(hence the average individual adjustment to make it zero in the aggregate is of order K/N).

1.3 Integral Representations

There was an intensely mathematical but economically grounded and difficult topic on which Birgit worked, on and off, all her professional life, namely integral representations as order-homomorphisms on totally preordered function spaces. The economic motivation of this work comes from the need to determine conditions allowing the representation of suitably defined separable preferences by means of additive utility functions. The challenge that Grodal tackled was the extension of the theory, confined until then to a finite number of components, to an infinite number of components. She had to face, therefore, the need to seek the representation not simply as a sum of component utilities but as an integral of such. Her interest on this topic was aroused at CORE, and, in fact, there is a 1968 COREworking paper with Jean-François Mertens on this matter. Much later the research was gathered into two working papers of the Institute of Economics of the University of Copenhagen that appeared in 1990. It is to be desired that they do not remain unpublished. It is conceivable that with her usual very high self-imposed standards she had ideally planned to elaborate more and to polish further the working papers. Unfortunately, she will not be able to do so, but, in truth, it is not clear that the work needs it. Publication would seem fitting.

1.4 Equilibrium with Coordination

In her mature stage, the heart of the contributions of Grodal were to equilibrium theory, competitive and noncompetitive. For competitive theory we have— beyond the work on the core—the research on clubs and markets that will be reviewed shortly, and two papers that she wrote with Karl Vind on coordinated equilibrium (see Grodal and Vind 2001).We could qualify the latter concept as a Danish School item

(HansKeiding has also dwelt in it). It constitutes a remarkable extension of the usual competitive equilibrium notion, and it yields a powerful tool for equilibrium analysis.

Another contribution of Grodal to equilibrium theory is contained in her last published paper (see Dierker, Dierker, and Grodal 2002), which is the only one she ever wrote on incomplete markets, a topic that, in fact, she mastered. The paper shows her at her peak. It contains a very interesting example displaying nonexistence of constrained efficient equilibria, and it makes us regret with added intensity, if that was possible, that this will have to be her last word on the matter.

1.5 Clubs and Markets

It is well known that one of the main powers of the Walrasian model of equilibrium is the enormous variety of applications that can conceivably be obtained by a proper interpretation of what constitutes a commodity. The story with time and with uncertainty is familiar, as is the personalized commodities formulation associated with public goods. These interpretations may seem easy ex post, but, in fact, they are far from obvious and in some cases they can only be made after resetting the problem carefully and with much delicacy. This is precisely what Grodal, with Bryan Ellickson, Suzanne Scotchmer, and William Zame, did for "club membership" in one of her latest works (see Ellickson, Grodal, Scotchmer, and Zame 1999), thus generating an equilibrium theory of commodity allocations and club membership that, naturally enough, shares the usual optimality and core equivalence properties of any Walrasian equilibrium.

In a sense, clubs are defined as the characteristics of a commodity, the supply of which is the number of membership slots. To overcome the integer problem, economies with a continuum of agents are considered from the start, although the clubs all have finite membership (thus at equilibrium there is an idealized continuum of clubs). The paper shows how all this can be made to fit together in a demand-supply–like model. It is an impressive piece of work and, to repeat, a substantial increase in the conventional coverage of the Walrasian model.

1.6 General Equilibrium with Imperfectly Competitive Firms

In spite of all her work in Walrasian equilibrium theory, one suspects that this was not the most persistent intellectual love of Birgit Grodal. This attribution will have to go to imperfect competition theory. Taken together the ten or so papers that she published on the latter theory (most of them in collaboration with Egbert Dierker and/or Hildegard Dierker) from 1989 to 2002 constitute the most sustained research effort of Grodal and the work that displays her mature powers at their best. In fact, her interest in this area starts at the very beginning of her career. If I may be allowed a personal recollection, the session of the European summer meeting of the Econometric Society in 1970 (at Barcelona), where I first met Birgit, was devoted to this very topic and, as discussant, she made the point that later would constitute the motivating kernel for her research in the field.

This point could be called the "problem of the numeraire," and it makes a lot of sense for someone that, like Grodal, would be intellectually inclined to question and probe the foundations of theory. In conventional price-taking theory, shareholders (one or many) are unanimous in desiring that firms profit-maximize. Further, it does not matter in which unit we choose to measure profits. However, in imperfect competition theory, where firms can have market power, it can matter a lot for a profit-maximizing behavior of the firm if we measure profits in one unit or another (say, in terms of its inputs or its outputs). This is only an indication of a deeper problem: The profit maximization objective is itself questionable. For example, shareholders are also consumers and as such their interest on the price level may be at odds with the interest of a manager to charge high prices. As an illustration, think of the case of a single owner, where there is no social choice aggregation problem among owners. Then, obviously, the firm should maximize the surplus of the owner. Except in special situations (e.g., parametric prices, the owner does not supply to the firm or consume its products) this is not equivalent to a profit maximization rule.

Grodal was deeply dissatisfied that imperfect competition theory had glossed over this problem, and on this she was correct. Her (and her coauthors) research approach consisted then in asking which sort of behavior could be postulated on imperfectly competitive firms that did not clash with fundamentals, that is, with the reality of one or several underlying shareholders. In Dierker and Grodal (1999) an ingenious and deep answer is provided. Take the easy case where the owners generate and spend their income in the parametric sector, that is, from and in goods that have fixed prices. Suppose then that the firm chooses a profit maximizing production (with respect to any numeraire in the parametric sector).

Trivially this production will have the following property: "The aggregate budget set of shareholders at a different production of the firm does not include in its interior the aggregate consumption of the shareholders at the profit maximizing production." The Dierker and Grodal proposal, under the name of "real wealth maximization," consists of adopting for the objective of the firm, also in the general case (where prices matters), the previous bracketed statement. This objective is immune to the choice of numeraire problem and takes into account the interest of the shareholders. A real wealth maximization production may not exist, but it does under fairly general conditions (guaranteeing that the problem is suitably convex) and it relates well to notions derived from surplus, or compensated surplus, maximization. The concept is rich enough and robust enough to build on it a theory of imperfect competition, both pure and applied. Grodal and her coauthors have already initiated the way. No doubt this will be an influential line of research in the future.

In a typical European fashion, Grodal had deep academic roots in the institution at which she was affiliated all her life: the University of Copenhagen. All her work stems from there. In turn, she was recognized by Danish and European society. Thus, in the last five years of her life she was a member of the Danish Competition Council, and from 1998, the Academia Europeae. The dedication to both she enjoyed much. From her Danish base, Grodal has been instrumental in the institutional development of economics in Europe. She had, so to speak, an Econometric Society profile (she was a Fellow of the Society and a member of its Council and Executive Committee for several periods) and was part of the small group of people that promoted with much effort, but also with much satisfaction and much success, the growth of modern quantitative economics at European universities. She was particularly proud of the mathematical economics undergraduate program at the University of Copenhagen, which she helped to launch in 1985. But her mentoring went beyond Denmark. She was Head of the Nordic Doctoral Programme in Economics from 1992 to 1997, and for many years she was involved the European econometric winter meeting, a small conference where promising young economists where selected from all over Europe and given a double opportunity: to present and discuss theirwork with senior members of the profession and to discover that they were not alone, that there were many other researchers similarly aged and similarly professionally inclined at other European universities.

The continuity of the meetings has been preserved to this day. Birgit Grodal was also very active in the founding and the development of the European Economic Association. She was a member of its council from 1987 to 2000 (except for the years 1990–1991) and was elected vice president in 2002. She would have been its president in 2005. It is our loss that this will not came to pass. But it will always be a great satisfaction to know that she was elected. She was, moreover, instrumental in another momentous event in this institutionalization process of the economics academic profession in Europe: namely, the simultaneous and coordinated celebration of the (summer) European meetings of the European Economic Association and of the Econometric Society. To go ahead with this (in principle, experimentally) was a recommendation of the "Grodal report," the report of a committee that she chaired and which she steered with the consummate ability of a good science manager. Grodal was determined, even stubborn, and liked an argument. But she was also gracious, kind, and always friendly and full of enthusiasm. I first met Birgit in the August of 1970 European Summer meeting of the Econometric Society, to which I have already referred. I was included in a session chaired by Grodal, who also served as discussant. To the eyes of the second-year graduate student that was I, she was severe with the papers from seasoned members of the profession. But she was kind to me, the student, in spite of the fact that I was presenting a pretty sophomoric piece of research. It was typical of her attitude towards students. Birgit, we will miss you.

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2

On the Definition of Differentiated Products in the Real World*

Beth Allen

University of Minnesota

Summary. This paper proposes an abstract model of commodity differentiation that incorporates manufacturing imprecision and dimensioning and tolerancing standards. The potential consistency of such a model based on engineering consideration is analyzed. For a large pure exchange economy, competitive equilibria exist and are Pareto optimal. Production issues such as the derived demand for intermediate products, continuity of cost functions, and product selection and technology issues such as mass customization, agile manufacturing, and manufacturability are discussed.

Key words: Differentiated commodities, General equilibrium, Hausdorff metric topology.

JEL Classification Numbers: D51, L15, D21.

2.1 Introduction

This paper proposes a new way to formulate commodity spaces in microeconomic theory that is both more specific and more abstract than standard definitions of commodity spaces, including those for differentiated commodities, in the existing literature. I focus on uncertainties inherent in any production technology and aim for consistency with how commodities are actually purchased. The overall goal is to demonstrate that one can modify our standard model in microeconomic theory so that it reflects these concerns yet nevertheless remains tractable for economic analysis. Then the resulting economic properties can be examined and compared to those of the existing benchmark model of an economy.

* This work was supported by the National Science Foundation through research grants DMI-9816144 and DMI-0070257. This paper was presented at the Institute of Economics, University of Copenhagen in Fall 2000. As always, Birgit Grodal was an energetic and enthusiastic academic host during my month-long visit. I wish to thank Marcus Berliant for a long, pleasant, and helpful conversation. An anonymous referee read the paper carefully and provided helpful comments.

This research is motivated by theoretical models of engineering design and manufacture, especially the solid geometric modelling work underpinning computer aided design (CAD) and computer assisted manufacturing (CAM) tools. For concreteness and simplicity, I have chosen to focus on geometric forms such as precision metal parts and dies for plastic molding. This offers the advantage of easy visualization, but note that the same engineering principles would carry over to other types of commodities.

An important aspect of any manufacturing procedure is its level of precision – the closeness of the actual manufactured object to the desired object that is specified in the design and the reproducibility of the operation with the process remaining under control without further interference. The subfield of dimensioning and tolerancing (D & T) studies this uncertainty, how it can and should be measured, how it is modelled formally, how its specifications should be standardized (i.e., ANSI 14.5 in the U.S. and ISO 9000 internationally), and how a given level of uncertainty affects production costs and possible time-to-market delays in the introduction of new products.

Yet, for economics, it is essential that any useful formal model be analytically tractable and display the potential to yield interesting economic conclusions. Thus there must be a balance between increased generality and abstraction on the one hand and the prospects for obtaining interpretable economic results on the other hand. One fruitful approach is to delineate clearly the comparisons and contrasts between a benchmark model and the proposed novel approach, while a related research strategy consists of displaying exactly the sense in which one model encompasses the other. This analysis is performed here for my proposed model versus Mas-Colell's (1975) renowned model of abstract commodity differentiation with indivisibilities. As a bonus, the presence of indivisibilities in the differentiated commodities (geometric objects) here is natural and intuitive.

In economic theory, Debreu (1959) pointed out the necessity of formalizing the definition of the set of commodities present in an economy. His well-known, well-exposited, and well-reasoned statement on this matter appears as Chapter 2. There he argues that a commodity should be described in terms of its complete physical description, its location, and its date of delivery so that all units of a given single commodity would be viewed as completely equivalent by each consumer and each firm in the economy. This paper focuses on the physical description aspect of the definition of a commodity and suggests that how economists think about physical descriptions of goods can be improved. My proposed improvement is consistent with actual (incomplete) contracts to purchase and sell goods – for instance, defense procurement – and features contracts that are, in principle, legally enforceable as the basis for defining commodities. In addition, my framework respects realistic limits on information with respect to the physical characteristics of products in that economic agents are not hypothesized to take account of nonverifiable information about the production process.

Debreu's (1959) admonishment to pay careful attention to the specification of the commodity space in economic theory has been followed up by a long list of researchers – e.g., Bewley (1972) and many others who have examined various infinite-dimensional commodity spaces in general equilibrium theory and Prescott and Townsend (1984), who advocate randomizations as a convexification device (later utilized for different purposes by Hornstein and Prescott, 1991, and by Cole and Prescott, 1995). My paper builds on the seminal article by Mas-Colell (1975), which provides a state-of-the-art model of abstract commodity differentiation.

However, to incorporate engineering considerations of product design and manufacture, it is necessary to add several layers to the Mas-Colell (1975) approach so that it reflects the specific structure of the commodity space suggested by geometric design theory and by dimensioning and tolerance analysis. This involves much more than simply adding uncertainty or randomness.

Yet, for such an approach to have important implications for economic theory, it must yield the fundamental ingredients for constrained optimization (this is needed in engineering too!) and for consistency of the resulting economic system, where consistency of the model means that it has a suitable equilibrium. Suitability means that one can define well-behaved price systems under reasonable market conditions such that at least one of these price systems can clear all markets simultaneously, given that all individual agents optimize taking prices as given. Furthermore, one wants the resulting allocations corresponding to any equilibrium to be efficient. In other words, the goal is existence and Pareto optimality of equilibrium allocations in the model. If there were possibly no equilibria or if an equilibrium could fail to be efficient in situations which otherwise satisfy appropriate versions of the well-known conditions that usually suffice to guarantee these properties, then one would naturally question the reasonableness of the proposed model.

The remainder of this paper is organized as follows: Section 2 explains several areas of engineering considerations that motivate this paper. With this motivation, Section 3 presents the proposed set of differentiated products and proves that it has the mathematical structure of a compact metric space. Section 4 presents the economic environment in terms of the new commodity space, preferences, and endowments. Then Section 5 defines competitive equilibrium, establishes its existence, and demonstrates its efficiency by appealing to a core equivalence result. Section 6 examines an alternative possible definition of differentiated products in the set $\mathcal{C}_0$ of nonempty closed convex subsets of the closed unit cube in a Euclidean space subject to production imprecision given by probabilities and explains why this approach is not adopted here. Continuing in this vein, Section 7 explores the potential re-definition of geometric objects as equivalence classes under the equivalence relations of translation or translation and rotation. Section 8 discusses various issues involved in the extension from pure exchange economies to those with production. Finally, Section 9 contains concluding comments.

2.2 Real world considerations

Geometric objects must be closed and bounded subsets of some finite-dimensional Euclidean space. Obviously, the main cases of interest are subsets of the plane and especially three-space, but $\mathbb{R}^n$ is specified in this paper because this level of added generality does not increase the difficulty. Fix a positive integer n and let $\mathcal{S}_0$ denote the set of nonempty compact subsets of $\mathbb{R}^n$. Elements of $\mathcal{S}_0$ will be called geometric objects. [Where confusion with the notion of object classes in computer science could occur, the literature uses the terms geometric solid (for three-dimensional subsets) or, more generally, artifacts, although the later terminology can be applied to virtually anything that is designed or manufactured.] Determination of the subsets of $\mathcal{S}_0$ which can be considered the natural domains of geometric objects is postponed to Subsection 2.3, after a topological structure on $\mathcal{S}_0$ has been introduced.

2.2.1 Approximations

Two distinct notions of approximation of a subset in $\mathbb{R}^n$ by a sequence (or net) of subsets in $\mathbb{R}^n$ are commonly found in the literature: the one based on the generalized volume or n-dimensional Lebesgue measure of the symmetric difference of two sets and that based on the Hausdorff metric (or, more generally, closed convergence of sets). The second choice is more natural for engineering applications and, in fact, has appeared in the engineering design literature, as discussed below.

To see the difference between the two approximation concepts, consider the problem of approximating a x cm by y cm rectangle in the plane, where $0 \leq x \leq 100$ and $0 \leq y \leq 100$, by a rectangle with integer-valued length and width (i.e., by a $\hat{x}$ cm by $\hat{y}$ cm rectangle, where $\hat{x} \in \{0, 1, \ldots, 100\}$ and $\hat{y} \in \{0, 1, \ldots, 100\}$). Let A be our desired set or nominal object (the x cm by y cm rectangle) and let B denote the set (the $\hat{x}$ cm by $\hat{y}$ cm rectangle) that we actually obtain as described above. Note that A and B are compact. Then the error measure based on the Hausdorff metric can be written as $\delta(A, B) = \max\bigl\{\max_{b \in B} \min_{a \in A} \bigl|\bigl|\bigl|a - b\bigr|\bigr|\bigr|, \max_{a \in A} \min_{b \in B} \bigl|\bigl|\bigl|a - b\bigr|\bigr|\bigr|\bigr\}$ where, for $x = (x_1, x_2) \in \mathbb{R}^2$, $\bigl|\bigl|\bigl|x\bigr|\bigr|\bigr| = \max\{|x_1|, |x_2|\}$ instead of the familiar Euclidean norm $\|x\| = \sqrt{x_1^2 + x_2^2}$ (which gives an equivalent but not identical distance between the sets A and B). The alternative area-based error measure, $\mathrm{Area}(A \triangle B) = \mathrm{Area}\bigl((A \cup B) \backslash (A \cap B)\bigr)$ instructs one to find the volume (or area in the plane) of the symmetric difference between the sets. It's easy to check that $\delta(A, B) = \max\{|x - \hat{x}|, |y - \hat{y}|\}$ and $\mathrm{Area}(A \triangle B) = |x - \hat{x}| \max\{y, \hat{y}\} + |y - \hat{y}| \max\{x, \hat{x}\} - |x - \hat{x}| \cdot |y - \hat{y}|$. In this example, the $\delta(A, B)$ error measure tends to be independent of the approximate magnitudes of x and y; taking $x = y = 99.5$ and $x = y = 0.5$ both give minimum errors of 0.5. On the contrary, the area of the symmetric distance necessarily goes to zero as x and y become close to zero even though the relative errors (under either error measure) explode.

Yet another useful way to understand the differences between these two error measures is to constrast them for the following sequence of sets: the ideal desired set A is fixed and equals the square with vertices $(0, 0)$, $(0, 50)$, $(50, 50)$ and $(50, 0)$ while for each k, the set that we actually obtain is B_k, where B_k is the union of A and the rectangle with vertices $(0, 50)$, $(0, 100)$, $(1/k, 100)$, $(1/k, 50)$ so that each B_k equals A plus a vertical spike of width $1/k$. For all k, $\delta(A, B_k) = 50$ but $\mathrm{Area}(A \triangle B_k) = 50/k \to 0$ as $k \to \infty$. This example indicates that the error measure δ is likely to perform better for certain engineering design problems than the error measure given by the volume of the symmetric difference.

2.2.2 The Hausdorff metric

The Hausdorff distance is defined for every pair of nonempty subsets of $\mathbb{R}^n$.

First, define (open) ϵ-neighborhoods of nonempty subsets of $\mathbb{R}^n$ by $B_\epsilon(A) = \{x \in \mathbb{R}^n \mid \text{there exists } y \in A \text{ with } \|x - y\| < \epsilon\}$ where $A \neq \emptyset$, $A \subseteq \mathbb{R}^n$, and $\epsilon > 0$. For every two nonempty subsets E and F of $\mathbb{R}^n$, define the (extended) Hausdorff distance $\delta(E, F)$ by $\delta(E, F) = \inf\{\epsilon \in (0, \infty] \mid E \subseteq B_\epsilon(F) \text{ and } F \subseteq B_\epsilon(E)\}$. [Say that an extended distance function, extended semimetric, or extended metric is a distance function, semimetric or metric that may assume the value of ∞.] Let $\mathcal{F}$ denote the set of subsets of $\mathbb{R}^n$ and let $\mathcal{F}_0$ denote the set of nonempty subsets of $\mathbb{R}^n$. Then the function $\delta\colon \mathcal{F}_0 \times \mathcal{F}_0 \to [0, \infty]$ is an extended semimetric on $\mathcal{F}_0$; $\delta(E, F) = 0$ whenever $E = F$, $\delta(E, F) = \delta(F, E)$, and $\delta(E, F) \leq \delta(E, G) + \delta(G, F)$. However, δ fails to be an extended metric on $\mathcal{F}_0$ because $\delta(E, F) = 0$ does not imply $E = F$; indeed $\delta(E, F) = 0$ whenever $\mathrm{cl}(E) = \mathrm{cl}(F)$, where $\mathrm{cl}(A)$ denotes the closure of A. This can be "fixed" by considering equivalence classes of sets in $\mathcal{F}_0$, where two sets are equivalent if they have the same closure. A natural representative of each equivalence class is the (unique) closed subset which equals the closure of every set contained in the given equivalence class. Of course, it simplifies the discussion to work directly with the set of nonempty closed subsets of $\mathbb{R}^n$.

The Hausdorff metric was defined by Hausdorff (1962). A convenient reference is Hildenbrand (1974, pp. 15–21), while Nadler (1978) discusses convergence of sets in greater generality. Note that the Hausdorff metric topology is closely related to the concept of closed convergence of sets; see, for instance Hildenbrand (1974). The topology induced by the Hausdorff metric has been used extensively in economic theory.

Let $\mathcal{G}$ denote the set of closed subsets of $\mathbb{R}^n$ and let $\mathcal{G}_0$ denote the set of nonempty closed subsets of $\mathbb{R}^n$. Then $\delta\colon \mathcal{G}_0 \times \mathcal{G}_0 \to [0, \infty]$ is an extended metric (since $\delta(E, F) = 0$ if and only if $E = F$ whenever $E \in \mathcal{G}_0$ and $F \in \mathcal{G}_0$) and $(\mathcal{G}_0, \delta)$ is an extended metric space. Note that the topology on $\mathcal{G}_0$ induced by the (extended) Hausdorff metric is not determined by the topology of $\mathbb{R}^n$ but rather can depend on the metric used on $\mathbb{R}^n$ in the sense that two metrics d' and d'' can define the same topology on $\mathbb{R}^n$ but induce different topologies on $\mathcal{G}_0$ unless d' and d'' are uniformly equivalent (i.e., if they yield exactly the same class of uniformly continuous real-valued functions on $\mathbb{R}^n$). This is why the above discussion specified the metric derived from the Euclidean norm on $\mathbb{R}^n$.

As mentioned above, in the context of geometric design one is concerned with closed and bounded sets. In $\mathbb{R}^n$, the closed and bounded sets are the compact sets. To set notation, let $\mathcal{S}$ be the set of compact subsets of $\mathbb{R}^n$ and let $\mathcal{S}_0$ be the set of nonempty compact subsets of $\mathbb{R}^n$. Note that $(\mathcal{S}_0, \delta)$ is a metric space; δ is a metric rather than an extended metric on $\mathcal{S}_0$ because the Hausdorff distance between any two nonempty compact sets is finite.

By a result of Aubin (1977, p. 164, Theorem 1), δ is a complete extended metric on $\mathcal{G}_0$. This says that if $\{S_k\}$ is a Cauchy sequence of sets in $\mathcal{G}_0$, then there exists $S \in \mathcal{G}_0$ such that $\lim_{k\to\infty} S_k = S$. If, in fact, $S_k \in \mathcal{S}_0$ for all k and $\delta(S_k, S) \to 0$, then S must be compact also because $\delta(T', T'') = \infty$ whenever T' is compact and

T'' is unbounded (closed but noncompact). This proves that $(\mathcal{S}_0, \delta)$ is a complete metric space.

In geometric design, one frequently works with closed sets that are contained in a given compact set because such uniform boundedness captures the notion that a maximum size initial material is available or that a given machine or manufacturing process is constrained by an overall feasible size limitation. Without loss of generality, let K denote the closed unit cube in $\mathbb{R}^n$ ($K = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1 \text{ for all } i = 1, 2, \ldots, n\}$). Let $\mathcal{K}$ denote the set of closed subsets of K and let $\mathcal{K}_0$ denote the set of nonempty closed subsets of K so that $\mathcal{K}_0 = \{S \subseteq \mathbb{R}^n \mid S \neq \emptyset,\ S \text{ is closed, and } S \subseteq K\}$. Then $(\mathcal{K}_0, \delta)$ is a compact metric space. (See Hildenbrand (1974 Theorem 1, p. 17).) This property constitutes a major advantage of using the topology induced by the Hausdorff metric.

In geometric design theory, the Hausdorff topology is also applied to the boundaries of geometric solids. Implicitly, this yields another extended metric space $(\mathcal{G}_0 \backslash \{\mathbb{R}^n\}, \delta^\partial)$ and metric spaces $(\mathcal{S}_0, \delta^\partial)$ and $(\mathcal{K}_0, \delta^\partial)$. For $G, H \in \mathcal{G}_0 \backslash \{\mathbb{R}^n\}$, $G, H \in \mathcal{S}_0$, or $\mathcal{G}, H \in \mathcal{K}_0$, $\delta^\partial(G, H) = \delta(\partial G, \partial H)$, where $\partial S = \mathrm{cl}(S) \backslash \mathrm{int}(S)$ denotes the boundary of the set S. By definition, the boundary of any set in $\mathcal{S}_0$ belongs to $\mathcal{S}_0$ and the boundary of any set in $\mathcal{K}_0$ is a nonempty closed subset of the compact set K and hence belongs to $\mathcal{K}_0$. [To see that the boundary of any set in $\mathcal{S}_0$ or $\mathcal{K}_0$ must be nonempty, recall that a set is both open and closed if and only if its boundary is empty; the only subsets of the connected space $\mathbb{R}^n$ which are both open and closed are the empty set and $\mathbb{R}^n$ itself.] Note that there are sets in $\mathcal{K}_0$ that are not the boundary of any set in $\mathbb{R}^n$ (for instance, K itself). Note also that δ^∂ is not defined on all of $\mathcal{G}_0$ because $\partial\mathbb{R}^n = \emptyset$.

Observe that $(\mathcal{K}_0, \delta)$ and $(\mathcal{K}_0, \delta^\partial)$ are distinct topological spaces, although both are metric spaces. Convergence in the δ metric is not equivalent to convergence in the δ^∂ metric. To see this, for $k = 3, 4, 5, \ldots$, let $S_k = K \backslash B_{1/k}(1/2, \ldots, 1/2)$, where $B_\epsilon(x) = \big\{y \in \mathbb{R}^n \;\big|\; \|x - y\| < \epsilon\big\}$ denotes the open ϵ-ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ (for $\epsilon > 0$). Then $S_k \xrightarrow{\delta} K$ but $S_k \xrightarrow{\delta^\partial} K \backslash \{(1/2, \ldots, 1/2)\}$ where $K \backslash \{(1/2, \ldots, 1/2)\}$ fails to be a closed subset of K but its boundary $\partial K \cup \{(1/2, \ldots, 1/2)\}$ is closed. This example illustrates that the δ^∂-limit of a sequence of compact sets need not be a closed set. Hence $(\mathcal{K}_0, \delta^\partial)$ is not closed.

2.2.3 The domain of geometric objects

The previous subsection stated that $(\mathcal{K}_0, \delta)$ is a compact metric space when endowed with the topology induced by the Hausdorff metric. Recall that $\mathcal{K}_0$ denotes the set of nonempty closed (and automatically bounded, and therefore compact) subsets of the closed unit cube K in $\mathbb{R}^n$. Yet not all sets in $\mathcal{K}_0$ serve as appropriate geometric objects. Hence, the domain $\mathcal{D}$ of geometric objects must be a proper subset of $\mathcal{K}_0$. Of course, compactness of $\mathcal{D}$ is highly desirable for mathematical tractability.

A natural restriction on geometric objects is the requirement that they be connected sets. Indeed, if a potential geometric object is not connected, it should be considered as two or more separate geometric objects, where each one of the redefined individual geometric objects consists of a single connected component of the originally proposed geometric object. This insistence on connectedness reflects manufacturing processes and practices, in that each connected component could equally well be produced at a different facility. From an economics viewpoint, the connected components could be viewed as extreme complements in consumption if the specifics of the situation render this true for some or all consumers and, in addition, firms could consider selling the various connected components as a bundled commodity. For a familiar example, think of left gloves and right gloves.

In this paper, I proceed beyond connectedness to the stronger condition of convexity. Let $\mathcal{C}_0$ denote the set of nonempty closed (and hence compact) convex subsets of K, so that $\mathcal{C}_0 = \{S \in \mathcal{K}_0 \mid S \text{ is convex}\}$. Then $(\mathcal{C}_0, \delta)$ is a compact metric space and, in fact, $\mathcal{C}_0$ is itself a convex set under the operations of taking the (Minkowski) sum of sets and scalar multiplication; i.e., if $S, T \in \mathcal{S}_0$ and $\lambda \in \mathbb{R}$, define $S + T = \{s + t \mid s \in S \text{ and } t \in T\}$ and $\lambda S = \{\lambda s \mid s \in S\}$. See Allen (1999c) for an explicit proof.

Here I follow the research strategy of focusing on $\mathcal{C}_0$ as the domain of geometric objects because of not only the desirability of convex sets but also some problems with the interpretation of the Hausdorff metric topology when it is applied to nonconvex sets. To see an explicit example of the difficulty, define a sequence $\{S_k\}$ of nonempty compact subsets of K, where $S_1 = S_2 = K$ and for each $k = 3, 4, 5, \ldots$, $S_k = K \backslash B_{1/k}(1/2, \ldots, 1/2)$, where $B_\epsilon(x)$ denotes the open ball of radius $\epsilon > 0$ centered at x in $\mathbb{R}^n$. Then, as $k \to \infty$, $S_k \to K$ but each S_k fails to be contractable and "would not hold water" because it has a hole. Another example is provided by setting each S_k equal to the (finite) subsets of K defined by points with coordinates expressed as decimals with (at most) k digits, so that in $\mathbb{R}^2$, $S_0 = \{(0,0), (0,1), (1,0), (1,1)\}$, $S_1 = \{(s_1, s_2) \in K \mid s_1 = 0, 1/10, 2/10, \ldots 1, \text{and } s_2 = 0, 1/10, \ldots, 1\}$. Then, as $k \to \infty$, $S_k \to K$ even though $S_k \cap K$ does not contain an open set for any k. Clearly K and the S_k could not be viewed as close substitutes for most purposes.

One solution to this problem may be to modify or strengthen the Hausdorff topology so that it distinguishes between a set and the same set after a tiny piece has been removed. Berliant has proposed a modified Hausdorff metric for this purpose; see Berliant and Dunz (1995) and Berliant and ten Raa (1988, 1992) but note that these references alter the metric further to reflect a given set of utility functions. Current research is addressing these issues.

2.2.4 Dimensioning and tolerancing

To think about dimensioning and tolerancing (D & T), consider the goal of drilling a hole in a cube of homogeneous metal. [The hole is an example of a feature (see Shah and Mäntylä, 1995).] Three distinct criteria are involved:

(1) *Size tolerance*, which means that the radius of the hole – and its depth if it does not extend completely through the piece of metal – must be within an acceptable range, which would usually take the form of a requirement that the hole's circumference must stay entirely within an annulus defined by two concentric

circles having radii equal to the minimum value and the maximum value in the acceptable range,

(2) *Form tolerance*, which means that the hole is sufficiently circular, rather than polyhedral or oval-shaped [regardless of its size], which again is typically verified by checking that the circumference lies within an annular region, and

(3) *Position tolerance*, which requires the hole to be in approximately the correct location relative to the edges of the cube of metal or relative to the locations of other features.

The three tolerancing constraints would be tested independently and the metal would be reworked or discarded if any criterion is not satisfied. This defines a *tolerance zone* or set of acceptable geometric objects. In the literature, axioms for tolerance zones have been provided. One important aspect is that exact form cannot be required; each criterion must have some "wiggle room", which need not be symmetric.

Note that this discussion focuses on D & T standards for a single geometric object and not statistical tolerancing, in which deviations with respect to some criteria can be offset by enhanced precision in terms of other criteria. Also, statistical quality control, in which random items from a batch are inspected and then a decision is made to accept or reject the entire batch, is not considered here.

The Hausdorff metric topology has been advocated in the engineering literature (i.e., Boyer and Stewart, 1991, 1992; Requicha, 1993; Requicha and Rossignac, 1992; Stewart, 1993) as a first step toward capturing D & T standards in a mathematical model. In brief, a tolerance zone is basically defined as a (relatively) open subset of geometric objects or an open ball, in the Hausdorff metric topology, around the nominal (desired) geometric object (see also Srinivasin, 1998).

2.2.5 Some remarks on the literature

The approach taken in this paper starts from the framework of general design theory, as developed by Yoshikawa (1981), who studies topologies and filters on abstract spaces associated with engineering design. Boyer and Stewart (1991, 1992) and Stewart (1993) introduce a topology (specified by the δ^{∂} metric defined in an earlier subsection) that is related to the one studied here. Requicha (1993) and Requicha and Rossignac (1992) discuss the Boyer and Stewart metric; see also the related papers by Requicha (1980, 1983), Tilove (1980) and Tilove and Requicha (1980) that focus on regular subsets in the context of dimensioning and tolerancing. (Recall that by definition, a set is regular if it equals the closure of its interior.) My paper does not focus on regular sets; this research strategy was chosen because of the difficulties associated with using the Hausdorff metric on the space of regular sets – lack of closure and the corresponding loss of compact subsets of geometric objects – that are pointed out in Allen (1999b). Peters, Rosen, and Shapiro (1994) and Rosen and Peters (1992, 1996) propose a quite different feature-based metric space topology for spaces of regular geometric designs. [See Shah and Mäntylä (1995) for an overview of features in engineering design.] A recent article by Allen (1999a) uses the Hausdorff topology and argues that, to characterize the sets of geometric objects that are manufacturable by some process or processes, one must take limits (and this involves a convergence concept or a topology). Mathematical properties of various subspaces of geometric objects are examined in Allen (1999c), based also on the topology induced by the Hausdorff metric.

2.3 Differentiated products

Section 2 argued that, as a first approach, one could take $\mathcal{C}_0$ to be the domain of geometric objects. For reasons of intuition, consistency with dimensioning and tolerancing standards, and technical tractability, $\mathcal{C}_0$ is endowed with its topology induced by the Hausdorff metric so that $(\mathcal{C}_0, \delta)$ becomes a convex compact metric space.

Thus, subsets of $\mathcal{C}_0$ become the basic differentiated products. Notice that the statement reads "subsets *of* $\mathcal{C}_0$" rather than "subsets *in* $\mathcal{C}_0$" because a commodity is some geometric object that belongs to a specified set of geometric objects.

Let $\mathcal{D}_0$ denote the set of nonempty closed subsets of $\mathcal{C}_0$, and give $\mathcal{D}_0$ the Hausdorff metric topology derived from the Hausdorff metric topology on $\mathcal{C}_0$. Note that $\mathcal{D}_0$ is not a subset of $\mathcal{C}_0$ but rather is a collection of subsets of $\mathcal{C}_0$ so that $\mathcal{D}_0$ is a set of sets. Note also that the Hausdorff distance is invoked twice in the definition of $\mathcal{D}_0$, first in the definition of $\mathcal{C}_0$ and then in a second layer involving the convergence of nonempty closed sets of nonempty convex compact subsets of $\mathbb{R}^n$. Write $(\mathcal{D}_0, \delta)$ where no confusion can occur.

Proposition 3.1 *$\mathcal{D}_0$ is a compact metric space.*

Proof.. This follows from Theorem 1 in Hildenbrand (1974, p. 17), since $(\mathcal{C}_0, \delta)$ is a compact metric space. □

However, the discussion in Section 2.4 suggests that not all elements of $\mathcal{D}_0$ are appropriate differentiated products. For example, a set consisting of a single geometric object (a set containing just one closed convex subset of K) is obviously nonempty and closed, but it violates the principle that exact form cannot be required in dimensioning and tolerancing.

To solve this problem, $\mathcal{D}_0$ will be restricted further and a proper subset of $\mathcal{D}_0$ will be taken to be the space of differentiated products. A consequence of its definition is compactness, so that tractability is not lost. Fix $\epsilon > 0$ and let $\mathcal{D}_\epsilon$ be the subset of $\mathcal{D}_0$ such that every element of $\mathcal{D}_\epsilon$ contains an open ϵ-ball.

Proposition 3.2 *For any sufficiently small $\epsilon > 0$, $\mathcal{D}_\epsilon$ is a nonempty proper compact subset of $(\mathcal{D}_0, \delta)$.*

Proof.. If $S_k \to S$ in $(\mathcal{D}_0, \delta)$ and each S_k is a compact set containing an open ϵ-ball for the given fixed $\epsilon > 0$, then so also does S contain an open ϵ-ball. The set $\mathcal{D}_\epsilon$ is a proper subset of $\mathcal{D}_0$ whenever $\epsilon > 0$ because, for instance, singletons belong to $\mathcal{D}_0$ but not to $\mathcal{D}_\epsilon$. The set $\mathcal{D}_\epsilon$ is nonempty whenever ϵ is sufficiently small relative to the size of K. □

Notice that $\mathcal{D}_\epsilon$ is not simply the collection of closed ϵ-balls, but rather contains all subsets that contain ϵ-balls. The mapping $\{\epsilon\} \times \mathcal{C}_0 \to \mathcal{D}_0$ defined by $(\epsilon, S) \mapsto \bar{B}_\epsilon(S)$ maps onto some proper subset of $\mathcal{D}_0$. However, note that (for $\underline{\epsilon} > 0$ and $\bar{\epsilon}$ sufficiently small) the map $[\underline{\epsilon}, \bar{\epsilon}] \times \mathcal{C}_0 \to \mathcal{D}_0$ (defined as above by $(\epsilon, S) \mapsto \bar{B}_\epsilon(S)$) is continuous for the product topology derived from the topologies on $\mathbb{R}$ and $(\mathcal{C}_0, \delta)$ and the "two layer" Hausdorff metric topology on $(\mathcal{D}_0, \delta)$.

2.4 The economic environment

This section lays out the economic model. It features the set $\mathcal{D}_\epsilon$ (for some sufficiently small $\epsilon > 0$) of differentiated products defined in the previous section, where $\mathcal{D}_0 \supset \mathcal{D}_\epsilon$ was endowed with a topology.

2.4.1 The commodity space

One aspect of the economic model which has not yet been emphasized is the hypothesis that commodities in $\mathcal{D}_0$ or $\mathcal{D}_\epsilon$ are indivisible. Differentiated products are assumed to be available only in integer amounts. This is a natural assumption for geometric objects, as fraction amounts – as well as irrational quantities – are difficult to interpret in an economic context.

These indivisibilities imply that, in order to enable equilibria possibly to exist, the presence of at least one perfectly divisible good is needed. This phenomenon would arise even if $\mathcal{D}_\epsilon$ were a finite set – the effects are unrelated to the fact that the model features infinitely many distinct commodities. Desirability assumptions for the divisible good are imposed in Subsection 4.3 below (for a further discussion, see Mas-Colell, 1975, 1977).

Accordingly, let h denote the perfectly divisible (homogeneous) good. For simplicity, only one divisible good is postulated; the extension to ℓ divisible goods that are priced in equilibrium simultaneously with the pricing of the differentiated commodities is a technical exercise. See Allen (1986b) for a discussion of the mathematical difficulties and an explicit proof in the context of a more complicated model with differentiated information that can be traded on markets.

Then the *set of commodities* is $\mathcal{D}_\epsilon \cup \{h\}$, for some fixed sufficiently small $\epsilon > 0$. The *commodity space* is taken to be the set of ordered pairs of bounded integer-valued Borel (signed) measures a on $\mathcal{D}_\epsilon$ such that $|a(\mathcal{D}_\epsilon)| < \infty$ [i.e., finite sums and differences of Dirac measures on $\mathcal{D}_\epsilon$] and scalars $b \in \mathbb{R}$, where, for $d \in \mathcal{D}_\epsilon$, $a(d)$ denotes the number of units of good d in the commodity bundle, for each $d \in \mathcal{D}_\epsilon$, and $b \in \mathbb{R}$ denotes the quantity of the perfectly divisible good h. Write $c = (a, b) \in \mathcal{M}^\circ(\mathcal{D}_\epsilon) \times \mathbb{R}$ where $\mathcal{M}^\circ(\mathcal{D}_\epsilon)$ denotes the set of finite integer-valued Borel measures a on $\mathcal{D}_\epsilon$.

Let $\mathcal{M}^M(\mathcal{D}_\epsilon) = \big\{a \in \mathcal{M}^\circ(\mathcal{D}_\epsilon) \;\big|\; |a(\mathcal{D}_\epsilon)| \leq M\big\}$ and let $\mathcal{M}^M_+(\mathcal{D}_\epsilon) = \{a \in \mathcal{M}^M(\mathcal{D}_\epsilon) \mid a(d) \geq 0 \text{ for all } d \in \mathcal{D}_\epsilon\}$. Then the *consumption set* for each trader in the economy is taken to be $\mathcal{M}^M_+(\mathcal{D}_\epsilon) \times \mathbb{R}_+$ for some fixed $\epsilon > 0$ and some fixed positive finite $M \in \mathbb{R}_{++}$.

Endow $\mathcal{M}^{\circ}(\mathcal{D}_{\epsilon})$ with its weak* topology or the topology of weak convergence of measures on $\mathcal{D}_{\epsilon} \subset \mathcal{D}_0$. This is the topology of pointwise convergence on the set $\mathcal{C}(\mathcal{D}_{\epsilon})$ of continuous real-valued functions on $\mathcal{D}_{\epsilon}$; i.e., $a_k \to a$ if for every $f\colon \mathcal{D}_{\epsilon} \to \mathbb{R}$ which is continuous (and bounded because $\mathcal{D}_{\epsilon}$ is compact when endowed with the Hausdorff topology), $\int f(d)\, da_n(d) \to \int f(d)\, da(d)$. Then $\mathcal{M}^M(\mathcal{D}_{\epsilon})$ becomes a compact metric space because the weak* topology is compact and metrizable on bounded subsets. Let $d_{\bullet}$ denote a metric for $\mathcal{M}^M(\mathcal{D}_{\epsilon})$.

2.4.2 Initial endowments

Recall that $c = (a, b) \in \mathcal{M}_+^M(\mathcal{D}_{\epsilon}) \times \mathbb{R}_+$ is a commodity bundle. Designate individual endowments by the subscript zero and write $c_0 = (a_0, b_0) \in \mathcal{M}_+^M(\mathcal{D}_{\epsilon}) \times \mathbb{R}_{++}$ for an initial endowment.

To set notation, define the set of all finite integer-valued nonnegative Borel measures on $\mathcal{D}_{\epsilon}$ by $\mathcal{M}_+^{\circ}(\mathcal{D}_{\epsilon}) = \bigcup\{\mathcal{M}_+^M(\mathcal{D}_{\epsilon}) \mid M \text{ is a finite integer}\}$. The difference between these sets is that $\mathcal{M}_+^M(\mathcal{D}_{\epsilon})$ is uniformly bounded by M (i.e., $|a(\mathcal{D}_{\epsilon})| \leq M$ for all $a \in \mathcal{M}_+^M(\mathcal{D}_{\epsilon})$), while $\mathcal{M}_+^{\circ}(\mathcal{D}_{\epsilon})$ consists of measures that are bounded but not uniformly so.

Where no confusion can result, the notation $c = (a, b)$ or $c_0 = (a_0, b_0)$ is used to designate either individual allocations and individual endowments or economy-wide allocations and economy-wide endowments, where "economy-wide" does not mean total or aggregate. When needed, explicit arguments are appended to c or c_0 so that $c(\cdot) = \big(a(\cdot), b(\cdot)\big)$ and $c_0(\cdot) = \big(a_0(\cdot), b_0(\cdot)\big)$ denote economy-wide allocations and endowments while, for instance, $c(i) = \big(a(i), b(i)\big)$ and $c_0(i) = \big(a_0(i), b_0(i)\big)$ refer to the allocations and endowments of some particular individual agent $i \in I$.

2.4.3 Preferences

In this economy, a preference relation $\preceq$ is a complete preorder on $\mathcal{M}_+^M(\mathcal{D}_{\epsilon}) \times \mathbb{R}_+$ [i.e., the graph $\mathrm{Gr}(\preceq)$ of $\preceq$ is a subset of $(\mathcal{M}_+^M(\mathcal{D}_{\epsilon}) \times \mathbb{R}_+) \times (\mathcal{M}_+^M(\mathcal{D}_{\epsilon}) \times \mathbb{R}_+)$] satisfying the following conditions:

(a) $\preceq$ is closed (continuity of preferences),
(b) If $c' = (a', b')$ and $c'' = (a'', b'')$ are such that $c'' \geq c'$ and $b'' > b'$, then $c'' \succ c'$ (monotonicity with strict desirability of the perfectly divisible commodity),
(c) If $c' = (a', b')$ and $c'' = (a'', b'')$ are such that $b' > 0$ and $b'' = 0$, then $c' \succ c''$ (any allocation with none of the perfectly divisible good is strictly dominated by any allocation with a positive amount of the perfectly divisible good),
(d) For any $c' = (a', b')$, there is $c'' = (a'', b'')$ with $a'' = 0$ such that $c'' \succ c'$ (yet another desirability condition for the perfectly divisible good),
(e) There is $\zeta \in \mathbb{R}$ such that if $c' = (a', b')$ and $c'' = (a'', b'')$ are such that $b' = b''$ and $d(a', a'') < 1/\zeta$ (where d denotes a metric for the weak* topology on $\mathcal{M}_+^M(\mathcal{D}_{\epsilon})$), then $(a', b' + \zeta) \succ (a'', b'')$.

Conditions (d) and (e) may be replaced by the condition (f), which is easier to understand.

(f) There exists $\zeta > 0$ such that $(0, b+\zeta) \succ (a,b)$ for all $c = (a,b) \in \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_+$.

Endow the space $\mathcal{P}$ of complete continuous preference preorders with the topology of closed convergence and let $d_{\preceq}$ be a metric for it (see Hildenbrand, 1974, for details).

The interpretation of continuity of preferences may be troublesome here, given the earlier arguments about "acceptable" sets of geometric objects and D&T notions. However, upper semicontinuity is all that is really needed, which allows for situations in which slight perturbation of a set of geometric objects results in a much worse set of geometric objects. (For example, imagine that the perturbed set contains geometric objects which must undergo costly reworking before they can be installed in an assembly line operation.)

Observe that convexity of preferences could be defined because convexity makes sense in the space $\mathcal{D}$, although convex combinations of sets in $\mathcal{D}_\epsilon$ are not the same as convex combinations of measures in $\mathcal{M}_+^\circ(\mathcal{D}_\epsilon)$ or $\mathcal{M}_+^M(\mathcal{D}_\epsilon)$. In any event, convexity is not required for the results in this paper, since a continuum of agents is needed to deal with the nonconvexities that inherently arise from the presence of indivisibilities.

2.4.4 The economy

This paper deals exclusively with large economies – those having an atomless continuum of agents. An economy then is defined to be a probability (joint) distribution on the space of preferences and endowments.

Definition 4.1 An *economy* is a Borel probability measure ν on $\big(\mathcal{P} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_{++}, \mathcal{B}(\mathcal{P} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_{++})\big)$, for some $\epsilon > 0$ sufficiently small, such that the following conditions hold: ν has compact support, $\operatorname{supp}\big(\int a_0(\cdot)\, d\nu(\cdot)\big) = \mathcal{D}_\epsilon$, and condition (e) in the definition of preferences holds uniformly for $\preceq$ in the support of the marginal distribution of ν on $\mathcal{P}$ [i.e., there is $\zeta > 0$ such that for all $\preceq$, if $c' = (a', b')$ and $c'' = (a'', b'')$ are such that $b' = b''$ and $d_\bullet(a', a'') < 1/\zeta$, then $(a', b'' + \zeta) \succ (a'', b'')$.

Remark 4.2 If, in the definition of $\mathcal{P}$, conditions (d) and (e) are replaced by condition (f), then the last requirement in Definition 4.1 can be replaced as follows: there is $\zeta > 0$ such that for every $\preceq$ in the support and every $c = (a,b) \in \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_{++}$, $(a, b+\zeta) \succ (a,b)$. This is just a uniform version of condition (f).

Remark 4.3 For the existence of competitive equilibrium result in Section 5, the last condition in Definition 4.1 can be dropped whenever each trader is hypothesized to own at most one total unit of all indivisible commodities (in $\mathcal{D}_\epsilon$) in his or her initial endowment.

Remark 4.4 Observe that the initial endowments of the perfectly divisible good are assumed to lie in some compact interval $[\underline{b}_0, \overline{b}_0]$ in $\mathbb{R}_{++}$ for almost all consumers, where $\underline{b}_0 > 0$ and $\overline{b}_0 < \infty$.

Remark 4.5 The condition that $\operatorname{supp}\big(\int a_0(\cdot)\, d\nu(\cdot)\big) = \mathcal{D}_\epsilon$ in Definition 4.1 says that all differentiated products in $\mathcal{D}_\epsilon$ (for the given ϵ) are actually available in the economy. All results remain valid if $\mathcal{D}_\epsilon$ is replaced by some smaller compact subset of $\mathcal{D}_\epsilon$. In this case, all allocations involve only differentiated products on the smaller set and only goods in the smaller set can be priced in equilibrium.

2.5 Equilibrium

As usual, an equilibrium is defined to be a price system and a feasible allocation such that each consumer's allocation is maximal (with respect to his or her preferences) on the budget set defined by the initial endowment and the price system. In this model, price systems must first be defined because the presence of infinitely many commodities usually means that, in principle, more than one candidate is available for the price space.

Accordingly, let $P = \{(p, p_b) \in \mathcal{C}(\mathcal{D}_\epsilon) \times \mathbb{R} \mid p(\cdot) \geq 0 \text{ and } p_b > 0\} = C^+(\mathcal{D}_\epsilon) \times \mathbb{R}_{++}$ define the set of price systems. This means that the price of each good is nonnegative, the price of the perfectly divisible good is strictly positive, and prices depend continuously on differentiated commodities in $(\mathcal{D}_\epsilon, \delta)$. Some zero prices for differentiated products could well arise in equilibrium because large sets in $\mathcal{D}_\epsilon$ may not be very attractive to consumers.

Definition 5.1 A Borel probability measure τ on $\big(\mathcal{P} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_{++} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_+, \mathcal{B}(\mathcal{P} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_{++} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_+)\big)$ is an *equilibrium distribution* for the economy ν if there is $p^* \in P = C^+(\mathcal{D}_\epsilon) \times \mathbb{R}_{++}$ such that:

(i) $\tau_{1,2,3} = \nu$, where $\tau_{1,2,3}$ denotes the (joint) marginal distribution of τ restricted to its first three components (the set $\mathcal{P} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_+$),
(ii) $\int c_0(\cdot)\, d\nu_{2,3}(\cdot) = \int c^*(\cdot)\, d\tau_{4,5}(\cdot)$, and
(iii) $\tau\big(\{(\succeq, c_0, c^*) \in \mathcal{P} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_{++} \times \mathcal{M}_+^M(\mathcal{D}_\epsilon) \times \mathbb{R}_+ \mid p^*c^* \leq p^*c_0$ and if c' is such that $p^*c' \leq p^*c_0$, then $c' \preceq c^*\}\big) = 1$.

Condition (i) says that τ is a distribution corresponding to the given economy ν and condition (ii) is aggregate feasibility of the equilibrium allocation $c^*(\cdot)$. Condition (iii) requires that almost all agents maximize their preferences over their budget sets defined by p^*.

Theorem 5.2 *Any economy satisfying the assumptions in this paper has an equilibrium distribution.*

Proof.. All of the assumptions in Mas-Colell (1975) are satisfied. The proof technique involves first approximating $\mathcal{D}_\epsilon$ by an increasing sequence of finite sets (in the topology of $\mathcal{D}_\epsilon$) and obtaining a corresponding sequence of equilibrium prices and equilibrium allocations for the finite restrictions. This involves checking that individual demands are upper hemicontinuous correspondences and that the aggregate demand, for the finite restrictions, is a convex-valued upper hemicontinuous correspondence, so that Kakutani's fixed point theorem applies. Along the sequence of finite approximations, a subsequence of equilibrium distributions converges weakly (by compactness) to a distribution which one can verify is an equilibrium distribution for the original economy ν with respect to the subsequential limit of the restricted equilibrium price systems. See Allen (1986a, b) for additional details. □

Remark 5.3 The technique of examining finite approximations and taking suitable (subsequential) limits is originally due to Bewley (1972) and permeates the literature on existence of competitive equilibrium with infinitely many commodities.

Remark 5.4 Equilibrium price systems necessarily satisfy certain no arbitrage conditions. In equilibrium, the price of a set in $\mathcal{D}_\epsilon$ can never exceed the sum of disjoint sets with union equal to the original set. However, similar inequalities do not apply to set-theoretic containment.

The next goal is to obtain the First Welfare Theorem in this model. To avoid the introduction of much additional technical notation, definition of the standard concept of an efficient (or Pareto optimal) distribution for a large pure exchange economy is *not included in this version of the paper. Similarly the core is not defined formally* because its introduction requires a standard representation for the economy (see Mas-Colell, 1975).

Proposition 5.5 *A distribution τ belongs to the core of an atomless economy ν if and only if τ is an equilibrium distribution for ν.*

Proof.. See Mas-Colell (1975, Theorem 2). □

Corollary 5.6 *Any equilibrium distribution τ for an economy ν is such that the equilibrium allocation distribution $\tau_{4,5}$ is Pareto optimal for ν.*

Proof.. Core allocations are necessarily Pareto optimal. □

Remark 5.7 A direct proof of Corollary 5.6 should be possible, but one must carefully check that local nonsatiation is not violated in this model. This would avoid the necessity of introducing the mathematical concept of a standard representation.

2.6 An alternate model with probabilities

Despite the arguments in Section 2 that sets of sets are the appropriate commodities with product differentiation, one may wonder about the potential formulation and consequences of a model in which the commodities are defined to be probability distributions over some space of product characteristics or precise physical descriptions. In the context of this paper, one would replace the sets in $\mathcal{D}_\epsilon$ by probability distributions on $\mathcal{C}_0$. Note that both sets and probabilities constitute natural generalizations of singletons, which can equally well be specified by Dirac probability measures.

Let $\mathcal{Q}(\mathcal{C}_0)$ denote the space of Borel probability measures on the compact metric space $\mathcal{C}_0$. Give $\mathcal{Q}(\mathcal{C}_0)$ the weak* topology of weak convergence of probability measures. Then it becomes a compact metric space; see Parthasarathy (1967). Thus $\mathcal{Q}(\mathcal{C}_0)$ has the same mathematical properties as $\mathcal{D}_0$.

However, one problem is that deletion of the Dirac measures or, more generally the measures with atoms, results in a subset which is not closed. This implies that the D & T axiom precluding exact form cannot be accommodated easily in a probabilistic framework.

Putting aside this problem, one can proceed to consider $\mathcal{M}^{\circ}\big(\mathcal{Q}(\mathcal{C}_0)\big)$ as the space of probabilistically-specified differentiated commodity bundles, where again I use convex subsets of K as the domain of geometric objects for specificity. The economic interpretation is that traders buy and sell known lotteries on geometric objects.

Verification that some random realization of a geometric object was drawn from the specified probability distribution is problematic. Appealing to reputation or random testing of drawings from a given distribution and a given seller would seem to be necessary in order to justify the implicit supposition that traders know the distribution or at least have subjective distributions that are consistent and cannot be contradicted.

This approach would have the advantage of avoiding defining preferences in a derived space such as $\mathcal{D}_{\epsilon}$ rather than on the space $\mathcal{C}_0$ of underlying geometric objects. Continuity properties of preferences thus become more natural and intuitive. However, with uncertainty, preference relations should be replaced by cardinal utilities, as is done in Allen (1986a,b). Continuity of derived ordinal preferences for probability distributions when traders maximize expected utility should follow when the distributions are suitably dispersed, which requires more than that they be atomless.

Note that, as earlier in this paper, probability distributions are not needed for convexification since the model features an atomless continuum of agents. Moreover, the differentiated commodities defined by probabilities would still be assumed to be indivisible.

2.7 Equivalence classes of geometric objects

One might wish to refine the definition of geometric objects as differentiated products so that it reflects affine invariance. Unlike buildings and bridges, the location of a geometric object – as opposed to its delivery location – is inessential and, similarly, the orientation of a geometric object when it is delivered generally doesn't matter. This suggests that, at least for the case of geometric objects, a basic differentiated commodity should be an equivalence class (under translation and rotation in $\mathbb{R}^n$) of nonempty compact subsets of $\mathbb{R}^n$. This idea is the basis of continuing research.

2.8 Production issues

The examples of geometric objects (precision-machined metal parts and dies for plastic injection molding) naturally serve as inputs to downstream production processes more than they would be expected to be purchased as final consumption goods. When these differentiated commodities are intermediate products, the resulting demand relations must be derived from the behavior of profit-maximizing or cost-minimizing firms. The requisite upper hemicontinuity should easily follow.

Deeper production issues are associated with the production of differentiated commodities in my model. The switch from a pure exchange economy to one with production requires lower semicontinuous cost functions or, more generally, well-behaved technology sets. The lower semicontinuity of cost functions when there are multiple production processes is derived in Allen (2000).

The modern issues of manufacturability, mass customization, dedicated versus flexible tools, and agility and flexibility require more attention to the specification of production technologies or cost functions, both for the short run and the long run. Effective answers to these questions also generally require the definition of a topology on the set of differentiated products, as is argued in Allen (1999a), where a simple manufacturability problem is formally posed and analyzed.

A microeconomic model with firms usually permits the possibility of strategic behavior. Hence, game theory is needed. The customary starting point is to postulate a static noncooperative game and inspect its Nash equilibria. Here the considerations of agility and flexibility might demand at least a two-stage game, with commitments on technology choice before actual production begins. Product selection decisions are also naturally placed in a game-theoretic model.

2.9 Conclusion

The main lesson of this paper is that, at least for some purposes, economic theorists should reformulate the basic microeconomic general equilibrium model to capture the notion that actual economic commodities are subject to manufacturing imprecision. In practice, this means that consumers and firms cannot guarantee that they purchase a product satisfying a complete and exact physical description, but rather they purchase an item that belongs to some specified set of products, where the set must permit some nontrivial range of all aspects of the product. The standard model can be modified and extended to take into account these considerations and nevertheless remain useful for economic theory in the sense that major results on the existence and efficiency of competitive equilibrium stay valid in the proposed reformulation.

The same principles can be applied to situations in which the underlying basic differentiated commodities are not restricted to be geometric objects. One simply requires a compact metric space of base commodities where the metric is compatible not only with consumers' notions of the substitution possibilities among goods but also with the relevant dimensioning and tolerancing standards and technological feasibilities.

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3

Equilibrium Pricing of Derivative Securities in Dynamically Incomplete Markets*

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Summary. We develop a method of assigning unique prices to derivative securities, including options, in the continuous-time finance model developed in Raimondo [47]. In contrast with the martingale method of valuing options, which cannot distinguish among infinitely many possible option pricing processes for a given underlying securities price process when markets are dynamically incomplete, our option prices are uniquely determined in equilibrium in closed form as a function of the underlying economic data.

Key words: Option pricing, General equilibrium, Dynamically incomplete markets.

JEL Classification Numbers: G13, D52

3.1 Introduction

Assuming that the price of a stock follows a geometric Brownian motion, Black and Scholes [11] developed the Black-Scholes formula for pricing options on that stock. Merton [42] showed that if markets are dynamically complete, there is a trading strategy that replicates the payoff of the option; as a consequence, the Black-Scholes formula for the price of the option can be obtained by arbitrage considerations.

Raimondo [47] proved existence of equilibrium in a continuous-time finance model with one agent; the theorem covers both models with dynamically incomplete

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as well as dynamically incomplete markets. The paper made two additional contributions: it obtained the pricing formula in closed form, resulting in specific pricing predictions, and it established methodology that has the potential to prove existence in the multi-agent, dynamically incomplete case; see Anderson and Raimondo [7].

Bick [9] provided a strong necessary condition for stock prices to follow geometric Brownian motion.[3] In Raimondo's representative agent model [47], the price of a stock follows generalized geometric Brownian motion only when all three of the following conditions are satisfied:

1. the agent has a CRRA utility function;
2. there is only one stock;
3. there are no endowments in the terminal period, so the only source of wealth in that period is the stock.

In all other cases, the equilibrium price of the stock does not follow geometric Brownian motion, and hence the original Black-Scholes formula for pricing options on the stock does not apply.

The main contribution of this paper is that we give explicit equilibrium pricing formulas for derivatives in settings in which the stock price is *not* geometric Brownian motion and in which markets may be dynamically incomplete. Specifically, we show that we can obtain an explicit formula, in closed form, for pricing options and other derivatives in the same setting as Raimondo [47]. The derivative price process is uniquely determined in closed form by the underlying economic data—the endowment process and utility of the agent, and the dividends of the securities. The pricing of the securities and derivatives exhibits a number of significant properties; in particular, there are specific correlations among the prices of the underlying securities which are not characteristic of geometric Brownian motion,[4] and the price of an option on one security depends on the price of all securities, not just the price of the security underlying the option.

The standard technique for pricing options and other derivatives in situations where the underlying securities price process is not geometric Brownian motion is the martingale method; Nielsen [44] provides an excellent exposition of this method. This method was initiated by Harrison and Kreps [26] in the complete markets case, and has since been extended to dynamically incomplete markets. The idea is to construct a martingale whose terminal value is the payoff of the option or other deriva-

[3] Bick showed that, in a single stock model with no endowment, CRRA utility is necessary for the stock price to follow geometric Brownian motion; Bick also provides references to earlier authors who had made the same observation, but not formally established it.

[4] For example, even if the terminal dividends (and hence the terminal prices) of stocks are independent, the prices of the stocks exhibit specific correlations due to wealth effects. If the terminal dividends and prices of securities are correlated, for example if the terminal prices are given by

$$S(T) = e^{S(0)+\mu T+\sigma W}$$

where W is a K-dimensional Brownian motion, $\mu \in \mathbf{R}^N$ and $\sigma \in \mathbf{R}^{N\times K}$ is a matrix, the covariance matrix of the securities at times $t < T$ is not equal to σ. The covariance matrix can be calculated explicitly from the closed-form pricing formula.

tive. If markets are dynamically complete, this martingale is uniquely determined by the underlying price process, and determines the pricing of the option by arbitrage. However, if markets are dynamically incomplete, the martingale is not uniquely determined by the underlying securities prices; indeed, the method allows infinitely many different pricing processes for the option.

This is not the first paper to address security and option pricing through equilibrium considerations. For example, Cox, Ingersoll and Ross [13] provide a very widely used characterization (via a partial differential equation) of equilibrium prices in a general equilibrium model, with dynamically complete markets and multiple (identical) agents, stochastic production and multiple Brownian motions. Their existence theorem depends on some strong endogenous assumptions.

In this paper and the companion papers cited below, we assert that equilibrium imposes more structure on finance models than that implied by the absence of arbitrage alone. It has been argued that equilibrium has no predictive power on securities and derivatives prices beyond that contained in the absence of arbitrage. The somewhat imprecise argument goes as follows: if prices are free of arbitrage, then under some hypotheses the prices will be martingales with respect to a probability measure mutually absolutely continuous with respect to the true probability measure. One can then extract a state-dependent felicity function for a single agent which supports the given arbitrage-free prices as an equilibrium.

Notice that this argument requires that the state-dependent felicity be chosen very carefully to produce the given pricing process. Suppose instead we require that the single agent's utility function be the expected utility generated by some state-*independent* felicity function. In that case, the argument just cited that any arbitrage-free pricing system can be justified as an equilibrium will not hold; state-dependence is essential to the proof. Of course, one might object that state-dependence is commonly observed in practice. It is difficult to argue against this objection, but the implication of this objection is not that one should consider a pricing process justified if it can be supported as equilibrium with respect to a felicity function whose state-dependence is carefully chosen to match the peculiarities of the pricing process. The implication is that the state-dependence should be specified as part of the model, and the pricing process should be required to be an equilibrium with respect to the exogenously given state-dependent utility function. Allowing arbitrary state-dependence is a virtue in a result of the form "for all state-dependent felicity functions, ... ;" is is not a virtue in a result of the form "there exists a state-dependent felicity function ... ."

One of the reasons that state-independence appears natural in some models is that the models are partial equilibrium. If a significant portion of household wealth is held in housing, a model that includes stocks but not housing is a partial equilibrium model. Since changes in the value of housing induce wealth effects that alter individuals' willingness to hold stocks, changes in housing values seem, in a stock-only model, to be instances of state-dependent felicity. But in a general equilibrium model which includes both stocks and housing, the state-dependence disappears. In particular, we argue that the relationship between stock pricing and housing can only be properly studied in a general equilibrium model which includes both. More generally, in this and the companion papers, we take the position that all assets and securities should be included in the model, and that felicity functions (and in particular any state-dependence of felicity functions) should be taken as exogenously specified.

This approach has real economic and financial consequences. If one's sole criterion for validating a price process is the absence of arbitrage, then the economic characteristics (endowments, utility functions, and information) of the agent(s) can play no role in determining prices: absence of arbitrage is a property of the pricing process alone, not of the underlying economic data. The pricing process in which stock prices are given by geometric Brownian motion, and option prices by the Black-Scholes formula, is arbitrage-free; hence, if one's sole criterion for price processes is the absence of arbitrage, one can never reject this pricing process on theoretical grounds. If one assumes geometric Brownian motion–Black-Scholes pricing, a specific prediction follows. One can deduce the implied volatility σ of the geometric Brownian motion from the stock price, the strike price of the option, the price of the option, and the time left until the expiration date of the option; the implied volatility σ must be independent of the strike price, so the graph of σ in terms of the strike price must be a horizontal line. Our model provides a specific prediction on the shape of the implied volatility curve in terms of the economic primitives, and it is certainly *not* a horizontal line except in very special cases. Empirically, the implied volatility curve is in the shape of a "smile," with the implied volatility being higher for strike prices well below the current stock price as well as for strike prices well above the current stock price (see Campbell, Lo and McKinley [12]). In future work, we hope to explore assumptions on the underlying primitives which generate a "smile" that matches the empirically-measured implied volatility curve.

Other papers (see for example Heston [27]) have used the martingale method to provide closed-form prices for options outside the geometric Brownian Motion context. However, the papers of which we are aware assume a particular stochastic process underlying the stock price. There is no guarantee that these stochastic processes are consistent with market clearing unless one carefully chooses a state-dependent utility function which carefully matches the peculiarities of the given stochastic process; indeed, just as Bick [9] showed that equilibrium security prices can be geometric Brownian Motion only under very special circumstances, Raimondo [47] makes it clear that geometric Lévy Processes and other generalizations of geometric Brownian motion that have been studied are equilibrium processes only under very special circumstances. Second, as noted above, the papers of which we are aware either assume dynamically complete markets, or in the case of dynamically incomplete markets provide one example of an arbitrage-free price system; there are infinitely many other arbitrage-free price systems, and there is no guarantee that the arbitrage-free price system they identify is consistent with market clearing with respect to a state-*independent* utility function. Third, closed forms solutions are obtained only in special cases.[5]

Although this paper only explicitly discusses the representative agent model of Raimondo [47], the method extends readily to the multi-agent model of Anderson and Raimondo [7]. In the multi-agent setting, equilibrium prices for the underlying securities need not be uniquely defined. Each equilibrium determines the terminal wealth of the agents, and the underlying security prices are given in closed form as a function of the underlying data and those terminal wealths. Equilibrium derivative prices are uniquely determined as a function of the equilibrium prices of the underlying securities, but the function, which involves the terminal wealths, is very complex. What is still true is that the derivative prices are expressed in closed form as a function of the underlying data, the underlying security prices and the terminal wealths.

In the multi-agent model, the terminal wealths depend on the whole history of prices, not just the terminal prices. As a consequence, the current price of an option is not determined as a function of the current prices of the securities, let alone a function of the current price of the stock on which it is written; instead, the current *equilibrium* price of an option depends on the entire history of securities prices up to that time. As a result, the usual formula for the trading strategy that replicates an option does not apply, even when markets are dynamically complete.

Anderson and Raimondo [5] provided general formulas for equilibrium pricing of derivative securities, and provided specific formulas in a number of examples. It appeared in a special journal issue with serious space constraints. As a result, it relied heavily on the exposition and proofs in Raimondo [47]. This paper provides the first self-contained exposition of those results, along with some extensions.

- In Anderson and Raimondo [5], all derivative securities paid off at the terminal date of the model, when the underlying securities paid their dividends. Here, we consider also derivative securities that pay off at intermediate dates, such as options with an exercise date before the terminal date in the model. This allows us to study equilibrium pricing of multiple options with varying exercise dates. Our formulas for equilibrium prices of options with exercise dates before the terminal date exhibit certain qualitative features, such as the volatility smile, of actual option prices (see Anderson, Diasakos and Raimondo [4]).
- We develop further the effect of wealth held outside of the financial markets, such as housing and human capital, on the pricing of derivatives. In this setting, markets that contain only the underlying financial assets are dynamically incomplete. However, the pricing of derivatives on the underlying financial assets necessarily reflects the price fluctuation of the nonfinancial assets, and thus creates the possibility of hedging the nonfinancial assets through trades involving the financial assets and their derivatives. We conjecture that, in a simple model with two sources of uncertainty, one driving a stock and the second driving the nonfinan-

[5] For example, Heston [27] obtains a closed-form solution with a deterministic interest rate, but not with a stochastic interest rate. We obtain a closed form solution with an endogenously determined stochastic interest rate.

cial asset, the addition of an option on the stock serves to dynamically complete the market.

As in Raimondo [47], our method makes use of nonstandard analysis, and in particular the nonstandard theory of stochastic processess as developed in Anderson [1], Keisler [34], and Lindström [35, 36, 37, 38]. These methods of nonstandard stochastic analysis have previously been applied to the theory of option pricing in Cutland, Kopp and Willinger [15, 16, 17, 18, 19, 20, 21]. Those papers primarily concern convergence of discrete versions of options to continuous-time versions, and their methods can likely be used to establish convergence results for the pricing formulas considered in this paper.

In nonstandard analysis, a hyperfinite set is an infinite set which possesses all the formal properties of finite sets; in particular, the Radner [45] and Duffie and Shafer [23, 24] existence results ensure that a hyperfinite incomplete markets economy has an equilibrium. We begin with a standard continuous-time model, construct a nonstandard hyperfinite economy, obtain an equilibrium for the hyperfinite economy, then use the nonstandard theory of stochastic processes to induce an equilibrium in the standard continuous-time model. For further comments on the methodology, see Raimondo [47].

3.2 The Model

The model we consider is a generalization of that in Raimondo [47].

1. Trade and consumption occur over a compact time interval $[0,T]$, endowed with a measure λ which agrees with Lebesgue measure on $[0,T)$ and such that $\lambda(\{T\}) = 1$.
2. The information structure is represented by a filtration $\{\mathcal{F}_t : t \in [0,T]\}$ on a probability space $(\Omega, \mathcal{F}, \mu)$. There is a standard K-dimensional Brownian motion $\beta = (\beta_1, \ldots, \beta_K)$ such that β_i is independent of β_j if $i \neq j$ and such that the variance of $\beta_i(t,\cdot)$ is t and $\beta_i(t,\cdot) = E(\beta_i(T,\cdot)|\mathcal{F}_t)$.
3. There is exactly one representative agent. The endowment of the agent satisfies
$$e(\omega,t) = 1 \text{ for all } (\omega,t) \in \Omega \times [0,T)$$
The endowment $e(\omega,T)$ in period T satisfies
$$e(\omega,T) = \rho(\beta_1(\omega,T), \ldots, \beta_K(\omega,T))$$
where $\rho : \mathbf{R}^K \to \mathbf{R}$ is continuous and satisfies
$$0 \leq \rho(x) \leq r + e^{r|x|}$$
for some $r \in \mathbf{R}_+$. The endowment in period $t \in [0,T)$ is interpreted as a rate of flow of endowment, while the endowment in period T is interpreted as a stock or lump. Given a measurable consumption function $c : \Omega \times [0,T] \to \mathbf{R}$, the utility function of the agent is

$$U(c) = E_\mu \left[\int_0^T \varphi_1(c_t)dt + \varphi_2(c_T) \right]$$

where the twice differentiable functions $\varphi_i : \mathbf{R}_{++} \rightarrow \mathbf{R}$ $(i = 1, 2)$ satisfy

$$\begin{cases} \varphi_i'(c) > 0 \text{ for } i = 1, 2 \\ \varphi_i''(c) < 0 \text{ for } i = 1, 2 \\ \varphi_i(z) \quad \text{ is bounded below} \end{cases}$$

Examples of utility functions satisfying the conditions on φ_i are the CARA utilities $\varphi_i(z) = \gamma e^{\alpha z}$ for $\alpha, \gamma < 0$ and the CRRA utilities $\varphi_i(z) = \gamma x^\alpha$ $(0 < \alpha < 1, \gamma > 0)$. The assumption that φ_i is bounded below is used at only point in the proof; we conjecture that it can be weakened to $\varphi_2'(z) = O\left(\frac{1}{z^r}\right)$ as $z \rightarrow 0$ for some $r \in \mathbf{R}$. If so, the CRRA utility function $\varphi_i(z) = \gamma \ln z$ $(\gamma > 0)$ and the CARA utility $\varphi_i(z) = \gamma x^\alpha$ $(\alpha < 0, \gamma < 0)$ would be covered by the theorem.

4. There are $J + 1$ tradable assets, with $0 \leq J \leq K$: J stocks $A_1, \ldots, A_J$ and a bond $B = A_0$. The payoffs of the securities are related to the Brownian motion via a vector $\mu \in \mathbf{R}^{J+1}$ and a $(J + 1) \times K$ matrix σ, whose j^{th} row is denoted by σ_j:

$$A_j(\omega, t) = \begin{cases} 0 & \text{if } t \neq T \\ e^{\mu_j T + \sum_{k=1}^K \sigma_{jk} \beta_k(\omega, T)} = e^{\mu_j T + \sigma_j \cdot \beta(\omega, T)} & \text{if } t = T \end{cases}$$

with $\mu_0 = \sigma_0 = 0$. [6]

The agent is initially endowed with security holdings $z(\omega, 0) = (0, (1, \ldots, 1))$: zero units of the bond and one unit of each stock. If $J = 0$ (i.e. there are no stocks in the model), we assume that $\rho(x) \geq e^{\alpha \cdot x}$ for some $\alpha \in \mathbf{R}^K$; this will ensure that the income in the terminal period T is not too small.

5. There are M derivative securities $D_m (1 \leq m \leq M)$ with exercise dates $T_m (1 \leq m \leq M) \in [0, T]$. The payoff of D_m is determined by the prices of the securities at date T_m, after normalizing the price of the bond to be 1. For example, D_1 could be an option to buy one unit of stock A_1 for X_1 units of the bond at time T_1; this yields a monetary payoff of

$$\max \{p_{A_1}(\omega, T_1) - X_1 p_B(\omega, T_1), 0\}$$

Each derivative security D_m is in zero net supply. The *monetary* payoff of D_m is given by[7]

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[6] The functional form $A_j(\omega, T) = e^{\mu_j T + \sigma_j \cdot \beta_j(\omega, T)}$ is not essential. All but one portion of the proof works if $A_j(\omega, T)$ is an arbitrary continuous function of $\beta_j(\omega, T)$ satisfying an exponential growth condition, and that one part works for a large class of functions of β_j, but we have not identified the exact class. Of course, changing the payoff will alter the pricing formula.

[7] In Anderson and Raimondo [5], derivative securities paid out at time T, with the payout (measured in units of the consumption good) equalling

$$H_m\left(p_B(\omega,T_m),p_{A_1}(\omega,T_m),\ldots,p_{A_J}(\omega,T_m)\right)$$

for some continuous function $H_m : \mathbf{R}^{J+1}_{++} \to \mathbf{R}$ satisfying $|H_m(x)| \leq \max\{S_m, |x|^r\}$ for some $S_m \in \mathbf{R}$ and $r \in \mathbf{R}_+$.

6. In order to define the budget set of an agent, we need to have a way of calculating the capital gain the agent receives from a given trading strategy. In other words, we need to impose conditions on prices and strategies that ensure that the stochastic integral of a trading strategy with respect to a price process is defined. The essential requirements are that the trading strategy at time t not depend on information which has not been revealed by time t, and the trading strategy times the variation in the price yields a finite integral. Specifically,
   a) The σ-algebra of predictable sets, denoted $\mathcal{P}$, is the σ-algebra generated by the sets $\{0\} \times F_0$ and $(s,t] \times F_s$ where $s < t \in \mathbf{R}_+$, $F_0 \in \mathcal{F}_0$, $F_s \in \mathcal{F}_s$ (see Metivier [43]). A stochastic process is said to be predictable if it is measurable with respect to $\mathcal{P}$.
   b) A security price process is a pair of stochastic processes $p = (p_A, p_B)$, where $p_A = (p_{A_1},\ldots,p_{A_J})$, and p_{A_j} and p_B are continuous square-integrable martingales with respect to $\{\mathcal{F}_t\}$. p_{A_j} and p_B are priced *cum dividend* at time T. A consumption price process is a stochastic process $p_C(\omega,t)$.
   c) Given a security price process p, there are unique measures q_{A_j} and q_B on the σ-algebra of predictable sets, which measure the quadratic variation of the components of p; they are generated by

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$$G_m(A_1(\omega,T),\ldots,A_J(\omega,T))$$

Thus, the payout of the derivatives was expressed as a function of the consumption payouts of the stocks, not the prices of the stocks. Since the bond pays out one unit of consumption good in all nodes at time T, there was no need to include it as an argument of H_m. Here, we allow for the possibility that $T_m < T$. We can't talk about D_m paying off in consumption, since consumption at time T_m is a flow, while the derivative payoff is a lump. Moreover, the ultimate payoff (in goods) at time T of the stocks is unknown at time T_m. Thus, it is necessary that we express the arguments of H_m in terms of the prices of securities (including the bond), and the payoff of H_m in monetary terms. If $T_m = T$, we have

$$\begin{aligned}
&G_m(A_1(\omega,T),\ldots,A_J(\omega,T))\\
&= \frac{H_m\left((p_B(\omega,T),p_{A_1}(\omega,T),\ldots,p_{A_J}(\omega,T))\right)}{p_C(\omega,T)}\\
&= \frac{H_m\left(p_C(\omega,T)\,(1,A_1(\omega,T),\ldots,A_J(\omega,T))\right)}{p_C(\omega,T)}
\end{aligned}$$

units of the consumption good. Thus, the formulation in Anderson and Raimondo [5] implicitly requires that H_m be homogenous of degree 1; this is a natural assumption, but we do not need it and do not impose it here.

$$q_{A_j}((s,t] \times F_s) = \int_{F_s} (p_{A_j}(\omega,t))^2 - (p_{A_j}(\omega,s))^2 d\mu$$
$$q_B((s,t] \times F_s) = \int_{F_s} (p_B(\omega,t))^2 - (p_B(\omega,s))^2 d\mu$$

for $s < t$ and $F_s \in \mathcal{F}_s$ and $q(\Omega \times \{0\}) = 0$.

d) Given a security price process p, a trading strategy is a pair (z_A, z_B) : $\Omega \times [0,T] \to [-M,\infty) \times [-M,\infty)^K$ such that z_A and z_B are predictable processes and $z_{A_j} \in L^2(\Omega \times [0,T], \mathcal{P}, q_{A_j})$, $z_B \in L^2(\Omega \times [0,T], \mathcal{P}, q_B)$.

7. Given a security price process p and a consumption price process p_C, the budget set is the set of all consumption plans c which satisfy the budget constraint

$$\mathbf{1} \cdot p_A(0) + \int_0^t zdp + \int_0^t p_C(\omega,s)(e(\omega,s) - c(\omega,s))ds = p(\omega,t) \cdot z(\omega,t)$$

for almost all ω and all $t < T$

$$\mathbf{1} \cdot p_A(0) + \int_0^T zdp + \int_0^T p_C(\omega,s)(e(\omega,s) - c(\omega,s))ds$$
$$+(e(\omega,T) + z_A(\omega,T)e^{\mu T + \sigma\beta(\omega,T)} + z_B(\omega,T) - c(\omega,T))p_C(\omega,T)$$
$$= p(\omega,T) \cdot z(\omega,T) \text{ for almost all } \omega$$

for some trading strategy z. We follow standard notation in writing $\mathbf{1} = (1,\ldots,1)$ and

$$\int zdp = \sum_{j=1}^{J} \int z_{A_j} dp_{A_j} + \int z_B dp_B$$

Observe that it is implicit in the definition that $p_C(\omega,\cdot)(e(\omega,\cdot) - c(\omega,\cdot)) \in L^1([0,T])$ for almost all ω.

8. Given a price process p, the demand of the agent is a consumption plan and a trading strategy which satisfy the budget constraint and such that the consumption plan maximizes utility over the budget set.
9. An equilibrium for the economy is a price process p, a trading strategy z and a consumption plan c which lies in the demand set so that the securities and goods markets clear, i.e. for almost all ω

$$z_A(\omega,t) = \mathbf{1} \text{ for all } t \in [0,T]$$
$$z_B(\omega,t) = 0 \text{ for all } t \in [0,T]$$
$$c(\omega,t) = 1 \text{ for all } t \in [0,T)$$
$$c(\omega,T) = e(\omega,T) + \mathbf{1} \cdot e^{\beta(\omega,T)}$$

where $e^{\mu T + \sigma\beta(\omega,t)}$ denotes the vector

$$\left(e^{\mu_1 T + \sigma_1 \cdot \beta_1(\omega,t)}, \ldots, e^{\mu_K T + \sigma_K \cdot \beta_K(\omega,t)}\right)$$

Theorem 1. *There is a standard probability space $(\Omega, \mathcal{F}, \mu)$, a filtration $\mathcal{F}_t$, and a K-dimensional Brownian motion $\beta = (\beta_1, \ldots, \beta_K)$ such that the continuous time finance model just described has an equilibrium. The pricing process is given by*

$$\begin{aligned}
p_{A_j}(\omega,t) &= e^{\mu_j T+\sigma_j\cdot\beta(\omega,t)} \int_{\mathbf{R}^K} \varphi_2'\left(F(t,\omega,x)\right) e^{\sqrt{T-t}\sigma_j\cdot x} d\Phi(x)\\
p_B(\omega,t) &= \int_{\mathbf{R}^K} \varphi_2'\left(F(t,\omega,x)\right) d\Phi(x)\\
p_C(\omega,t) &= \varphi_1'(1) \textit{ for } t<T\\
p_C(\omega,T) &= \varphi_2'\left(F(T,\omega,0)\right)\\
p_{D_m}(\omega,t) &= E\left(H_m\left(p_B(\omega,T_m), p_{A_1}(\omega,T_m),\ldots,p_{A_J}(\omega,T_m)\right) \mid \mathcal{F}_t\right)\\
&\quad \textit{if } t \le T_m\\
&= 0 \textit{ if } t > T_m\\
\frac{p_{A_j}(\omega,t)}{p_B(\omega,t)} &= e^{\mu_j T+\sigma_j\cdot\beta(\omega,t)} \frac{\int_{\mathbf{R}^K} \varphi_2'(F(t,\omega,x)) e^{\sqrt{T-t}\sigma_j\cdot x} d\Phi(x)}{\int_{-\infty}^{\infty} \varphi_2'(F(t,\omega,x)) d\Phi(x)}
\end{aligned} \tag{3.1}$$

where

$$F(t,\omega,x) = \rho\left(\beta(\omega,t)+\sqrt{T-t}x\right) + \sum_{j=1}^{J}\left(e^{\mu_j T+\sigma_j\cdot\left(\beta(\omega,t)+\sqrt{T-t}x\right)}\right)$$

and Φ is the cumulative distribution function of the standard K-dimensional normal.

Remark 1. We have chosen to write the pricing formula for the derivative securities D_m as an expectation. It is straightforward, but a little tedious, to write out the expression as integrals with respect to Gaussian distributions. Doing this requires substituting the Gaussian integral formulas for $p_{A_j}(\cdot, T_m)$ and $p_B(\cdot, T_m)$ into the expectation, and replacing the expectation itself with a Gaussian integral. The resulting formula is long, but in practice can easily be calculated numerically by a simple Fortran program.

Remark 2. It is well known that stochastic differential equations need not have solutions on the probability space and filtration on which they are defined. Roughly speaking, there may not be enough measurable sets to define the solution. As a consequence, existence theorems for solutions of stochastic differential equations take one of two forms: either they assume that the probability space and filtration have certain nice properties to start with and demonstrate the existence of so-called strong solutions, or they work with an arbitrary probability space and filtration and prove the existence of so-called "weak solutions" which can be defined on a richer probability space and filtration. We have chosen to take the former route and assume our probability space and Brownian motion have nice properties to begin with, but our methods will also show that if we begin with an arbitrary probability space and Brownian motion, one can obtain a "weak equilibrium," an equilibrium on a different probability space, filtration, and Brownian motion. Note that any two K-dimensional Brownian motions are essentially the same from the standpoint of probability theory; for example, they have the same finite-dimensional distributions, and they induce the same measure on $C([0,1], \mathbf{R}^K)$. Similarly, every weak equilibrium of the model just described will satisfy the pricing formula given in Equation (3.1).

Before we turn to the proof, it may be helpful to give several examples of models included in Theorem 1 and the pricing processes that prevail in those models.

Example 1. There is one Brownian motion and a single stock based on it, one bond, and a European call option on the stock, with strike price X and exercise date $T_1 = T$. The strike price is expressed in units of the bond, so that the option conveys the right to purchase 1 share of stock for X units of the bond. We have

$$\begin{aligned} H(p_B(\omega,T),p_A(\omega,T)) &= \max\{p_A(\omega,T) - Xp_B(\omega,T), 0\} \\ &= p_C(\omega,T)\max\left\{\frac{p_A(\omega,T)}{p_C(\omega,T)} - X\frac{p_B(\omega,T)}{p_C(\omega,T)}, 0\right\} \\ &= p_C(\omega,T)\max\{A(\omega,T) - X, 0\} \end{aligned}$$

The agent has no endowment in the terminal period T, so ρ is identically zero. In period T, the agent has the CRRA utility function $\phi_2(c) = \sqrt{c}$. Markets are dynamically complete. Observe that

$$\begin{aligned} \int_a^b e^{\alpha x} d\Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_a^b e^{\alpha x} e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_a^b e^{(2\alpha x - x^2)/2} dx \\ &= \frac{e^{\alpha^2/2}}{\sqrt{2\pi}} \int_a^b e^{(-\alpha^2 + 2\alpha x - x^2)/2} dx \\ &= \frac{e^{\alpha^2/2}}{\sqrt{2\pi}} \int_a^b e^{-(x-\alpha)^2/2} dx \\ &= \frac{e^{\alpha^2/2}}{\sqrt{2\pi}} \int_{a-\alpha}^{b-\alpha} e^{-x^2/2} dx \\ &= e^{\alpha^2/2}\left(\Phi(b-\alpha) - \Phi(a-\alpha)\right) \end{aligned}$$

The pricing process is

$$\begin{aligned} p_A(\omega,t) &= e^{\mu T + \sigma\beta(\omega,t)} \int_{-\infty}^{\infty} \varphi_2'\left(F(t,\omega,x)\right) e^{\sigma\sqrt{T-t}x} d\Phi(x) \\ &= e^{\mu T + \sigma\beta(\omega,t)} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{e^{\mu T + \sigma\beta(\omega,t) + \sigma\sqrt{T-t}x}}} e^{\sigma\sqrt{T-t}x} d\Phi(x) \\ &= \frac{e^{(\mu T + \sigma\beta(\omega,t))/2}}{2} \int_{-\infty}^{\infty} e^{\sigma\sqrt{T-t}x/2} d\Phi(x) \\ &= \frac{e^{(\mu T + \sigma\beta(\omega,t))/2 + \sigma^2(T-t)/8}}{2} \end{aligned}$$

$$
\begin{aligned}
p_B(\omega,t) &= \int_{-\infty}^{\infty} \varphi_2'\left(F(t,\omega,x)\right) d\Phi(x)\\
&= \int_{-\infty}^{\infty} \frac{1}{2\sqrt{e^{\mu t+\sigma\beta(\omega,t)+\sigma\sqrt{T-t}x}}} d\Phi(x)\\
&= \frac{1}{2e^{(\mu T+\sigma\beta(\omega,t))/2}} \int_{-\infty}^{\infty} e^{-\sigma\sqrt{T-t}x/2} d\Phi(x)\\
&= \frac{e^{-(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}\\
\frac{p_A(\omega,t)}{p_B(\omega,t)} &= \frac{\frac{e^{(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}}{\frac{e^{-(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}}\\
&= e^{\mu T+\sigma\beta(\omega,t)}
\end{aligned}
$$

$$
\begin{aligned}
p_D(\omega,t) &= \int_{-\infty}^{\infty} H\left(p_B(\omega,T), p_A(\omega,T)\right)\, d\Phi(x)\\
&= \int_{-\infty}^{\infty} p_C(\omega,T)\max\left\{A(\omega,T)-X,0\right\}\, d\Phi(x)\\
&= \int_{-\infty}^{\infty} \frac{\max\{0, e^{\mu T+\sigma\beta(\omega,t)+\sigma\sqrt{T-t}x}-X\}}{2\sqrt{e^{\mu T+\sigma\beta(\omega,t)+\sigma\sqrt{T-t}x}}} d\Phi(x)\\
&= \int_{\frac{\ln X-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}}^{\infty} \frac{e^{\mu T+\sigma\beta(\omega,t)+\sigma\sqrt{T-t}x}}{2\sqrt{e^{\mu T+\sigma\beta(\omega,t)+\sigma\sqrt{T-t}x}}} d\Phi(x)\\
&\quad -X\int_{\frac{\ln X-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}}^{\infty} \frac{1}{2\sqrt{e^{\mu T+\sigma\beta(\omega,t)+\sigma\sqrt{T-t}x}}} d\Phi(x)\\
&= \frac{1}{2}\int_{\frac{\ln X-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}}^{\infty} \sqrt{e^{\mu T+\sigma\beta(\omega,t)+\sigma\sqrt{T-t}x}}\, d\Phi(x)\\
&\quad -\frac{X}{2e^{\mu T+\sigma\beta(\omega,t)/2}}\int_{\frac{\ln X-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}}^{\infty} e^{-\sigma\sqrt{T-t}x/2} d\Phi(x)\\
&= \frac{e^{(\mu T+\sigma\beta(\omega,t))/2}}{2}\int_{\frac{\ln X-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}}^{\infty} e^{\sigma\sqrt{T-t}x/2} d\Phi(x)\\
&\quad -\frac{Xe^{-(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}\Phi\left(-\frac{\sigma\sqrt{T-t}}{2}-\frac{\ln X-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}\right)\\
&= \frac{e^{(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}\Phi\left(\frac{\sigma\sqrt{T-t}}{2}-\frac{\ln X-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}\right)\\
&\quad -\frac{Xe^{-(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}\Phi\left(-\frac{\sigma\sqrt{T-t}}{2}-\frac{\ln X-\mu T-\sigma\beta(\omega,t)}{\sigma\sqrt{T-t}}\right)\\
&= \frac{e^{(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}\Phi\left(\frac{\ln(e^{\mu T+\sigma\beta(\omega,t)}/X)+\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right)\\
&\quad -\frac{Xe^{-(\mu T+\sigma\beta(\omega,t))/2+\sigma^2(T-t)/8}}{2}\Phi\left(\frac{\ln(e^{\mu T+\sigma\beta(\omega,t)}/X)-\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right)
\end{aligned}
$$

$$\begin{aligned}\frac{p_D(\omega,t)}{p_B(\omega,t)} &= e^{\mu T+\sigma\beta(\omega,t)}\Phi\left(\frac{\ln(e^{\mu T+\sigma\beta(\omega,t)}/X)+\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right)\\ &\quad -X\Phi\left(\frac{\ln(e^{\mu T+\sigma\beta(\omega,t)}/X)-\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right)\\ &= \frac{p_A(\omega,t)}{p_B(\omega,t)}\Phi\left(\frac{\ln((p_A(\omega,t)/p_B(\omega,t))/X)+\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right)\\ &\quad -X\Phi\left(\frac{\ln((p_A(\omega,t)/p_B(\omega,t))/X)-\sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right)\end{aligned}$$

Note that the ratio $\frac{p_D(\omega,t)}{p_B(\omega,t)}$ follows the standard Black-Scholes formula. Note also that the ratio $\frac{p_A(\omega,t)}{p_B(\omega,t)}$ follows geometric Brownian motion. The reader may be surprised to see that the term μT rather than μt appears in the formula, i.e. $\frac{p_A(\omega,t)}{p_B(\omega,t)}$ follows a geometric Brownian motion with no drift. The reason is that the factor $e^{\mu T}$ in the payoff of the stock at time T just multiplies the value of the stock at all time periods by a constant factor. If we modified our utility function to incorporate time-discounting, identical drift factors would appear in p_A and p_B, so the ratio would be unchanged. We could obtain a nonzero drift in an infinite horizon model in which the agent discounts, and dividends are paid at all times and follow a geometric Brownian motion with drift.

Example 2. In this example, we modify Example 1 by changing the utility function in period T to the CARA function $\phi_2(c) = -e^{-\alpha c}$, for $\alpha > 0$. For simplicity, we also take $\mu_1 = 0$ and $\sigma_1 = 1$.

$$\begin{aligned}p_D(\omega,t) &= \int_{-\infty}^{\infty} \alpha e^{-\alpha e^{\beta(\omega,t)+\sqrt{T-t}x}} \max\left\{e^{\beta(\omega,t)+\sqrt{T-t}x}-X,0\right\} d\Phi(x)\\ &= \int_{\frac{\ln X-\beta(\omega,t)}{\sqrt{T-t}}}^{\infty} \alpha e^{-\alpha e^{\beta(\omega,t)+\sqrt{T-t}x}} e^{\beta(\omega,t)+\sqrt{T-t}x} d\Phi(x)\\ &\quad -X\int_{\frac{\ln X-\beta(\omega,t)}{\sqrt{T-t}}}^{\infty} \alpha e^{-\alpha e^{\beta(\omega,t)+\sqrt{T-t}x}} d\Phi(x)\\ &= \int_{\frac{\ln X-\beta(\omega,t)}{\sqrt{T-t}}}^{\infty} \alpha\left(\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}e^{(n+1)\beta(\omega,t)}e^{(n+1)\sqrt{T-t}x}\right)d\Phi(x)\\ &\quad -X\int_{\frac{\ln X-\beta(\omega,t)}{\sqrt{T-t}}}^{\infty} \alpha\left(\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}e^{n\beta(\omega,t)}e^{n\sqrt{T-t}x}\right)d\Phi(x)\\ &= \alpha\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}e^{(n+1)\beta(\omega,t)}\int_{\frac{\ln X-\beta(\omega,t)}{\sqrt{T-t}}}^{\infty} e^{(n+1)\sqrt{T-t}x}d\Phi(x)\\ &\quad -X\alpha\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}e^{n\beta(\omega,t)}\int_{\frac{\ln X-\beta(\omega,t)}{\sqrt{T-t}}}^{\infty} e^{n\sqrt{T-t}x}d\Phi(x)\\ &= \alpha\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}e^{(n+1)\beta(\omega,t)}e^{\frac{(n+1)^2(T-t)}{2}}\Phi\left(\frac{(n+1)\sqrt{T-t}}{2}-\frac{\ln X-\beta(\omega,t)}{\sqrt{T-t}}\right)\\ &\quad -X\alpha\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}e^{n\beta(\omega,t)}e^{\frac{n^2(T-t)}{2}}\Phi\left(\frac{n\sqrt{T-t}}{2}-\frac{\ln X-\beta(\omega,t)}{\sqrt{T-t}}\right)\\ &= \alpha\sum_{n=0}^{\infty}\frac{(-\alpha)^n}{n!}e^{n\beta(\omega,t)}\left[e^{\frac{(n+1)^2(T-t)}{2}}e^{\beta(\omega,t)}\Phi(z_n)-Xe^{\frac{n^2(T-t)}{2}}\Phi\left(z_n-\frac{\sqrt{T-t}}{2}\right)\right]\end{aligned}$$

$$p_A(\omega,t) = \alpha \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} e^{(n+1)\beta(\omega,t)} e^{\frac{(n+1)^2(T-t)}{2}}$$

$$p_B(\omega,t) = \alpha \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!} e^{n\beta(\omega,t)} e^{\frac{n^2(T-t)}{2}}$$

where

$$z_n = \frac{\ln e^{\beta(\omega,t)}/X}{\sqrt{T-t}} + \frac{(n+1)\sqrt{T-t}}{2}$$

The pricing formula is considerably more complex than in the Black-Scholes Geometric Brownian Motion case, but it can easily be evaluated numerically.

Example 3. In this example, we modify Example 1 by adding a stochastic endowment in the terminal period T. We can think of this stochastic endowment as arising from some asset, such as housing, which is outside the financial markets. We take $J = 1$, $K = 2$, and $e(\omega,T) = e^{\beta_2(\omega,T)}$, so the endowment is given by the terminal value of an exponential Browian motion for which there is no corresponding stock. The pricing formula becomes

$$\begin{aligned}
p_A(\omega,t) &= e^{\mu_1 T + \sigma_1\beta_1(\omega,t)} \int_{-\infty}^{\infty} \varphi_2'\left(F(t,\omega,x)\right) e^{\sigma_1\sqrt{T-t}x_1} d\Phi(x) \\
&= e^{\mu_1 T + \sigma_1\beta_1(\omega,t)} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{F(t,\omega,x)}} e^{\sigma_1\sqrt{T-t}x_1} d\Phi(x) \\
p_B(\omega,t) &= \int_{-\infty}^{\infty} \varphi_2'\left(F(t,\omega,x)\right) d\Phi(x) \\
&= \int_{-\infty}^{\infty} \frac{1}{2\sqrt{F(t,\omega,x)}} d\Phi(x) \\
\frac{p_A(\omega,t)}{p_B(\omega,t)} &= \frac{e^{\mu_1 T + \sigma_1\beta_1(\omega,t)} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{F(t,\omega,x)}} e^{\sigma\sqrt{T-t}x_1} d\Phi(x)}{\int_{-\infty}^{\infty} \frac{1}{2\sqrt{F(t,\omega,x)}} d\Phi(x)} \\
p_D(\omega,t) &= \int_{-\infty}^{\infty} \frac{\max\{0, e^{\mu_1 T + \sigma_1\beta_1(\omega,t) + \sigma_1\sqrt{T-t}x_1} - X\}}{2\sqrt{F(t,\omega,x)}} d\Phi(x) \\
\frac{p_D(\omega,t)}{p_B(\omega,t)} &= \frac{\int_{-\infty}^{\infty} \frac{\max\{0, e^{\mu_1 T + \sigma_1\beta_1(\omega,t) + \sigma_1\sqrt{T-t}x_1} - X\}}{2\sqrt{F(t,\omega,x)}} d\Phi(x)}{\int_{-\infty}^{\infty} \frac{1}{2\sqrt{F(t,\omega,x)}} d\Phi(x)}
\end{aligned}$$

where

$$F(t,\omega,x) = e^{\beta_2(\omega,t) + \sqrt{T-t}x_2} + \left(e^{\mu_1 T + \sigma_1\beta_1(\omega,t) + \sigma_1\sqrt{T-t}x_1}\right)$$

Notice that the inclusion of any nonzero endowment in the final period, whether stochastic or deterministic, precludes the simplification that was possible in Example 1. In particular, $\frac{p_A}{p_B}$ is *not* geometric Brownian motion and the price of the option is *not* given by the conventional Black-Scholes formula. Note also that the price of the stock A and the option D depend not only on the Brownian motion β_1 that determines the final stock dividend, but also the Brownian motion β_2, which is independent of the dividends of the stock.

In a market in which only the stock and the bond are traded, markets are clearly dynamically incomplete. In this setting, the option is *not* redundant. Because the stock and its derivative both fluctuate in response to changes in β_2 as well as to changes in β_1, this creates the possibility of replicating β_2 by making trades among the the bond, the stock, and the option. We conjecture that the inclusion of the option dynamically completes the market.

Note that the option price is given in closed form that can easily be evaluated numerically. Assuming one knows $\beta(\omega, t)$, one simply has to do two numerical integrations with respect to the Gaussian distribution; it is not necessary to simulate a random walk process. Alternatively, one can compute the stock price $p_A(\omega, t)/p_B(\omega, t)$ numerically as a function of $\beta(\omega, t)$. One can then compute the option price as a function of the price ratio p_A/p_B, the strike price of the option, and $T - t$, given the utility function and the final period endowment.

Example 4. Suppose we modify Example 1 by eliminating the bond. Thus, we consider a model in which there is one stock, no bond, and a European call option on the stock. This does not fall under Theorem 1 as stated, but essentially the same argument guarantees that equilibrium exists.[8] The equilibrium trading strategy prescribes that the agent holds zero units of the bond at all (ω, t); accordingly, the equilibrium consumptions and stock and option holdings in this example are the same as those in Example 1. However, in this example, there is no market for the bond. If we consider the market with the stock but not the bond or the option present, markets are dynamically incomplete and there is no trading strategy involving only the stock which will replicate the option. It is not possible to price the option by a pure arbitrage argument on the stock; equivalently, the martingale method identifies infinitely many possible pricing processes. Nonetheless, in the market with the stock and the option both present, the option has a unique *equilibrium* price. The prices are exactly the same as in Example 1. The pricing formula is

$$\begin{aligned}
p_A(\omega, t) &= \frac{e^{(\mu T + \sigma\beta(\omega,t))/2 + \sigma^2(T-t)/8}}{2} \\
p_D(\omega, t) &= \frac{e^{(\mu T + \sigma\beta(\omega,t))/2 + \sigma^2(T-t)/8}}{2} \varPhi\left(\frac{\ln(e^{\mu T + \sigma\beta(\omega,t)}/X) + \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right) \\
&\quad - \frac{X e^{-(\mu T + \sigma\beta(\omega,t))/2 + \sigma^2(T-t)/8}}{2} \varPhi\left(\frac{\ln(e^{\mu T + \sigma\beta(\omega,t)}/X) - \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

where $\varPhi$ is the cumulative distribution function of the standard 1-dimensional normal.

[8] The only change, other than notation, is that in the construction of the consumption plan $\hat{\hat{c}}$ and trading strategy $\hat{\hat{y}}$, one must buy or sell units of the stock rather than units of the bond.

Example 5. Here, we modify Example 1 by considering an option with exercise date $T_1 < T$. The strike price of the option is X_1 units of the bond, so we have

$$H_1(p_B(\omega,T_1),p_A(\omega,T_1)) = \max\{p_A(\omega,T_1) - X_1 p_B(\omega,T_1)\}$$

The pricing processes p_A, p_B and p_A/p_B are unchanged. For $t \leq T_1$, we have

$$\begin{aligned} p_{D_1}(\omega,t) &= E\left(\max\{p_A(\omega,T_1) - X_1 p_B(\omega,T_1),0\}|\,\mathcal{F}_t\right) \\ \frac{p_{D_1}(\omega,t)}{p_B(\omega,t)} &= \frac{E\left(\max\{p_A(\omega,T_1) - X_1 p_B(\omega,T_1),0\}|\,\mathcal{F}_t\right)}{p_B(\omega,t)} \end{aligned}$$

Thess can easily be evaluated as two-dimensional numerical integrations. Notice that

$$\begin{aligned} \beta(\omega,T) &= \beta(\omega,T_1) + \sqrt{T-T_1}x \\ &= \beta(\omega,t) + \sqrt{T_1 - t}y + \sqrt{T-T_1}x \end{aligned}$$

where x and y are independent standard normals. Thus, if we fix $\beta(\omega,t)$, then each value of y determines a value of $\beta(\omega,T_1)$, so that the price ratio $\frac{p_A(\omega,T_1)}{p_B(\omega,T_1)}$ and the option payout can be computed as a numerical integration with respect to x. Thus, the value of the option at time t can be computed as a two-dimensional numerical integration over the system normal pair (x,y). Note that this is not a *simulation* of the option price; it is a numerical calculation of the *exact* equilibrium price of the option. The equilibrium option prices show some of the qualitative features of actual option prices, including a volatility smile (Anderson, Diasakos and Raimondo [4]).

3.3 Proof

Up to now, all of our definitions and results have been stated without any reference to nonstandard analysis. Our proof makes extensive use of nonstandard analysis, in particular Anderson's construction of Brownian Motion and the Itô Integral ([1]) and Lindström's extension of that construction to stochastic integrals with respect to L^2 martingales [35, 36, 37, 38]. It is beyond the scope of this paper to develop these methods; Anderson [3] and Hurd and Loeb [33] are suitable references.

We construct our probability space, filtration and Brownian Motion following Anderson's construction [1]. Specifically, we construct a hyperfinite economy as in Raimondo [47], with the following changes to incorporate the derivatives $D_1,\ldots,D_M$:

1. For all $\omega \in \hat{\Omega}$, define $\hat{A}(\omega,t) = \hat{B}(\omega,t) = 0$ for all $t < \hat{T}$, $\hat{A}(\omega,\hat{T}) = e^{\mu\hat{T}+\sigma\cdot\hat{\beta}(\omega,\hat{T})}$ (i.e. $\hat{A}_j(\omega,\hat{T}) = e^{\mu_j\hat{T}+\sigma_j\cdot\hat{\beta}(\omega,\hat{T})}$, $j=1,\ldots,J$) and $\hat{B}(\omega,\hat{T}) = 1$. Define $A(\omega,t) = B(\omega,t) = 0$ for $T \in [0,T)$, $A(\omega,T) = e^{\mu T+\sigma\cdot\beta(\omega,T)}$, $B(\omega,T) = 1$. Note that $A(\omega,T) = {}^\circ\hat{A}(\omega,\hat{T})$ for μ-almost all ω.
2. In Anderson and Raimondo [5], the derivatives all paid off only at time T. They were treated like conventional securities, paying off in units of time-T consumption. Here, the derivative D_m pays off at time T_m in monetary units of the

bond, rather than units of consumption. Moreover, the payoff at time T_m is a function of the endogenously determined equilibrium prices. Let $\hat{T}_m = \frac{\lfloor nT_m \rfloor}{n}$. Let

$$\hat{D}_m(\omega,t) = \begin{cases} 0 & \text{if } t \neq \hat{T}_m \\ {}^*H_m\left(\left(\hat{p}_B(\omega,\hat{T}_m),\hat{p}_{A_1}(\omega,\hat{T}_m),\ldots,\hat{p}_{A_m}(\omega,\hat{T}_m)\right)\right) & \text{if } t = \hat{T}_m \end{cases}$$

where this payoff is measured in monetary units. Our existence proof will need to be adjusted to deal with this situation.

3. A security price is an internal function $\hat{p} = (\hat{p}_A,\hat{p}_B,\hat{p}_D) : \mathcal{T}\times\hat{\Omega} \to {}^*\mathbf{R}^J_+ \times {}^*\mathbf{R}_+ \times {}^*\mathbf{R}^M_+$ such that $\hat{p}(t,\cdot)$ is $\hat{\mathcal{F}}_t$-measurable. A consumption price is an internal function $\hat{p}_C : \mathcal{T}\times\hat{\Omega} \to {}^*\mathbf{R}_+$.
4. A trading strategy is $\hat{z} = (\hat{z}_A,\hat{z}_B,\hat{z}_D) : \mathcal{T}\times\hat{\Omega} \to {}^*\mathbf{R}^J \times {}^*\mathbf{R} \times {}^*\mathbf{R}^M$ which is $\hat{\mathcal{F}}_t$-measurable.
5. An equilibrium for the economy is a security price process $\hat{p}$, a consumption price process $\hat{p}_C$, a trading strategy $\hat{z}$ and a consumption plan $\hat{c}$ which lies in the demand set so that the securities and goods markets clear, i.e. for all ω

$$\begin{aligned} \hat{z}_A(\omega,t) &= \mathbf{1} \text{ for all } t \in \mathcal{T} \\ \hat{z}_B(\omega,t) &= 0 \text{ for all } t \in \mathcal{T} \\ \hat{z}_D(\omega,t) &= 0 \text{ for all } t \in \mathcal{T} \\ \hat{c}(\omega,t) &= 1 \text{ for all } t < \hat{T} \\ \hat{c}(\omega,\hat{T}) &= \hat{e}(\omega,\hat{T}) + \sum_{j=1}^{J} e^{\mu\hat{T}+\sigma_j\cdot\hat{\beta}(\omega,\hat{T})} \end{aligned}$$

Theorem 2. *The hyperfinite economy just described has an equilibrium. The pricing process is given by*

$$\begin{aligned} \hat{p}_{A_j}(\omega,t) &= e^{\mu_j\hat{T}+\sigma_j\cdot\hat{\beta}(\omega,t)} \int_{{}^*\mathbf{R}^K} {}^*\varphi_2'\left(\hat{F}(t,\omega,x)\right) e^{\sqrt{\hat{T}-t}\sigma_j\cdot x} d\hat{\Phi}(x) \\ \hat{p}_B(\omega,t) &= \int_{{}^*\mathbf{R}^K} {}^*\varphi_2'\left(\hat{F}(t,\omega,x)\right) d\hat{\Phi}(x) \\ \hat{p}_{D_m}(\omega,t) &= E\left({}^*H_m\left(\hat{p}_B(\omega,T_m),\hat{p}_{A_1}(\omega,T_m),\ldots,\hat{p}_{A_J}(\omega,T_m)\right) \middle| \hat{\mathcal{F}}_t\right) \\ &\quad \textit{if } t \leq \hat{T}_m \\ &= 0 \textit{ if } t > \hat{T}_m \\ \hat{p}_C(\omega,t) &= \varphi_1'(1) \textit{ for } t < \hat{T} \\ \hat{p}_C(\omega,\hat{T}) &= {}^*\varphi_2'(\hat{F}(\hat{T},\omega,0)) \end{aligned}$$

where

$$\hat{F}(t,\omega,x) = {}^*\rho\left(\hat{\beta}(\omega,t) + \sqrt{\hat{T}-t}x\right) + \sum_{j=1}^{J} e^{\mu_j\hat{T}+\sigma_j\cdot\left(\hat{\beta}(\omega,t)+\sqrt{\hat{T}-t}x\right)}$$

and $\hat{\Phi}$ is the cumulative distribution function of the K-dimensional normalized binomial distribution, each of whose components is distributed as

$$\sqrt{\frac{4\Delta T}{\hat{T}-t}}{}^{*}b\left(\left(\frac{\hat{T}-t}{\Delta T},\frac{1}{2}\right)-\frac{\hat{T}-t}{2\Delta T}\right)$$

Proof.. The analogous results in Raimondo [47] (Theorem 3.1) and Anderson and Raimondo [5] (Theorem 3.1) relied on Radner's [45] theorem on the existence of equilibrium in incomplete markets with a short-sale constraint. Because the derivative payoffs depend on the endogenously determined prices, our hyperfinite model does not satisfy the assumptions of Radner's Theorem. Accordingly, we make strong use of the fact that our economy has a single agent. This allows us to drop the short-sale condition, at the cost of making a generalization to the multi-agent case more difficult.

Because we have a single-agent economy, we know that the agent's consumption at each node (ω, t) with $t < \hat{T}$ must just equal the agent's endowment; moreover, the agent's consumption at each node $(\omega, \hat{T})$ is just $\hat{e}(\omega, \hat{T}) + \sum_{j=1}^{J} \hat{A}_j(\omega, \hat{T})$. Since the bond and the derivatives are in zero net supply, they do not affect consumption at equilibrium. Set the price of consumption at each node to equal the marginal utility of consumption at that node, computed at the eqilibrium consumption level, and value all securities at node (ω, t) as the expected value of their dividends at that node and all successor nodes. Since the stocks and the bond pay off in units of consumption, this completely defines their prices and determines the payoffs of the derivatives. Since the derivatives pay off in monetary units, we value each derivative at a node as the expected value of its monetary payoff at that node and all successor nodes.

These prices satisfy the equations in the statement of Theorem 2. The proof of Theorem 3.1 in Raimondo [47] showed that equilibrium prices must satisfy specific first-order conditions that yield these same equations. Turning that argument around, at the prices we have specified, the equilibrium security holdings ($\hat{z}_A = \mathbf{1}$, $\hat{z}_B = 0$, $\hat{z}_D = 0$) satisfy the first-oder conditions to maximize the representative agent's utility function over his/her budget set. Since the representative agent's utility function is globally concave, the first-order conditions are sufficient for utility maximization. This shows that we have an equilibrium with the prices specified in the statement of the Theorem and completes the proof.

Proposition 1. . *Suppose* $a \in \mathrm{ns}({}^*\mathbf{R}^K)$, $r \in \mathbf{R}$, $r \geq 1$. *Let*

$$f_a(x) = e^{a \cdot x} = e^{a_1 x_1 + \cdots + a_K x_K}$$

Then $f_a \in SL^r({}^*\mathbf{R}^K, d\hat{\Phi})$.

Proof.. Raimondo [47] proves this in the case $r = 1$. For the general case, note that

$$(f_a(x))^r = e^{r a_1 x_1 + \cdots + r a_K x_K}$$

Theorem 3. *Suppose that* $\varphi_2'(c) = O(1/c^r)$ *as* $c \to 0$ *and* $|H_m(x)| \leq \max\{S_m, |x|^r\}$, *for some* $r \in \mathbf{R}$. *Then for* μ*-almost all* ω, *the equilibrium pricing process satisfies*

$$\begin{aligned}
{}^\circ\left(\hat{p}_{A_j}(\omega,t)\right) &= e^{\mu_j T+\sigma_j\cdot\beta(\omega,{}^\circ t)}\int_{-\infty}^{\infty}\varphi_2'\left(F({}^\circ t,\omega,x)\right)e^{\sqrt{T-{}^\circ t}\sigma_j\cdot x}d\Phi(x)\\
{}^\circ\left(\hat{p}_B(\omega,t)\right) &= \int_{-\infty}^{\infty}\varphi_2'\left(F({}^\circ t,\omega,x)\right)d\Phi(x)\\
{}^\circ\left(\hat{p}_C(\omega,t)\right) &= \varphi_1'(1)\\
{}^\circ\left(\hat{p}_C(\omega,\hat{T})\right) &= \varphi_2'\left(F(T,\omega,0)\right)\\
{}^\circ\left(\hat{p}_{D_m}(\omega,t)\right) &= E\left(H_m\left({}^\circ\hat{p}_B(\omega,\hat{T}_m),{}^\circ\hat{p}_{A_1}(\omega,\hat{T}_m),\ldots,{}^\circ\hat{p}_{A_J}(\omega,\hat{T}_m)\right)\middle|\,\mathcal{F}_{{}^\circ t}\right)\\
&\quad \textit{if}\; t\le\hat{T}_m\\
&= 0\;\textit{if}\; t>\hat{T}_m
\end{aligned}$$

for all $t \in \mathcal{T}$, where Φ is the cumulative distribution function of the standard K-dimensional normal.

Proof.. The proofs for p_A and p_B are essentially the same as in Raimondo [47]. As part of that proof, Raimondo shows that

$$\hat{p}_A(\cdot,\hat{T}),\hat{p}_B(\cdot,\hat{T})\in SL^2({}^*\mathbf{R}^K,d\hat{\Phi})$$

Using Proposition 1 in that argument, we obtain that

$$\hat{p}_A(\cdot,\hat{T}),\hat{p}_B(\cdot,\hat{T})\in SL^{2r}({}^*\mathbf{R}^K,d\hat{\Phi})$$

$\hat{\Phi}$ is the measure induced on ${}^*\mathbf{R}^K$ by the internal measure $\hat{\mu}$ on $\hat{\Omega}$, so

$$\hat{p}_A(\cdot,\hat{T}),\hat{p}_B(\cdot,\hat{T})\in SL^{2r}(\hat{\Omega},\hat{\mu})$$

Since

$$\hat{p}_A(\cdot,\hat{T}_m)=E\left(\hat{p}_A(\cdot,\hat{T})\middle|\,\hat{\mathcal{F}}_{\hat{T}_m}\right)\text{ and }\hat{p}_B(\cdot,\hat{T}_m)=E\left(\hat{p}_B(\cdot,\hat{T})\middle|\,\hat{\mathcal{F}}_{\hat{T}_m}\right)$$

we have

$$\hat{p}_A(\cdot,\hat{T}_m),\hat{p}_B(\cdot,\hat{T}_m)\in SL^{2r}(\hat{\Omega},\hat{\mu})$$

Since $H_m(x)\le\max\{S_m,|x|^r\}$, we have

$${}^*H_m\left(\hat{p}_B(\cdot,\hat{T}_m),\hat{p}_{A_1}(\cdot,\hat{T}_m),\ldots,\hat{p}_{A_J}(\cdot,\hat{T}_m)\right)\in SL^2(\hat{\Omega},\hat{\mu})$$

Therefore, if $t\le\hat{T}_m$,

$$\begin{aligned}
{}^\circ\hat{p}_{D_m}(\omega,t) &= {}^\circ E\left({}^*H_m\left(\hat{p}_B(\omega,\hat{T}_m),\hat{p}_{A_1}(\omega,\hat{T}_m),\ldots,\hat{p}_{A_J}(\omega,\hat{T}_m)\right)\middle|\,\hat{\mathcal{F}}_t\right)\\
&= E\left(H_m\left({}^\circ\hat{p}_B(\omega,\hat{T}_m),{}^\circ\hat{p}_{A_1}(\omega,\hat{T}_m),\ldots,{}^\circ\hat{p}_{A_J}(\omega,\hat{T}_m)\right)\middle|\,\mathcal{F}_t\right)
\end{aligned}$$

Theorem 4. *$\hat{p}_A$, $\hat{p}_B$ and $\hat{p}_D$ are internal almost surely S-continuous SL^2 martingales with respect to the internal filtration $\{\hat{\mathcal{F}}_t\}$. If we define*

$$\begin{aligned}
p_A(\omega,t) &= {}^\circ\hat{p}_A(\omega,\hat{t})\\
p_B(\omega,t) &= {}^\circ\hat{p}_B(\omega,\hat{t})\\
p_D(\omega,t) &= {}^\circ\hat{p}_D(\omega,\hat{t})
\end{aligned}$$

for $t \in [0, T]$, then p_A, p_B and p_D are almost surely continuous square integrable martingales with respect to the filtration $\{\mathcal{F}_t\}$.[9] *Moreover, for $t \le T_m$,*

$$p_{D_m}(\omega, t) = E\left(H_m\left(p_B(\omega, T_m), p_{A_1}(\omega, T_m), \ldots, p_{A_J}(\omega, T_m)\right) \middle| \mathcal{F}_t\right)$$

Proof.. The proof that $\hat{p}_A$ is an S-continuous SL^2 martingale and p_A is a continuous square-integrable martingale is given in Raimondo [47]. The proofs for $\hat{p}_B$, p_B, and the proofs that $\hat{p}_D$ is an SL^2-martingale and p_D is a square-integrable martingale are essentially the same. The S-continuity of $\hat{p}_D$ and the continuity of p_D follow from the S-continuity of $\hat{p}_a$ and $\hat{p}_B$. From Theorem 3, if $t \le \hat{T}_m$,

$$\begin{aligned} p_{D_m}(\omega, t) &= {}^{\circ}\hat{p}_{D_m}(\omega, \hat{t}) \\ &= E\left(H_m\left({}^{\circ}\hat{p}_B(\omega, \hat{T}_m), {}^{\circ}\hat{p}_{A_1}(\omega, \hat{T}_m), \ldots, {}^{\circ}\hat{p}_{A_J}(\omega, \hat{T}_m)\right) \middle| \mathcal{F}_t\right) \\ &= E\left(H_m\left(p_B(\omega, T_m), p_{A_1}(\omega, T_m), \ldots, p_{A_J}(\omega, T_m)\right) \middle| \mathcal{F}_t\right) \end{aligned}$$

Proof of Theorem 1: The remainder of the proof is essentially identical to the proof of Theorem 2.1 in Raimondo [47]. ■

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[9] Since $\hat{p}_{D_m}(\omega, t) = 0$ for $t > \hat{T}_m$, $\hat{p}_{D_m}$ is an S-continuous martingale only on $\{t \in \mathcal{T} : 0 \le t \le \hat{T}_m\}$ and p_{D_m} is a continous martingale only on $[0, T_m]$.

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4

Adaptive Contracting: The Trial-and-Error Approach to Outsourcing*

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Summary. Adaptive contracting occurs when a principal experiments with the delegation of authority through leaving contracts incomplete. We highlight two potential benefits of adaptive contracting: First, the delegation of authority can be advantageous even if the agent acts opportunistically, since expected private benefits will be shared between the parties through price negotiation. Second, the principal extracts information from experimenting with delegation of authority and we identify a positive option value embodied in the principal's ability to extend or withdraw the delegated authority in future contracting periods.

Key words: Incomplete contracting, Trial and error, Authority, Outsourcing, Procurement.

JEL Classification Numbers: D72, L33, L97.

4.1 Introduction

Writing procurement contracts between a government institution and a private supplier is in general a complicated affair. The costs and benefits for the parties involved - including the public - have many components and a government institution normally has limited organizational resources and many political and economical objectives to satisfy.[3]

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[3] James Q. Wilson (1989) provides a classical study of the workings of local and national government institutions.

How detailed should such contracts be? How much time and effort should a public agency spend on analyzing all the contractual details before the contract is signed? What are the effects on the service provider and the service provided by not describing all possible details in the contract? When contracts are left incomplete, the agent will have the authority to decide on matters not specified in the contract. In this paper we provide a formal analysis of *adaptive contracting*, where the government experiments with the degree of contractual incompleteness.

We show that adaptive contracting can be optimal, because it entails an option value for the government when a relatively incomplete contract is written. The contractor's opportunistic response to the increased authority may increase or decrease the total value of the relationship in a way which is impossible to learn exactly without trying out this delegation in reality. The option value embodied in an adaptive contracting strategy arises from the government's ability to revise the contract, whenever the contractor's opportunistic behaviour is decremental to the total value of the relationship. Adaptive contracting functions like a *trial-and-error* approach.

Government agencies around the world frequently apply an adaptive strategy when making outsourcing contracts in practice (see e.g. Wilson (1995) and World Bank (1995)). An illustrative case is the outsourcing of local bus transportation in the Copenhagen region in Denmark.[4] The costs and benefits for the parties involved in a contract covering outsourcing of local bus operation are multidimensional including at least three central areas: First, there is the organization of the service provided, i.e. the routes to be serviced, the frequencies of buses, the number of stops, etc. etc. Secondly, there are a number of cost related factors, e.g. the age and standard of the buses used, the quality and service provided to the passengers in a given bus, the environmental standard of the buses and the service. Finally, an important issue is the length of contracting and the conditions for terminating a contract, e.g. to which degree should long lasting investment costs be reimbursed if a contract is ended.

In 1990 the first contracting round was based on small and very incomplete contracts for a limited number of bus operations and a limited number of years. The authorities and the bus operators learned from these trial contracts and the experience obtained from giving the bus operators large authority has been the foundation for later contract revisions.[5] Thus, essentially the authorities use an adaptive contracting strategy.

Adaptive contracting may seem an obvious way to save on transaction costs in a setting with limited resources to write contracts, which surely is the case for many public and private organizations (see e.g. Williamson (1979) and (1985) and Wil-

[4] Private bus operation on local routes has a long tradition in Denmark where public and private operated bus routes have coexisted for more than fifty years. The first significant outsourcing of previous public operated local bus operations was implemented in the Copenhagen area in 1990 partly triggered by new EC rules. Today, almost 80 pct. of the bus transportation is outsourced to private operators.

[5] A visible indicator of this trial and error approach is the number of pages in the contracting material that the bus operators make their offer on. In the first round in 1990, the contract were appr. 30 pages long, where as the number of pages exceeded 300 in the 12. and latest round in 2002 (see HUR 1990-2002).

son (1989)). However, our argument is different, we show that if certain production modes are non-verifiable ex-post, then adaptive contracting may be beneficial even in the absence of direct transaction costs, since it generates information about the agents' response to increased authority.

The existing theoretical contract literature has not focused on the option value of adaptive contracting. Aghion and Tirole (1997) analyzes how the incentives to collect information as well as the congruence of preferences of the procurer and the contractor determines the optimal allocation of the decision authority in a setting where contracting is done once. They make the important distinction between real and formal authority, both of which can be allocated freely among the contracting parties. In our model, formal authority is the right to decide which contract is offered to the agent. Formal authority cannot be delegated, it always belongs to the government. Real authority is the right to choose which action are implemented in situations where the contract is silent, we sometimes denote this decision authority.

They also discuss the possibility that a principal makes what they call "contingent delegation", i.e. makes clear that he may retain authority at a future date. They find that contingent delegation is intermediate in between delegation and no delegation in terms of initiative and loss of control for the principal. They do not consider the option value, which we focus upon.

Part of the contracting literature has focused on optimal contracting in view of informational asymmetries, see e.g. Laffont and Tirole 1991 and 1993 for contributions related to outsourcing of public goods or services. This approach assumes that all relevant observable details are built into the contract.

Alternatively, the *incomplete* contracting literature assumes that contracts are necessarily incomplete (see Hart 1995). This approach takes as a premise that certain actions are observable but non-verifiable and therefore cannot be built into a contract. Hence, the gabs in existing contracts reflect issues which are essentially non-contractible.[6] We extend this literature by showing that given the external restrictions on contracting, it may be optimal deliberately to leave gaps in contracts in order to explore the outcome of delegating authority.

One strand of the contracting literature has explicitly focused on the impact of renegotiating multi-period contracts. Dewatripont (1989) shows that renegotiation can be costly if it blocks the contracting parties' ability to commit to contracts which are ex-ante efficient but ex-post inefficient in settings with asymmetric information. Another cost identified with renegotiation is the *ratchet effect* which is present when information is revealed gradually over time (see e.g. Laffont and Tirole (1988) and Hart and Tirole (1988)). Our contribution in relation to this literature is to identify the option value embodied in using short term contracts that allows for renegotiation.

[6] Hart, Shleifer and Vishny (1997) shows that non-verifiability in contracting can have important consequences for outsourcing of public services. In Bennedsen and Schultz (2004) we extend Hart, Shleifer and Vishny (1997)'s incomplete contracting approach to outsourcing of public service and analyze the interaction between a government's incentive to outsource and the degree of competition in the private market for the good or service.

The paper also relates to a large organizational literature analyzing adaptive behavior in organizational design (see for instance the case study by Van de Ven and Polley (1992) and the special issue of *Organizational Design* Vol.2, No.1, (1990)). In economics, the focus has been on adaptive behavior in economic decision making. One approach to this has been to view optimal rational behavior as the outcome of a converging series of trial and errors as pointed out by e.g. Smith (1982) and Lucas (1986).[7] Finally, our analysis contributes to the literature on outsourcing and privatization. The welfare effects of outsourcing is well documented both theoretically (see Laffont and Tirole (1991), Shapiro and Willig (1993), Shleifer and Vishny (1994), Schmidt (1996) and Bennedsen (1999) for different theoretical models of the welfare consequences of privatization and Shleifer (1998) for a general survey) and empirically (see Vickers and Yarrow (1988), La Porta *et.al* (1997) and World Bank (1995) and (1997)).

The rest of the paper is organized as follows: Section 2 presents the model. Section 3 analyzes the model under the assumption that there are no inter-period contract renegotiation. Section 4 analyzes the optimal use of adaptive contracting when contract can be renegotiated. Finally, Section 5 concludes.

4.2 The model

We analyze a simple two period model. There are two players, a service buying government, G, and a private service provider, S. The price paid by the government for the service is p. We assume that G decides on contract length and type, but that the price is determined through bargaining, such that the parties split the generated surplus equally within each period.

The service provider can organize production in a number of ways. The first two are standard modes, yielding repectively high and low quality, and can be described ex-ante and verified ex-post in a court.[8] If G picks such a contract, we say that it keeps (decision) authority, since it then effectively decides the production mode.

The government's payoff, U^g, and service provider's payoff, U^s, from a complete contract with price, p, are given by:

[7] Lucas states that "Technically, I think of economics as studying decision rules that are steady states of some adaptive process, decision rules that are found to work over a range of situations and hence are no longer revised appreciably as more experience accumulates," (Lucas 1986, p.402).

[8] Notice, we assume that production modes can be verifiable to a court but not the quality level as such. However if a verifiable production mode has been used and it is known from prior experience which quality level is associated with this particular production mode, it is as if the court can verify the quality level directly. We will, thus, occasionally write the verifiable high and low quality contract as a shortcut for a contract specifying the verifiable standard production modes, generating a high or low quality service.

$$U^g = \begin{cases} B - p & \text{if a low quality production mode is chosen,} \\ B + \Delta B - p & \text{if a high quality production mode is chosen,} \end{cases}$$

$$U^s = \begin{cases} p - C & \text{if a low quality production organization is chosen,} \\ p - C - \Delta C & \text{if a high quality production organization is chosen.} \end{cases}$$

where benefits, $B \geq 0$, $\Delta B > 0$, and costs, $C \geq 0$ and $\Delta C > 0$.

The government can refrain from specifying the production mode precisely. Then *authority* - in the form of decision rights with respect to the production mode - is given to the service provider, S. S can then choose one of the two verifiable modes, or one of $n \geq 2$ other unverifiable modes. The first of these we denote the (expected) best alternative production mode and it has ex-ante unknown pay-off consequences for both G and S. This (expected) best alternative (which we henceforth denote the alternative production mode) can be to test the newest undocumented technology, some new materials or new ways of organizing the labor input. Since, the alternative technology is unproven and learned through implementation, the alternative approach cannot be described ex-ante or verified ex-post, and the pay-off consequences can only be learned through implementation in practice. In short, the alternative non-verifiable production mode gives rise to a third non-contractible, quality level. The payoffs associated with the alternative production mode are:

$$\begin{aligned} U^g &= B - p + R_g, \\ U^s &= p - C + R_s, \\ \text{where } R_g &= \alpha\mu + r_g \text{ with } r_g \sim UD[-\bar{r}_g, \bar{r}_g] \\ \text{and } R_s &= \mu + r_s \text{ with } r_s \sim UD[-\bar{r}_s, \bar{r}_s]. \end{aligned}$$

Here r_s is a stochastic variable, uniformly distributed on $[-\bar{r}_s, \bar{r}_s]$ and $\mu \geq 0$. To make the analysis below interesting we assume that $\bar{r}_s > \mu$. Similarly r_g is uniform on $[-\bar{r}_g, \bar{r}_g]$. The parameter $\alpha \in [-1, 1]$ measures the alignment of G's and S's *expected* utility with respect to the alternative service organization. The model is thus general enough to capture the cases of independent preferences (α is zero), positively aligned preferences in expectation (α is positive) or negatively aligned preferences in expectation (α is negative).[9] We assume that R_g and R_s are observable but non-verifiable ex-post and that if the parties negotiate in a future period, they cannot write a complete contract specifying that the alternative production mode should be used, even in the case where it has been used in the first period.[10]

[9] The combined restriction that $|\alpha| \leq 1$ and $\mu \geq 0$ implies that the alternative organization method in a one period version of the model always provides in expectation at least as much surplus as the basic quality service. This assumption reduces the number of cases below; however, it does not change the fundamental results of the paper and it is straightforward - but notationally cumbersome - to analyze cases where either $\mu < 0$ and/or $|\alpha| > 1$.

[10] Alternatively, we could have assumed that it is possible to contract on the alternative quality level if it has been used in a previous period. As we briefly discuss in Section 4 below, this would increase the incentives to use adaptive contracting and thus strengthen the key insight from our analysis.

The $n-1$ other non-verifiable production modes have expected surpluses which are less than the best alternative mode. To simplify we assume that they have known payoff consequences and the minimum cost of delivering the service using any of these other unverifiable production modes is C. Furthermore, we assume that the mode that delivers the service at cost C also yields a benefit B for G. In short they are unattractive.[11]

The timeline is as follows: First G decides on the length of the contract (one or two periods) and on whether to choose a complete contract specifying a (standard) quality level or to delegate authority through leaving the contract incomplete. If the contract is complete, the service provider then delivers the service using the desired production mode. If the contract is incomplete, the service provider then decides the production mode. If he chooses the alternative mode, he delivers the service and the associated payoffs are revealed.

To simplify we make the following assumption:

Assumption 1

$$\Delta H \equiv \Delta B - \Delta C \geq (1+\alpha)\mu.$$

Assumption 1 states that the extra social welfare generated from a high rather than low quality contract is higher than the expected extra social welfare from the alternative production mode. Since $(1+\alpha)\,\mu \geq 0$, Assumption 1 also implies that the high quality contract is better than the low quality contract. The assumption thus strengthen our focus on delegation of authority through adaptive contracting, since in the absence of any dynamic authority issues, G would write a complete contract specifying a high quality.

Finally, if contracts are complete, the service provider has no other incentives or possibilities to investigate the alternative production mode. The true values of r_s and r_g, can therefore only be learned through implementation.

The model has two periods as just described. No new relevant information is revealed to the parties in the period before the service provider delivers the service; hence, they have no incentive to renegotiate the contract within the period. In addition, notice that there is no private information in the model. The utility consequences of the different modes of production are common knowledge (if known to somebody). Hence there are no incentives for the service provider to act strategically in order to manipulate beliefs, i.e. *signaling* is not an issue in our model.

[11] These other production modes play no part in the analysis below, however, they secure that G cannot de facto make the alternative contract verifiable to a court by specifying in a contract that neither of the two verifiable standard modes must be used. Notice, none of the general insights of the paper is driven by this assumption, however, to conduct the analysis we need to know which production modes the agent prefer.

4.3 Delegating authority in the absence of inter-period contract revision

We begin the analysis by assuming that contracts can only be written at the start of period 1, so the government cannot change the contract in the start of period 2. This provide a benchmark to which we compare when later allowing for contract revision in the start of period 2. The Perfect Bayesian equilibrium outcome of this restricted outsourcing game is characterized in Proposition (1):

Proposition 1. *Let,*

$$\overline{\Delta H} \equiv \frac{1}{2}(1+\alpha)\mu + \frac{\bar{r}_s + \mu}{4\bar{r}_s}((\frac{1}{2}+\alpha)\mu + \frac{1}{2}\bar{r}_s) \tag{4.1}$$

The equilibrium of the restricted contracting game implies that
a) If $\Delta H \geq \overline{\Delta H}$, the government chooses a complete contract specifying high quality.
b) If $\Delta H \leq \overline{\Delta H}$, the government chooses an incomplete contract. The service provider chooses the alternative production mode in period 1. In period 2, he chooses the alternative production mode if R_s is non-negative and the low quality service otherwise.

Proof.. Since p is set such that the expected total surplus are distributed equally among G and S, G offers the contract that generates the highest expected total surplus.

First, we compare expected total surplus, ETS , from a contract specifying high quality in both periods with a contract specifying high quality in the first period and leaving authority to S in the second:

$$\begin{aligned} ETS_{hh} &= 2(B - C + \Delta H), \\ ETS_{ha} &= 2B - 2C + \Delta H + E\{R_g + R_s\} = 2B - 2C + \Delta H + (1+\alpha)\mu, \end{aligned} \tag{4.2}$$

where $E\{R_s + R_g\}$ denotes the expected value of $R_s + R_g$. By assumption $\Delta H > (1+\alpha)\mu \geq 0$, hence, $ETS_{hh} > ETS_{ha}$. Expected pay-offs from the high quality contract are $U^g_{hh} = U^s_{hh} = B - C + \Delta H$.

Second, if G delegates authority, S effectively chooses between providing low quality service in both periods, low quality in the first period and the alternative production mode in the second period or the alternative production mode in both periods. The expected total surplus generated from the first two options are,

$$\begin{aligned} ETS_{ll} &= 2(B - C), \\ ETS_{la} &= 2(B - C) + E\{R_s + R_g\} = 2(B - C) + (1+\alpha)\mu \geq ETS_{ll}. \end{aligned}$$

Comparing with equation (4.2) it is seen that both cases are dominated by writing a high quality contract in both periods.

If S picks the alternative service organization in period 1, he learns r_s and picks the alternative organization again in period 2 if R_s is non-negative and the low quality service otherwise. Notice, since r_s is uniformly distributed, the probability that $R_s \geq 0$ is $\frac{\bar{r}_s+\mu}{2\bar{r}_s}$. The expected total surplus in this case is,

$$\begin{aligned} ETS_{a\cdot} &= 2(B-C) + E\{R_g + R_s\} + \frac{\bar{r}_s+\mu}{2\bar{r}_s}E\{R_s + R_g | R_s \geq 0\} \\ &= 2(B-C) + (1+\alpha)\mu + \frac{\bar{r}_s+\mu}{2\bar{r}_s}((\frac{1}{2}+\alpha)\mu + \frac{1}{2}\bar{r}_s). \end{aligned}$$

Through price negotiation, S receives half this value, implying that S chooses the alternative production method whenever the contract allocates authority to him at date 1. Hence, G chooses a two period complete contract if and only if,

$$U^g_{hh} = B - C + \Delta H \geq B - C + \frac{1}{2}(1+\alpha)\mu + \frac{\bar{r}_s+\mu}{4\bar{r}_s}((\frac{1}{2}+\alpha)\mu + \frac{1}{2}\bar{r}_s) = U^g_{a.},$$

which is equivalent to

$$\Delta H \geq \frac{1}{2}(1+\alpha)\mu + \frac{\bar{r}_s+\mu}{4\bar{r}_s}((\frac{1}{2}+\alpha)\mu + \frac{1}{2}\bar{r}_s) \equiv \overline{\Delta H}.$$

From the proof of Proposition 1 it is clear that delegation always increases the service providers expected utility. Since the parties split the surplus equally in the initial bargaining, the government is also interested in delegation if the total surplus is increased. The condition for this is condition (1). To gain some intuition for the result, its helpful to look at the case were $\mu = 0$, i.e. were there is no expected benefit or loss from the alternative provision mode to either party relative to producing the basic service quality. In this case the threshold value, $\overline{\Delta H}$, reduces to:

$$\overline{\Delta H} \equiv \frac{1}{8}\bar{r}_s. \tag{4.3}$$

The government only wishes to delegate authority if the (per period) net benefit of high quality contract does not exceed half of S's expected gain from having authority over the production decision. The expected private gain for S arises from the option value in period two whenever he chooses the alternative production mode in period one: By assumption, there is no expected gain from using the alternative production mode in period one; however, by experimenting, S learns the value of R_s. This provides S with the option to choose the alternative production mode in period two iff $R_s \geq 0$ and choose a production mode that generates the low quality service otherwise. The expected value of this option is the expected value of R_s, conditioned on R_s being positive, times the probability that R_s is indeed positive. In total this equals $\frac{1}{2} \cdot \frac{1}{2}\bar{r}_s$ out of which the the government receives half through ex ante price negotiation. The option value implies, that more uncertainty about the service provider's utility from the alternative production mode makes it more likely that authority is delegated.

The additional terms in equation (4.1) reflect the further effects that arise when $\mu > 0$. Straightforward differentiation gives that $\frac{\partial\overline{\Delta H}}{\partial\alpha}$ and $\frac{\partial\overline{\Delta H}}{\partial\mu}$ are both positive.

Hence more aligned preferences or higher expected benefit to S makes delegation of authority more likely.

Notice that the condition given in Proposition 1 does *not* depend on the uncertainty about how the government's utility is affected by alternative production mode, as measured by $\bar{r}_g$. Since the government cannot use the information about its private benefit to revise the contract before period 2, from an ex-ante perspective there is no option value for the government.[12]

How, then, does the government's willingness to delegate authority depend on the amount of uncertainty about the service provider's private gain? The derivative $\frac{\partial \overline{\Delta H}}{\partial \bar{r}_s}$ is positive, if $\alpha < -\frac{1}{2}$. If $\alpha > -\frac{1}{2}$, the derivative is positive iff $\bar{r}_s > \sqrt{(1+2\alpha)}\mu$. In these cases, larger uncertainty about the service providers payoff increases the likelihood that the government prefers delegation. The intuition here is a little more involved. Still the option is more valuable for the service provider, and the government gets part of this in the initial negotiations. This tends to make delegation more attractive. However, the condition determining when the service provider chooses one and the other form for provision is also affected. The probability that R_s is positive equals $\frac{\bar{r}_s+\mu}{2\bar{r}_s}$. This probability decreases in $\bar{r}_s$ and this creates an offsetting effect, since the positive effects of delegation are realized more seldom.

These comparative static results are summarized in the following Corollary.

Corollary 1.

a) Increasing uncertainty about the government's payoff from the alternative production mode does not affect the government's incentive to delegate authority.

b) Assume $\mu = 0$: Increasing uncertainty about the service provider's payoff from the alternative production mode increases the government's incentive to delegate authority through leaving contract incomplete.

c) If $\mu > 0$, the government is more likely to prefer an incomplete contract the higher is μ and α. The effect of an increase in $\bar{r}_s$ is ambiguous. If $\alpha < -\frac{1}{2}$ or $\bar{r}_s > \sqrt{(1+2\alpha)}\mu$, then an increase in $\bar{r}_s$ tends to make delegation preferable.

4.4 Adaptive contracting

We now proceed to analyze inter-period contract revision of the following kind: The government can in period 1 choose between a two period long-term contract or a one-period short term contract, which will be followed by a new contract in period 2. This framework, therefore, opens the possibility of adaptive contracting. We solve the model for the Perfect Bayesian Equilibrium.

The following Lemma significantly reduces the relevant contracting strategies:

Lemma 1. *In equilibrium the government either offers:*
a) A complete long-term contract specifying high quality production mode in both

[12] Clearly, introducing risk aversion into the model would imply an independent channel through which uncertainty would affect the government's decision to delegate authority.

periods;
or,
b) A short-term incomplete contract, which is replaced in period 2 with either another short-term incomplete contract or a short-term complete contract specifying high quality production mode.

Proof.. We prove the Lemma backwards. If a short term contract specifies a standard production mode in period 1, nothing is learned about r_s and r_g, and the optimal contract in the second period specifies high quality, (Assumption 1). It follows that the incomplete contract can only be chosen in the second period if it was chosen in the first period.

By assumption, it is never optimal to specify low quality. If an incomplete contract is chosen in the first period, the government learns the realizations of the stochastic variables r_s and r_g. If it only writes a short time contract, it has the option to specify high quality in the second period. Hence, it never chooses an incomplete contract for both periods already in period 1.

If G specifies high quality in period one, it may as well specify it for the second period also, as it will be the choice anyway in the second period.

The Lemma states that there are two relevant contracting strategies to analyze. Either the extra value of having high quality is so high that the government prefers this outcome ex-ante,[13] or it is better to leave authority to the service provider, giving him incentives to pick the alternative production mode and then in the next period offer the best contract in view of the information gained in the first period - i.e. an *adaptive* contract.

We now analyze the parties' expected surplus from using the adaptive contracting strategy. When the contract is incomplete, and S has authority in period 1, S rationally chooses the alternative production mode. In period 2, there are three options: The government stipulates a high quality contract effectively removing the authority away from S; S keeps authority but has learned that R_s is negative, implying that he chooses the low quality; and, finally that S keeps authority and has learned that R_s is positive, so he chooses the alternative production mode again. Since the last case happens with a positive probability, S is strictly better off in expected terms choosing the alternative production mode in period 1.[14]

After S chooses the alternative production mode in period 1, G and S learn r_s and r_g. Both S and G have the option to avoid the alternative quality level again in

[13] Since none of the parties learn anything about their potential private benefits, it is obvious that this long-term contract could be replaced by two short-term contracts. However a small positive negotiating cost would make it strictly more beneficial to both parties to negotiate a long-term contract.

[14] Notice, since the alternative mode is non-verifiable by assumption, it is not possible to write a second period contact specifying that this mode *should* be chosen. Effectively, the only way to implement this mode in the second period is to write an incomplete contract, and the service provider therefore still has authority over whether this mode should be used. Below we briefly discuss the case, where the alternative production mode is contractible in the second period if and only if it has been used in the first period.

period 2. S opts for the alternative production mode in the second period if and only if $R_s \geq 0$, whereas, as we will show below, G only delegates authority if it is in the parties' joint interests, i.e. if $R_s \geq 0$ and $R_g + R_s \geq \Delta H$. We are therefore interested in the ex ante probability of the event $R_g + R_s \geq \Delta H$ given it is known that $R_s \geq 0$.

As $R_s + R_g = (1+\alpha)\mu + r_s + r_g$, and $R_s = \mu + r_s$, the probability that $R_s + R_g > \Delta H$ conditioned on $R_s \geq 0$ is equal to the probability that $r_s + r_g > \Delta H - (1+\alpha)\mu$ conditioned on $r_s \geq -\mu$. Let

$$T \equiv \Delta H - (1+\alpha)\mu.$$

T expresses the expected additional surplus from the high quality service rather than the alternative service quality. Let $\Phi(T)$ denote the probability that $r_s + r_g < T$ given $r_s \geq -\mu$.[15]. The distribution function Φ is fully characterized in the Appendix, here we present - without proof - Φ for the parameter values, we focus on below.

Lemma 2. *If* $-\bar{r}_g + \bar{r}_s \leq T \leq \bar{r}_g - \mu$, *then*

$$\Phi(T) = \frac{2\bar{r}_g - \bar{r}_s + 2T + \mu}{4\bar{r}_g}.$$

From Lemma 1 we know that there are only two possible equilibrium outcomes. Recall that the parties split the surplus in the negotiations. Hence, in order to determine which contract type the government prefers, it suffices to determine whether the long-term complete high quality or the adaptive contract yields the highest total surplus.

Consider first the long-term complete high quality contract. From Equation (4.2) we notice that the additional total surplus relative to a low quality contract over the two periods equals:

$$2\Delta H = 2(\Delta B - \Delta C). \tag{4.4}$$

Next, if G offers the adaptive contract, the expected additional surplus from period 1 is $(1+\alpha)\mu$. If $R_s \geq 0$ and $R_s + R_g \geq \Delta H$ then the second period contract will be left incomplete. Let this event be denoted IC_2. The probability that $R_s \geq 0$ is $\frac{\bar{r}_s + \bar{\mu}}{2\bar{r}_s}$, while the probability that $R_s + R_g \geq \Delta H$ equals $1 - \Phi(\Delta H - (1+\alpha)\mu) = 1 - \Phi(T)$. Hence, the incomplete contract is chosen again with probability $\frac{\bar{r}_s + \mu}{2\bar{r}_s}(1 - \Phi(T))$.

Let the expected additional surplus conditioned on the event IC_2 be denoted $E\{R_s + R_g | IC_2\}$. The expected surplus from the adaptive contracting strategy is:

$$(1+\alpha)\mu + \frac{\bar{r}_s + \mu}{2\bar{r}_s}(1 - \Phi(T))E\{R_s + R_g | IC_2\} + \\ \left(1 - \frac{\bar{r}_s + \mu}{2\bar{r}_s}(1 - \Phi(T))\right)\Delta H. \tag{4.5}$$

The incomplete contract will be chosen in the first period if the surplus in (4.5) exceeds the surplus in (4.4). Simplifying a bit proves the following,

[15] Hence, the probability that $T < r_s + r_g$ given $r_s \geq -\mu$, then equals $1 - \Phi(T)$.

Proposition 2. *The government delegates authority through offering an incomplete contract to the service provider in period 1 if and only if,*

$$E\{r_s + r_g | IC_2\} \geq \frac{T + \frac{\bar{r}_s+\mu}{2\bar{r}_s}(1-\Phi(T))\Delta H}{\frac{\bar{r}_s+\mu}{2\bar{r}_s}(1-\Phi(T))}. \tag{4.6}$$

Equation (4.6) provides the condition for when adaptive contracting is optimal. With probability $\frac{\bar{r}_s+\mu}{2\bar{r}_s}(1-\Phi(T))$ the service provider will be allowed to keep the authority in the second period which generates an expected surplus of $E\{R_s+R_g|IC_2\}$ to the government. The opportunity cost of delegating authority is the net benefit given up in the first period. In the second period, when private benefits from the alternative production is known, adaptive contracting is costless for the government.

It is interesting to know how the incentives to use adaptive contracting is affected by the degree of uncertainty about the agents' private benefit. Unfortunately, the formulas become rather lengthy and there are a number of cases to consider. We now derive a clear result in the important case where $\mu = 0$, i.e. where there is no (one period) expected surplus from the alternative process relative to the basic service provision. In this case, $T = \Delta H$, and condition (4.6) reduces to

$$E\{r_s + r_g | IC_2\} \geq \frac{(2 + (1-\Phi(\Delta H))\,\Delta H}{1-\Phi(\Delta H)}. \tag{4.7}$$

As is clear from the full expression for Φ given in the Appendix, there are quite a number of sub cases to consider. Rather than going tediously through all possible sub cases, we will focus first on the case, where $\bar{r}_g \geq \bar{r}_s$ and the high quality contract is not overwhelmingly attractive so $\Delta H < \bar{r}_g$. Then Φ is as given in Lemma 2.

Proposition 3. *Assume $\mu = 0$, $\bar{r}_g \geq \bar{r}_s$, and $\Delta H < \bar{r}_g$. In this case, the government offers a short term incomplete contract to the service provider in period 1 if and only if*

$$\Delta H \leq 5\bar{r}_g + \frac{1}{2}\bar{r}_s - 2\sqrt{6\bar{r}_g^2 + \bar{r}_g\bar{r}_s}. \tag{4.8}$$

This is more likely to be fulfilled, the smaller is the expected gain from the high quality contract, ΔH, and the larger is the uncertainty about the utility consequences of the alternative production mode for either of the parties, i.e. the larger is $\bar{r}_s$ and $\bar{r}_g$.

Proof.. We first evaluate the left hand side of the inequality (4.7).

$$\begin{aligned} E\{r_s + r_g | IC_2\} &= E\{r_s + r_g | r_s \geq 0 \text{ and } r_g + r_s \geq \Delta H\} \\ &= \int_0^{\bar{r}_s} \frac{1}{\bar{r}_s} \left(\int_{\Delta H - r_s}^{\bar{r}_g} \frac{r_s + r_g}{\bar{r}_g - (\Delta H - r_s)} dr_g \right) dr_s. \end{aligned}$$

This is true since, the conditional density, given $r_s \geq 0$, at a particular r_s equals $\frac{1}{\bar{r}_s}$. With this r_s, r_g has to be larger than $\Delta H - r_s$ for $r_g + r_s \geq \Delta H$ to hold. Since r_g is uniformly distributed, the conditional density at an r_g fulfilling $r_g \geq \Delta H - r_s$

is $\frac{1}{\bar{r}_g-(\Delta H-r_s)}$. We sum r_s and r_g, multiply with the relevant densities and integrate over the relevant ranges. Integrating yields

$$E\{r_s+r_g|IC_2\}=\frac{1}{2}\Delta H+\frac{1}{2}\bar{r}_g+\frac{1}{4}\bar{r}_s.$$

In the range of variables, we consider here, Φ is given in Lemma 2. Inserting into the right hand side of (4.7) , and manipulating a bit gives

$$\frac{(2+(1-\Phi(\Delta H))\,\Delta H}{1-\Phi(\Delta H)}=\frac{(10\bar{r}_g+\bar{r}_s-2\Delta H)\,\Delta H}{2\bar{r}_g+\bar{r}_s-2\Delta H}.$$

Therefore condition (4.7) becomes

$$\frac{1}{2}\Delta H+\frac{1}{2}\bar{r}_g+\frac{1}{4}\bar{r}_s\geq\frac{(10\bar{r}_g+\bar{r}_s-2\Delta H)\,\Delta H}{2\bar{r}_g+\bar{r}_s-2\Delta H}.$$

Solving for ΔH yields that in the relevant range (recall that we assume that $\Delta H\leq\bar{r}_g$), this is equivalent to condition (4.8) of the Proposition. Finally, differentiating the right hand side of (4.8) yields

$$\frac{\partial RHS}{\partial\bar{r}_s}=\frac{1}{2}\frac{\sqrt{\bar{r}_g\,(6\bar{r}_g+\bar{r}_s)}-2\bar{r}_g}{\sqrt{\bar{r}_g\,(6\bar{r}_g+\bar{r}_s)}}>0$$

as $r_g, r_s>0$. Similarly

$$\frac{\partial RHS}{\partial\bar{r}_g}=\frac{5\sqrt{\bar{r}_g\,(6\bar{r}_g+\bar{r}_s)}-12\bar{r}_g-\bar{r}_s}{\sqrt{\bar{r}_g\,(6\bar{r}_g+\bar{r}_s)}}>0\text{ iff }6\bar{r}_g^2+\bar{r}_g\bar{r}_s-\bar{r}_s^2>0$$

which is fulfilled under our assumption that $\bar{r}_g\geq\bar{r}_s$.

Under adaptive contracting the alternative production mode is only implemented in period 2 when it is privately beneficial for the service provider and jointly beneficial for both parties. Contrary to the case with no inter-period contracting, the government is now also insured against the downside realization of the uncertainty. The renegotiation in the current setting gives both parties insurance, and hence the option value is extended to both parties. This implies that more uncertainty, whether it concerns the government's utility or the service provider's utility, strengthen the incentives to use the adaptive approach. This is a qualitatively different result from the ones we obtained in the previous section.

When $\mu\neq 0$, the formulas becomes substantially more involved. We therefore restrict ourselves to consider a single case where $\mu>0$, and where the uncertainty is the same for both parties, so that $\bar{r}_s=\bar{r}_g=\bar{r}$. Again, we assume that the high quality contract is not very attractive, so that $T\leq\ \bar{r}-\mu$, which is equivalent to $\Delta H\leq\bar{r}+\alpha\mu$. Let

$$H\,(\mu,\alpha,\bar{r})\equiv\frac{1}{2\,(\mu+\bar{r})}(11\bar{r}^2+3\bar{r}\mu-$$
$$\sqrt{112\bar{r}^4+16\,(1-2\alpha)\,\bar{r}^3\mu+\left(4\alpha^2-20\alpha-23\right)\bar{r}^2\mu^2+\left(24\alpha+18+8\alpha^2\right)\bar{r}\mu^3+\left(12\alpha+9+4\alpha^2\right)\mu^4}).$$

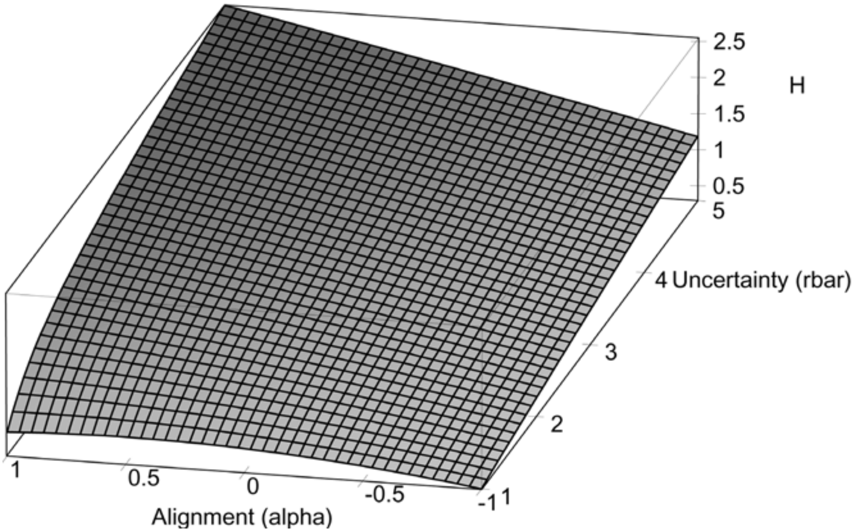

Fig. 4.1. $H(1, \alpha, \bar{r})$

Following the steps of the previous proof, inserting for Φ in condition (4.6) and solving the resulting inequality for ΔH yields that G use adaptive contracting if and only if:

$$\Delta H \leq H(\mu, \alpha, \bar{r}).$$

Unsurprisingly, a low expected value of the high quality complete contract tends to make the incomplete contract relatively more attractive. The expression for $H(\mu, \alpha, \bar{r})$ is unfortunately a little complicated. In order to gain further insight, Figure 1 below presents three dimensional plot of H, for the case $\mu = 1$.

Figure 1 reveals that increasing uncertainty makes the incomplete contract relatively more attractive, as was the case when $\mu = 0$. The effect of α is slightly more complicated. Straightforward differentiation gives that $\frac{\partial H}{\partial \alpha}$ is positive for $\alpha < \frac{1}{2}\frac{8r^2-3\mu^2-3r\mu}{\mu(\mu+r)}$ and negative otherwise. For $\mu = 1$, this imply that $\frac{\partial H}{\partial \alpha}$ is positive for all $\alpha \in [-1, 1]$ provided, $\bar{r} \geq \frac{5}{16} + \frac{1}{16}\sqrt{185} \approx: 1.16$. Hence, except for low degrees of uncertainty, we have the result, that increasing the degree of alignment of preferences makes delegation more attractive as would appear natural. For small degrees of uncertainty, the result may, however, be the opposite, as is also clear from figure 1.

Before we end this section it is worth discussing the assumption that the alternative production mode is unverifiable in period two, even if it has been applied in period one. Let us instead assume that if the alternative production mode has been used in period one, then it is feasible to write a contract specifying this production mode in period 2. This changes the condition for choosing the alternative quality in period 2 from the joint event of $R_s \geq 0$ and $R_s + R_g \geq 0$ to only $R_s + R_g \geq 0$. Obviously this would increase the government's incentive to choose adaptive contracting. Hence, in all the cases where the joint surplus is positive, but where the service provider's private utility is lower than what he obtains from delivering a low quality service, we observe more delegation in the second period under this alternative assumption. We conclude, that the alternative assumption strengthen our analysis since it makes adaptive contracting even more valuable.

4.5 Conclusion

Adaptive contracting is widely used around the world, in particular it is a standard approach when government agencies write outsourcing contracts with private firms. Many observers and participants argue that the trial and error approach is the only feasible way to write a contract for a bureaucratic public organization with its many limitations due to time, organizational and various political constraints. The main insight of the present analysis has been to show that the trial-and-error approach can be an optimal form of contracting even without such constraints.

Our theoretical analysis highlighted the connection between adaptive contracting and the optimal allocation of - decision - authority. The general principle resulting from our analysis is that authority should be allocated to the party that uses it for most surplus generation independently on how this surplus is distributed ex-post. The trial-and-error approach allocates authority initially to the service provider who uses this authority opportunistically. This is beneficial for both parties as long as the service provider's choices increase the total value generated in the relationship. The ex-post distribution of this rent is not important because it will be offset through the negotiated service price.

Further work has to be undertaken to obtain a comprehensive knowledge about which organizational settings that favor the adaptive relative to a comprehensive contracting approach. As shown in the previous sections, the more uncertainty about the consequences of the delegation of authority, the more incentive the parties have to apply adaptive contracting, since it increases the option value associated with experimenting in addition to increasing the cost of writing complete contracts. This observation is particular interesting in a multi-dimensional contract where the quality of the service may have many parameters each of which it is possible to delegate authority over.

Our analysis were restricted to a two period contracting setting. In reality many outsourcing decisions covers many periods. We conjecture that extending our model to a multi-period setting only increases the incentives to use adaptive contracting, since the embedded option value increases.

Appendix: Complete characterization of $\Phi\left(.\right)$.

Lemma 2. *The distribution function $\Phi\left(.\right)$ is given by:*

- *If $-\bar{r}_g+\bar{r}_s\leq\bar{r}_g+\mu$ then,*

$$\Phi\left(T\right)=\begin{cases}\frac{1}{4}\frac{T^2+2T\bar{r}_g+\bar{r}_g^2+2T\mu+\mu^2+2\bar{r}_g\mu}{(\bar{r}_s+\mu)\bar{r}_g} & \textit{if } -\bar{r}_g-\mu\leq T\leq-\bar{r}_g+\bar{r}_s,\\ -\frac{1}{4}\frac{\bar{r}_s-2\bar{r}_g-2T-\mu}{\bar{r}_g} & \textit{if } -\bar{r}_g+\bar{r}_s\leq T\leq\ \bar{r}_g-\mu,\\ -\frac{1}{4}\frac{-2T\bar{r}_s+\bar{r}_s^2-2\bar{r}_g\bar{r}_s+T^2+2T\bar{r}_g-3\bar{r}_g^2}{(\bar{r}_s+\mu)\bar{r}_g}+\frac{T-\bar{r}_g+\mu}{\bar{r}_s+\mu} & \textit{if } \ \bar{r}_g-\mu<T<\bar{r}_g+\bar{r}_s.\end{cases}$$

- *If $-\bar{r}_g+\bar{r}_s>\bar{r}_g+\mu$ then,*

$$\Phi\left(T\right)=\begin{cases}\frac{1}{4}\frac{T^2+2T\bar{r}_g+\bar{r}_g^2+2T\mu+\mu^2+2\bar{r}_g\mu}{\bar{r}_g\bar{r}_s} & \textit{if } \ -\bar{r}_g-\mu\leq T\leq\bar{r}_g-\mu,\\ \frac{\bar{r}_g}{\bar{r}_s+\mu}+\frac{T-\bar{r}_g+\mu}{\bar{r}_s+\mu} & \textit{if } \ \bar{r}_g-\mu\leq T\leq-\bar{r}_g+\bar{r}_s,\\ -\frac{1}{4}\frac{-2T\bar{r}_s+\bar{r}_s^2-2\bar{r}_g\bar{r}_s+T^2+2T\bar{r}_g-3\bar{r}_g^2}{(\bar{r}_s+\mu)\bar{r}_g}+\frac{T-\bar{r}_g+\mu}{\bar{r}_s+\mu} & \textit{if } -\bar{r}_g+\bar{r}_s<T<\bar{r}_g+\bar{r}_s.\end{cases}$$

Proof.. We are interested in the probability that the sum $R_g+R_s<\Delta H$, where $\Delta H=\Delta B-\Delta C-d$, given that $R_s\geq 0$.

This problem can be written

$$\begin{aligned}&\text{Prob}\left(\mu+r_s+\alpha\mu+r_g<\Delta H\ \mid\mu+r_s>0\right)\\ &=\text{Prob}\left(r_s+r_g<\Delta H-\left(1+\alpha\right)\mu\ \left|r_s>-\mu\right.\right)\\ &=\text{Prob}\left(r_s+r_g<T\ \left|r_s>-\mu\right.\right)\ \text{since}T\equiv\Delta H-\left(1+\alpha\right)\mu.\end{aligned}$$

Notice that given $r_s\geq-\mu$, the conditional density at any r_s is $\frac{1}{\bar{r}_s+\mu}$.

Recall that $\Phi\left(T\right)$ denotes the probability that $r_g+r_s\leq T$ given $r_s\geq-\mu$. Then $\Phi\left(T\right)$ equals the integral over positive r_s, given the conditional density of r_s, multiplied by the probability that $r_g\leq T-r_s$ at this particular r_s. We therefore have,

$$\Phi\left(T\right)=\int_{-\mu}^{\bar{r}_s}\frac{1}{\bar{r}_s+\mu}G\left(T-r_s\right)dr_s,$$

where $G\left(\cdot\right)$ denotes the cdf of r_g.

Remember

$$G\left(r_g\right)=\begin{cases}0\text{ for }r_g<-\bar{r}_g,\\ \frac{r_g-(-\bar{r}_g)}{2\bar{r}_g}\text{ for }-\bar{r}_g\leq r_g\leq\bar{r}_g,\\ 1\text{ for }\bar{r}_g<r_g.\end{cases}\tag{4.9}$$

This feature implies that Φ will be pasted together from three different parts. The support of Φ is $\left[-\bar{r}_g-\mu,\bar{r}_g+\bar{r}_s\right]$.

We need to distinguish between two cases according to whether

$$-\bar{r}_g+\bar{r}_s\lesseqgtr\bar{r}_g-\mu.\tag{4.10}$$

First we consider the case

$$-\bar{r}_g+\bar{r}_s<\bar{r}_g-\mu.\tag{4.11}$$

Notice that if

$$\bar{r}_g > \bar{r}_s,$$

then we for sure have that (4.11) is fulfilled.

In the present case, the real line looks like this

Let us consider $G\left(T - r_s\right)$.

When $-\bar{r}_g + \bar{r}_s < \bar{r}_g - \mu$, then the interval $T \in [-\bar{r}_g + \bar{r}_s, \bar{r}_g - \mu]$, is non-empty. For T in this interval, we have that $T - r_s \in [-\bar{r}_g, \bar{r}_g]$ for all realizations of r_s in $[-\mu, \bar{r}_s]$. In this case the relevant formula for $G\left(T - r_s\right)$ is the second line in (4.9) above - this explains the second line in (4.12) below.

When $T \in [-\bar{r}_g - \mu, -\bar{r}_g + \bar{r}_s]$, then high realizations of r_s will imply that $T - r_s < -\bar{r}_g$, in which case the first line in (4.9) is relevant. For a given $T \in [-\bar{r}_g - \mu, -\bar{r}_g + \bar{r}_s]$ the probability that $T - r_s < -\bar{r}_g$ equals

$$\text{Prob}\left(\, r_s > T + \bar{r}_g\right) = \frac{\bar{r}_s - \left(T + \bar{r}_g\right)}{\bar{r}_s + \mu}.$$

On the other hand the second line in (4.9) is relevant for small realizations of r_s, i.e. for all realizations $r_s < T + \bar{r}_g$. This explains the first line in (4.12) below.

Then consider high T, $T \in [\bar{r}_g - \mu, \bar{r}_g + \bar{r}_s]$. Low realizations of r_s imply that $T - r_s > \bar{r}_g$, in which case the relevant part of G is given by the third line in (4.9). This is true for $r_s < T - \bar{r}_g$. The probability of such an r_s is

$$\text{Prob}\left(r_s < T - \bar{r}_g\right) = \frac{T - \bar{r}_g + \mu}{\bar{r}_s + \mu}.$$

For high realizations of r_s ($r_s > T - \bar{r}_g$), the relevant part of G is the second line in (4.9). This explains the third line in (4.12).

We therefore have that the cdf of the sum is

$$\Phi\left(T\right) = \begin{cases} \int_{-\mu}^{T+\bar{r}_g} \frac{1}{\bar{r}_s+\mu} \frac{T-r_s+\bar{r}_g}{2\bar{r}_g} dr_s \ + 0 \frac{\bar{r}_s-(T+\bar{r}_g)}{\bar{r}_s+\mu} \text{ for } -\bar{r}_g - \mu \leq \ T \leq -\bar{r}_g + \bar{r}_s, \\ \int_{-\mu}^{\bar{r}_s} \frac{1}{\bar{r}_s+\mu} \frac{T-r_s+\bar{r}_g}{2\bar{r}_g} dr_s \text{ for } -\bar{r}_g + \bar{r}_s \leq T \leq \ \bar{r}_g - \mu, \\ \int_{T-\bar{r}_g}^{\bar{r}_s} \frac{1}{\bar{r}_s+\mu} \frac{T-r_s+\bar{r}_g}{2\bar{r}_g} dr_s \ + 1 \frac{T-\bar{r}_g+\mu}{\bar{r}_s+\mu} \text{for } \bar{r}_g - \mu < T < \bar{r}_g + \bar{r}_s. \end{cases} \tag{4.12}$$

Performing the integrations yields,

$$\Phi\left(T\right) = \begin{cases} \frac{1}{4} \frac{T^2+2T\bar{r}_g+\bar{r}_g^2+2T\mu+\mu^2+2\bar{r}_g\mu}{(\bar{r}_s+\mu)\bar{r}_g} \text{ for } -\bar{r}_g - \mu \leq \ T \leq -\bar{r}_g + \bar{r}_s, \\ -\frac{1}{4} \frac{\bar{r}_s-2\bar{r}_g-2T-\mu}{\bar{r}_g} \text{ for } -\bar{r}_g + \bar{r}_s \leq T \leq \ \bar{r}_g - \mu, \\ -\frac{1}{4} \frac{-2T\bar{r}_s+\bar{r}_s^2-2\bar{r}_g\bar{r}_s+T^2+2T\bar{r}_g-3\bar{r}_g^2}{(\bar{r}_s+\mu)\bar{r}_g} + \frac{T-\bar{r}_g+\mu}{\bar{r}_s+\mu} \text{for } \bar{r}_g - \mu < T < \bar{r}_g + \bar{r}_s. \end{cases}$$

The density function is

$$\phi(T) = \begin{cases} \frac{1}{2}\frac{T+\bar{r}_g+\mu}{(\bar{r}_s+\mu)\bar{r}_g} \text{ for } -\bar{r}_g-\mu \leq T \leq -\bar{r}_g+\bar{r}_s, \\ \frac{1}{2\bar{r}_g} \text{ for } -\bar{r}_g+\bar{r}_s \leq T \leq \bar{r}_g-\mu, \\ -\frac{1}{2}\frac{-\bar{r}_s+T-\bar{r}_g}{(\bar{r}_s+\mu)\bar{r}_g} \text{for } \bar{r}_g-\mu < T < \bar{r}_g+\bar{r}_s. \end{cases}$$

Next, consider the case where,

$$-\bar{r}_g+\bar{r}_s > \bar{r}_g-\mu \tag{4.13}$$

In this case, the real line looks like,

As above we have that , when $T \in [-\bar{r}_g-\mu, -\bar{r}_g+\bar{r}_s]$, then high realizations of r_s will imply that $T - r_s < -\bar{r}_g$, in which case the first line in (4.9) is relevant. For a given $T \in [-\bar{r}_g-\mu, -\bar{r}_g+\bar{r}_s]$ the probability that $T - r_s < -\bar{r}_g$ equals

$$\text{Prob}\,(\, r_s > T+\bar{r}_g) = \frac{\bar{r}_s-(T+\bar{r}_g)}{\bar{r}_s+\mu}.$$

On the other hand when $T \in [\bar{r}_g-\mu, \bar{r}_g+\bar{r}_s]$, then low realizations of r_s imply that $T - r_s > \bar{r}_g$, in which case the relevant part of G is given by the third line in (4.9). This is true for $r_s < T - \bar{r}_g$. The probability of such an r_s is

$$\text{Prob}\,(r_s < T-\bar{r}_g) = \frac{T-\bar{r}_g+\mu}{\bar{r}_s+\mu}.$$

For the rest of the realizations of r_s, i.e. for $r_s \in [T-\bar{r}_g, T+\bar{r}_g]$, then $T - r_s \in [-\bar{r}_g, \bar{r}_g]$. Hence for $T \in [-\bar{r}_g-\mu, -\bar{r}_g+\bar{r}_s]$, the cdf is given by the second line in (4.14) below.

For small $T \in [-\bar{r}_g-\mu, \bar{r}_g-\mu]$, only the probability that high realizations of T imply that $T - r_s < -\bar{r}_g$ is relevant. This explains the first line in (4.14) below.

Finally for high $T \in [-\bar{r}_g+\bar{r}_s, -\bar{r}_g+\bar{r}_s]$ only the probability that low realizations of T imply that $T - r_s > \bar{r}_g$ is relevant. This explains the third line in (4.14)

$$\Phi(T) = \begin{cases} \int_{-\mu}^{T+\bar{r}_g} \frac{1}{\bar{r}_s}\frac{T-r_s+\bar{r}_g}{2\bar{r}_g} dr_s + 0\frac{\bar{r}_s-(T+\bar{r}_g)}{\bar{r}_s+\mu} \text{ for } -\bar{r}_g-\mu \leq T \leq \bar{r}_g-\mu, \\ \int_{T-\bar{r}_g}^{T+\bar{r}_g} \frac{1}{\bar{r}_s+\mu}\frac{T-r_s+\bar{r}_g}{2\bar{r}_g} dr_s + 0\frac{\bar{r}_s-(T+\bar{r}_g)}{\bar{r}_s+\mu} + 1\frac{T-\bar{r}_g+\mu}{\bar{r}_s+\mu} \text{ for } \bar{r}_g-\mu \leq T \leq -\bar{r}_g+\bar{r}_s, \\ \int_{T-\bar{r}_g}^{\bar{r}_s} \frac{1}{\bar{r}_s+\mu}\frac{T-r_s+\bar{r}_g}{2\bar{r}_g} dr_s + 1\frac{T-\bar{r}_g+\mu}{\bar{r}_s+\mu} \text{for } -\bar{r}_g+\bar{r}_s < T < \bar{r}_g+\bar{r}_s. \end{cases} \tag{4.14}$$

Integrating yields,

$$\Phi(T) = \begin{cases} \frac{1}{4}\frac{T^2+2T\bar{r}_g+\bar{r}_g^2+2T\mu+\mu^2+2\bar{r}_g\mu}{\bar{r}_g\bar{r}_s} \text{ for } -\bar{r}_g-\mu \leq T \leq \bar{r}_g-\mu, \\ \frac{\bar{r}_g}{\bar{r}_s+\mu}+\frac{T-\bar{r}_g+\mu}{\bar{r}_s+\mu} \text{ for } \bar{r}_g-\mu \leq T \leq -\bar{r}_g+\bar{r}_s, \\ -\frac{1}{4}\frac{-2T\bar{r}_s+\bar{r}_s^2-2\bar{r}_g\bar{r}_s+T^2+2T\bar{r}_g-3\bar{r}_g^2}{(\bar{r}_s+\mu)\bar{r}_g}+\frac{T-\bar{r}_g+\mu}{\bar{r}_s+\mu} \text{ for } -\bar{r}_g+\bar{r}_s < T < \bar{r}_g+\bar{r}_s. \end{cases}$$

The density becomes

$$\phi(T) = \begin{cases} \frac{1}{2}\frac{T+\bar{r}_g+\mu}{\bar{r}_g\bar{r}_s} \text{ for } -\bar{r}_g-\mu \leq T \leq \bar{r}_g-\mu, \\ \frac{1}{\bar{r}_s+\mu} \text{ for } \bar{r}_g-\mu \leq T \leq -\bar{r}_g+\bar{r}_s, \\ -\frac{1}{2}\frac{-\bar{r}_s+T-\bar{r}_g}{(\bar{r}_s+\mu)\bar{r}_g} \text{ for } -\bar{r}_g+\bar{r}_s < T < \bar{r}_g+\bar{r}_s. \end{cases}$$

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5

Monetary Equilibria over an Infinite Horizon*

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Summary. Money provides liquidity services through a cash-in-advance constraint. The exchange of commodities and assets extends over an infinite horizon under uncertainty and a sequentially complete asset market. Monetary policy sets the path of rates of interest and accommodates the demand for balances through open market operations or loans. A public authority, which, most pertinently, inherits a strictly positive public debt, raises revenue from taxes and seignorage, and it distributes possible budget surpluses to individuals through transfers. Competitive equilibria exist, under mild solvency conditions. But, for a fixed path of rates of interest, there is a non-trivial multiplicity of equilibrium paths of prices of commodities. Determinacy requires that, subject to no-arbitrage and in addition to rates of interest, the prices of state-contingent revenues be somehow determined.

Key words: Money, Equilibrium, Indeterminacy, Monetary policy, Fiscal policy.

JEL Classification Numbers: D50, E40, E50.

5.1 Introduction

We prove the existence of general competitive equilibria in a monetary economy under interest rate pegging; and we show that they display indeterminacy that we characterize.

We modify the canonical Walrasian model by introducing money balances that facilitate transactions. The economy extends over an infinite horizon under uncertainty. Elementary securities make for a sequentially complete asset market. The transaction technology takes the simple form of the cash-in-advance constraint. Money balances are supplied by a central bank, which produces these at no cost and lends them at set short-term nominal rates of interest, meeting demand. The profits

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of the central bank, seignorage, accrue to a public authority. The primitives include initial nominal claims held by individuals that, in the aggregate, are the counterpart of an initial public debt of the public authority.

The public authority covers its expenditures, including initial debt, through public revenues, which consist of taxes and seignorage. Taxes are lump-sum commodity (*i.e.*, wealth) taxes collected from individuals at predetermined real levels. Importantly, the public authority distributes its eventual budget surpluses as lump-sum transfers to individuals, while no further instruments are available to correct eventual budget deficits.

Over a finite horizon, with no public debt and no taxes, Drèze and Polemarchakis [7] proved the existence of competitive equilibria, for arbitrarily set nominal rates of interest and price levels at all terminal nodes. Alternatively stated, the overall price level is arbitrary and the variability of short-term rates of inflation is unrestricted. This important indeterminacy feature reflects the intuitive property that the rate of interest at every date-event pins down expected inflation, but not inflation variability. We extend these results over an infinite horizon. The introduction of a public debt requires some qualifications.

Initial public debt must be met by public revenues from taxes and seignorage. If predetermined tax levels are positive, a suitable lower bound on the overall price level guarantees public solvency. Otherwise, the public authority must rely on seignorage, the yield of which is, roughly speaking, proportional to the overall price level. Public solvency then requires positive nominal rates of interest and positive transactions, and, hence, demand for money balances, reflecting gains to trade. The condition, due to Dubey and Geanakoplos [8, 9], requires that nominal rates of interest do not exceed gains to trade.

Our main results assert

1. the existence of equilibria at all overall price levels above a lower bound, provided that conditions on gains to trade, if needed, are satisfied, and
2. the indeterminacy of rates of inflation, up to no-arbitrage conditions.

Our work extends that of Woodford [17] to the case of heterogenous individuals and multiple commodities, which is in turn similar to cash-in-advance economies with a representative individual of Wilson [16] and Lucas and Stokey [13]. Differently from this literature, our more general formulation provides a framework that is suitable for the study of incomplete asset markets.

Recent literature (Woodford [17, 18] and Cochrane [6]) proposes a fiscal theory of price determination. This asserts that the price level is determined so as to balance the initial public debt and public revenues from taxes and seignorage. We here obtain indeterminacy of equilibria since, differently from that literature, we assume that the public authority can redistribute its eventual budget surpluses.

Similarly, Dubey and Geanakoplos [8, 9] consider the case of a given initial stock of outside money and an additional injection of inside money, which allows for an unambiguous determination of the nominal rate of interest: seignorage revenue should absorb the outside money. The analogy with the fiscal theory is strong and, indeed, Dubey and Geanakoplos [8, 9] obtain a determinate equilibrium as they do not allow for a distribution of public budget surpluses.[4]

5.2 A Monetary Economy

5.2.1

As far as notation is concerned, we shall use $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$ to denote the vector space of real-valued maps on $\mathcal{S}\times\mathcal{N}$, where $\mathcal{S}$ ($\mathcal{N}$) is a countable (finite) set. A typical element of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$ is $x=(\ldots,x_\sigma,\ldots)$, where each $x_\sigma=(\ldots,x_{\sigma\nu},\ldots)$ is a vector in $\mathbb{R}^{\mathcal{N}}$. An element x of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$ is positive if $x_{\sigma\nu}\geq 0$ for every (σ,ν) in $\mathcal{S}\times\mathcal{N}$; it is strictly positive if $x_{\sigma\nu}>0$ for every (σ,ν) in $\mathcal{S}\times\mathcal{N}$; it is uniformly strictly positive if there is $\epsilon>0$ such that $x_{\sigma\nu}\geq\epsilon$ for every (σ,ν) in $\mathcal{S}\times\mathcal{N}$; it is bounded if there is $\epsilon>0$ such that $\epsilon\geq x_{\sigma\nu}\geq-\epsilon$ for every (σ,ν) in $\mathcal{S}\times\mathcal{N}$. For (x,z) in $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}\times\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$,

$$x_\sigma\cdot z_\sigma=\sum_{\nu\in\mathcal{N}}x_{\sigma\nu}z_{\sigma\nu}\text{ and }\|x_\sigma\|=\sum_{\nu\in\mathcal{N}}|x_{\sigma\nu}|\,.$$

For an element x of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$, its positive and negative parts are, respectively,

$$x^+=(\ldots,(\ldots,\max\{x_{\sigma\nu},0\},\ldots),\ldots)\text{ and }x^-=x^+-x.$$

Similar definitions apply to the vector space $\mathbb{R}^{\mathcal{S}}$. Notice that a typical element of $\mathbb{R}^{\mathcal{S}}$ is $x=(\ldots,x_\sigma,\ldots)$, where each x_σ lies in $\mathbb{R}$. Notice that, throughout the paper, positive means greater than or equal to zero.

5.2.2

Time and the resolution of uncertainty are described by an event-tree, a countable set, $\mathcal{S}$, endowed with a (partial) order, $\succeq$. For every date-event, σ, an element of $\mathcal{S}$, t_σ denotes its date. The unique initial date-event is ϕ, with $t_\phi=0$. For a given date-event, σ, $\sigma_+=\{\tau\succ\sigma:t_\tau=t_\sigma+1\}$ denotes the set of its immediate successors, a finite set; $\mathcal{S}_\sigma=\{\tau\in\mathcal{S}:\tau\succeq\sigma\}$ the set of all its (weak) successors, a subtree; $\mathcal{S}^t=\{\sigma\in\mathcal{S}:0\leq t_\sigma\leq t\}$ the set all date-events up to date t; $\mathcal{S}_t=\{\sigma\in\mathcal{S}:t_\sigma=t\}$ the set all date-events at date t. Date-events are points in time. For accounting purposes, all values are defined as of the beginning of the time interval separating a date-event from its successors.

[4] Our work also contributes to a long debate on general equilibrium with incomplete financial markets. Following the demonstration of real indeterminacy with nominal assets in [2, 5, 11], some argued that the very notion of nominal assets is a misconception and only real assets should be considered as fruitful for economic analysis. The argument goes further: nominal assets are meaningful only if money is somehow introduced; if money were introduced, however, the real value of money would be determined, roughly speaking, by some sort of quantity theory equations, which would make real any asset initially described as nominal. In this perspective, our conclusions cast doubt of the cogency of the above argument.

5.2.3

Markets are sequentially open for commodities, assets and balances, that are numéraire. At every date-event, there is a finite set $\mathcal{N} = \{\ldots, \nu, \ldots\}$ of tradable commodities, which are perfectly divisible and perishable. The commodity space coincides with the space of all bounded elements of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$. Prices of commodities p are a positive element of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$. These are spot nominal prices.

The asset market is sequentially complete. It simplifies, at no loss of generality, to assume that all securities that are traded at a date-event deliver a payoff only at the immediately succeeding date-events. A security plan v is an element of $\mathbb{R}^{\mathcal{S}}$. At a date-event, $(v_\tau : \tau \in \sigma_+)$ represents the deliveries (or payoffs) of the security plan at the immediately following date-events. State prices a are a strictly positive element of $\mathbb{R}^{\mathcal{S}}$, normalized so that $a_\phi = 1$. At a date-event σ, the market value of a security plan, with payoffs $(v_\tau : \tau \in \sigma_+)$ across its immediately succeeding date-events, is

$$\frac{1}{a_\sigma} \sum_{\tau \in \sigma_+} a_\tau v_\tau,$$

where $a_\sigma^{-1} a_\tau$ is the spot price at σ of an elementary (Arrow) security with payoff at $\tau \in \sigma_+$.

At given state prices, (one-period) nominal rates of interest r are a positive element of $\mathbb{R}^{\mathcal{S}}$. By the absence of arbitrage opportunities, they satisfy

$$\frac{1}{1 + r_\sigma} = \frac{1}{a_\sigma} \sum_{\tau \in \sigma_+} a_\tau. \tag{5.1}$$

Nominal rates of interest are positive because, balances being storable, arbitrage opportunities would otherwise emerge. Given nominal rates of interest r, state prices a are consistent with those nominal rates of interest if they fulfill the above no arbitrage conditions (5.1) at all date-events.

5.2.4

There is a finite set $\mathcal{I} = \{\ldots, i, \ldots\}$ of individuals. An individual is described by preferences $\succeq^i$ over the consumption space, the space of all positive bounded elements of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$, and an endowment e^i of commodities, a positive bounded element of $\mathbb{R}^{\mathcal{S}\times\mathcal{N}}$. The choice of the consumption space fits in a well-established tradition, beginning with Bewley [3]. Preferences and endowments of commodities are restricted by two common assumptions.

(P) Preferences. *Preferences $\succeq^i$ are continuous in the (relative) Mackey topology, convex and strictly monotone.*

(E) Endowments. *The endowment e^i is uniformly strictly positive.*

Continuity of preferences in the Mackey topology, introduced in Bewley [3], is a strong requirement. It encompasses, for example, preferences that are represented by an additively separable utility function,

$$\sum_{\sigma \in \mathcal{S}} \mu_\sigma \beta^{t_\sigma} u^i \left(x^i_\sigma \right),$$

where μ_σ is the probability of σ, $0 < \beta < 1$ is the discount factor, and $u^i : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}$ is bounded, continuous, increasing and concave. In particular, the continuity of preferences in the Mackey topology implies that the individual is impatient: sufficiently distant modifications of consumption plans do not reverse the order of preference. Uniform impatience across individuals would be a stronger requirement. The much stronger assumption of a uniform rate of impatience across date-events, in some recent literature on incomplete asset markets over an infinite horizon, as in Hernández and Santos [12] and Magill and Quinzii [14], is not needed here.

Individuals are also characterized by shares $\left(\ldots, \zeta^i, \ldots\right) \geq 0$, with $\sum_i \zeta^i = 1$, and initial nominal wealths $\left(\ldots, \delta^i, \ldots\right) \geq 0$, with $\sum_i \delta^i = \delta$. The former will be used to distribute transfers across individuals. The latter represent given initial claims in terms of the numéraire.

Fundamentals are thus $\left(\ldots, \left(\succeq^i, e^i, \zeta^i, \delta^i\right), \ldots\right)$.

5.2.5

A public authority (or a government, or a central bank) sets nominal rates of interest, possibly contingent on date-events. Nominal rates of interest r are, thus, a positive element of $\mathbb{R}^{\mathcal{S}}$. The supply of balances m is a positive element of $\mathbb{R}^{\mathcal{S}}$. As nominal rates of interest are given, the supply of balances accommodates the demand. Although our analysis could be adapted to cope with all arbitrarily set nominal rates of interest, we impose a restriction that facilitates presentation.

(M) Nominal Rates of Interest. *Nominal rates of interest r are bounded.*

The public authority also sets a fiscal plan. Taxes $\left(\ldots, g^i, \ldots\right)$ are a positive bounded element of $\mathbb{R}^{\mathcal{S} \times \mathcal{N} \times \mathcal{I}}$. It is interpreted as establishing that, at a date-event, an individual is required to deliver $p_\sigma \cdot g^i_\sigma$ units of account to the public authority. In the aggregate, taxes are $g = \sum_i g^i$, a positive bounded element of $\mathbb{R}^{\mathcal{S} \times \mathcal{N}}$. We restrict fiscal plans so as to avoid problems of solvency and, more importantly, to carry out a limit argument in the proof of existence of equilibria.

(F) Taxes. *The net endowment $e^i - g^i$ is uniformly strictly positive.*

Notice that, in particular, it can be assumed that $g^i = \theta^i e^i$ for some $0 \leq \theta^i < 1$, with $\left(\ldots, \theta^i, \ldots\right) \geq 0$ being tax rates across individuals. Our commodity taxes then reduce to a wealth tax. An alternative would introduce taxes on net supplies $\left(x^i - e^i\right)^-$, corresponding to VAT or income taxes, with interesting implications for public revenue.

The public authority also issues transfers and trades in securities, subject to sequential budget constraints. Transfers h are a positive element of $\mathbb{R}^{\mathcal{S}}$. It is interpreted as positive deliveries of units of account from the public authority to individuals.

(T) Transfers. *Transfers h are distributed to individuals according to the given shares. Thus, $h^i = \zeta^i h$.*

Public liabilities w are an element of $\mathbb{R}^{\mathcal{S}}$, with a given initial value $w_\phi = \delta$. Notice that the initial public liability δ corresponds to initial nominal claims $\left(\ldots, \delta^i, \ldots\right)$ of individuals, that is, $\delta = \sum_i \delta^i$. To simplify the presentation, at no loss of realism, we assume that there is a strictly positive initial public liability.

(L) Initial Public Liability. *The initial public liability δ is strictly positive.*

Given nominal rates of interest and taxes, a public plan consists of transfers h and public liabilities w. A public plan (h, w) is subject, at every date-event, to a sequential public budget constraint,

$$\left(\frac{r_\sigma}{1+r_\sigma}\right) m_\sigma + \frac{1}{a_\sigma} \sum_{\tau \in \sigma_+} a_\tau w_\tau = w_\sigma + h_\sigma - \left(\frac{1}{1+r_\sigma}\right) p_\sigma \cdot g_\sigma. \tag{5.2}$$

The interpretation of budget constraint (5.2) is the following: The public authority enters a date-event σ with a given public liability w_σ. This requires a delivery of units of account due to past investments in securities and balances. The public authority issues a transfer h_σ and supplies securities $(v_\tau : \tau \in \sigma_+)$, so as to balance its budget

$$m_\sigma + \frac{1}{a_\sigma} \sum_{\tau \in \sigma_+} a_\tau v_\tau = w_\sigma + h_\sigma, \tag{5.3}$$

given that balances m_σ are supplied so as to accommodate the market demand. At every immediately succeeding date-event $\tau \in \sigma_+$, public liabilities amount to

$$w_\tau = v_\tau + m_\sigma - p_\sigma \cdot g_\sigma. \tag{5.4}$$

The convention that the value of taxes $p_\sigma \cdot g_\sigma$ is delivered by individuals at the end of date-event σ (or, better, at the immediately following date-event $\tau \in \sigma_+$) is only made to simplify notation. Substituting (5.4) into (5.3) and using (5.1), one obtains (5.2), with all terms evaluated at the beginning of date-event σ. In particular, v is a portfolio of securities, whereas w consolidates securities and balances. The latter is termed public liabilities since it is the amount of units of account that must be covered by issuing balances and supplying securities.

It is assumed that the public authority only trades is (one-period) safe bonds. Thus, an additional constraint requires that, at a date-event, $v_{\tau'} = v_{\tau''}$ for all (τ', τ'') in $\sigma_+ \times \sigma_+$. Equivalently,

$$w_{\tau'} = w_{\tau''} \text{ for all } (\tau', \tau'') \in \sigma_+ \times \sigma_+. \tag{5.5}$$

Indeed, under such a restriction, there is an element b of $\mathbb{R}^{\mathcal{S}}$ such that, at a date-event,

$$(w_\tau - m_\sigma : \tau \in \sigma_+) = b_\sigma (\ldots, 1, \ldots)$$

is interpreted as the stock of (one-period) safe bonds issued by the public authority.

5.2.6

An individual formulates a plan $\left(x^i, m^i, w^i\right)$. The demand for balances m^i is a positive element of $\mathbb{R}^{\mathcal{S}}$. The wealth plan w^i is an element of $\mathbb{R}^{\mathcal{S}}$, with a given initial value $w^i_\phi = \delta^i$. At a date-event, such a plan is subject to a budget constraint,

$$\left(\frac{r_\sigma}{1+r_\sigma}\right) m^i_\sigma + \frac{1}{a_\sigma} \sum_{\tau \in \sigma_+} a_\tau w^i_\tau + p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right) \leq w^i_\sigma + h^i_\sigma - \left(\frac{1}{1+r_\sigma}\right) p_\sigma \cdot g^i, \tag{5.6}$$

a liquidity (or cash-in-advance) constraint,

$$p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^- - m^i_\sigma \leq 0, \tag{5.7}$$

and a solvency constraint,

$$-\frac{1}{a_\sigma} \sum_{\tau \in \mathcal{S}_\sigma} a_\tau \left(h^i_\tau + \left(\frac{1}{1+r_\tau}\right) p_\tau \cdot \left(e^i_\tau - g^i_\tau\right)\right) \leq w^i_\sigma. \tag{5.8}$$

In the budget constraint (5.6), the nominal interest rate represents the opportunity cost of collecting proceeds of sales with a one-period lag. Solvency constraints serve to eliminate Ponzi schemes, as in Santos and Woodford [15]. They are equivalent to the restriction that an individual can incur any amount of nominal debt that can be repayed in finite time. The value of the endowment in commodities at a date-event is taxed at the nominal interest rate, since revenues from sales are carried over in the form of balances that do not earn interest. Equivalently, one could restrict wealth plans through some sort of transversality conditions.

By imposing sequential constraints (5.6)-(5.7), we faithfully reproduce the sequence of trades that is described by Woodford [17] in a cash-in-advance economy with a representative individual. At a date-event σ, an individual inherits some wealth w^i_σ from previous transactions and receives a transfer h^i_σ. The individual demands securities $\left(v^i_\tau : \tau \in \sigma_+\right)$ and balances n^i_σ so as to satisfy a budget constraint of the form

$$n^i_\sigma + \frac{1}{a_\sigma} \sum_{\tau \in \sigma_+} a_\tau v^i_\tau \leq w^i_\sigma + h^i_\sigma. \tag{5.9}$$

Notice that such a budget constraint only refers to transactions in financial instruments. The individual then trades in commodities, under a cash-in-advance constraint,

$$p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^+ - n^i_\sigma \leq 0. \tag{5.10}$$

Transactions in commodities modify the holdings of balances, which now amount to

$$m^i_\sigma = n^i_\sigma - p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^+ + p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right)^- = n^i_\sigma - p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right). \tag{5.11}$$

At an immediately following date-event $\tau \in \sigma_+$, the inherited wealth, net of taxes, is

$$w^i_\tau = v^i_\tau + m^i_\sigma - p_\sigma \cdot g^i_\sigma. \tag{5.12}$$

Substituting (5.12) into (5.9) and using (5.1) and (5.11), one obtains (5.6). Also, because of (5.11), (5.7) is equivalent to (5.10). Finally, notice that, under market clearing for commodities, $\sum_i m^i = \sum_i n^i$, so that one can express the aggregate demand for balances in terms of final holdings after transactions in commodities, namely, m^i_σ.

5.3 Equilibrium

Given nominal rates of interest r and taxes $(\ldots, g^i, \ldots)$, an equilibrium consists of prices p, state prices a, consistent with set nominal rates of interest r, a collection of plans for individuals $(\ldots, (x^i, m^i, w^i), \ldots)$ and a public plan (h, w) such that the following conditions are satisfied:

(a) For every individual i, the plan (x^i, m^i, w^i) is $\succeq^i$-maximal subject to sequential budget, cash-in-advance and solvency constraints (5.6)-(5.8).
(b) The public plan (h, w) satisfies sequential public budget constraint (5.2), at the supply of balances $m = \sum_i m^i$, with trades only in safe bonds (5.5).
(c) Markets clear for commodities, $\sum_i x^i = \sum_i e^i$, and assets, $\sum_i w^i = w$.

An equilibrium is said to be with no transfers if $h = 0$. It is said to be with unrestricted public portfolio if public liabilities do not consist of safe bonds only (that is, condition (5.5) is omitted).

Equilibria with no transfers are those studied by the literature on the fiscal theory of price determination. Allowing for positive transfers corresponds to the hypothesis that eventual public budget surpluses can be distributed to individuals, though no instruments are available to correct eventual public budget deficits. Apart from this natural assumption of public budget surplus disposability, our notion of an equilibrium is exactly that of Woodford [17] extended to a monetary economy with multiple commodities and heterogenous individuals.

5.4 Consolidation

Since the asset market is complete, the sequence of budget constraints faced by an individual reduces to a single constraint at the initial date-event.

Lemma 1. *At equilibrium, $\sum_{\sigma \in \mathcal{S}} a_\sigma \|p_\sigma\|$ is finite.*

At equilibrium, therefore, the intertemporal budget constraint of an individual,

$$\sum_{\sigma \in \mathcal{S}} \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m^i_\sigma + \sum_{\sigma \in \mathcal{S}} a_\sigma p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right) \leq \delta^i + \zeta^i \sum_{\sigma \in \mathcal{S}} a_\sigma h_\sigma - \sum_{\sigma \in \mathcal{S}} \left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot g^i_\sigma, \tag{5.13}$$

is well-defined.

Lemma 2. *At equilibrium, a consumption plan is attainable under sequential budget, cash-in-advance and solvency constraints (5.6)-(5.8) if and only if it is attainable under the unique intertemporal budget constraint (5.13) and sequential cash-in-advance constraints (5.7). Optimality of a consumption plan requires that, at every date-event,*

$$a_\sigma w^i_\sigma = \sum_{\tau\in\mathcal{S}_\sigma}\left(\frac{r_\tau}{1+r_\tau}\right)a_\tau m^i_\tau + \sum_{\tau\in\mathcal{S}_\sigma} a_\tau p_\tau\cdot\left(x^i_\tau - e^i_\tau\right) + \sum_{\tau\in\mathcal{S}_\sigma}\left(\frac{1}{1+r_\tau}\right)a_\tau p_\tau\cdot g^i_\tau - \zeta^i\sum_{\tau\in\mathcal{S}_\sigma} a_\tau h_\tau$$

and

$$\left(\frac{r_\sigma}{1+r_\sigma}\right)a_\sigma p_\sigma\cdot\left(x^i_\sigma - e^i_\sigma\right)^- = \left(\frac{r_\sigma}{1+r_\sigma}\right)a_\sigma m^i_\sigma.$$

The transversality condition takes the form

$$\lim_{t\to\infty}\sum_{\sigma\in\mathcal{S}_t} a_\sigma w^i_\sigma = 0.$$

As the cash-in-advance constraint is binding whenever the nominal rate of interest is strictly positive, the intertemporal budget constraint of an individual reduces to

$$\sum_{\sigma\in\mathcal{S}}\left(\frac{r_\sigma}{1+r_\sigma}\right)\pi_\sigma\cdot\left(x^i_\sigma - e^i_\sigma\right)^- + \sum_{\sigma\in\mathcal{S}}\pi_\sigma\cdot\left(x^i_\sigma - e^i_\sigma\right) \le \delta^i + \zeta^i\sum_{\sigma\in\mathcal{S}} a_\sigma h_\sigma - \sum_{\sigma\in\mathcal{S}}\left(\frac{1}{1+r_\sigma}\right)\pi_\sigma\cdot g^i_\sigma,$$

where

$$\pi = (\ldots,(\ldots,\pi_{\sigma\nu},\ldots),\ldots) = (\ldots,(\ldots,a_\sigma p_{\sigma\nu},\ldots),\ldots)$$

are present value prices of commodities. Given present value prices of commodities, the optimal consumption plan of an individual is affected by state prices only through modifications of the outside (nominal) claims of such an individual,

$$\delta^i + \zeta^i\sum_{\sigma\in\mathcal{S}} a_\sigma h_\sigma.$$

This is a consequence of a complete asset market. Similarly, any proportional alteration of present value prices of commodities induces an adjustment in consumption plans because of a redistribution of the real value of outside nominal claims across individuals.

At equilibrium, aggregation across individuals yields

$$\sum_{\sigma\in\mathcal{S}}\left(\frac{r_\sigma}{1+r_\sigma}\right)a_\sigma p_\sigma\cdot\sum_i\left(x^i_\sigma - e^i_\sigma\right)^- + \sum_{\sigma\in\mathcal{S}}\left(\frac{1}{1+r_\sigma}\right)a_\sigma p_\sigma\cdot g_\sigma = \delta + \sum_{\sigma\in\mathcal{S}} a_\sigma h_\sigma,$$

which is the intertemporal public budget 'constraint'. Such a constraint emerges only as an equilibrium restriction, as fiscal plans are considered to be given exogenously. In particular, it is a consequence of intertemporal Walras' Law.

5.5 Gains to Trade

At equilibrium, the overall public revenue must (weakly) exceed the initial public liability, that is,

$$\sum_{\sigma\in\mathcal{S}}\left(\frac{r_\sigma}{1+r_\sigma}\right)a_\sigma p_\sigma\cdot\sum_i\left(x^i_\sigma-e^i_\sigma\right)^-+\sum_{\sigma\in\mathcal{S}}\left(\frac{1}{1+r_\sigma}\right)a_\sigma p_\sigma\cdot g_\sigma\geq\delta>0.$$

The initial public liability $\delta>0$ is a given nominal magnitude. The sources of public revenue are seignorage and taxes, respectively, the first and the second term in the left-hand side of the above inequality. If taxes are strictly positive, any high enough overall price level suffices to balance an intertemporal public budget, possibly by the exhaustion of a budget surplus through transfers. Otherwise, the initial public liability is to be honored by means of seignorage. This requires strictly positive nominal rates of interest and, more relevantly, that individuals trade at set nominal rates of interest, so as to hold strictly positive quantities of balances in aggregate. Trade is guaranteed by an assumption on the gains to trade.

Following Dubey and Geanakoplos [8, 9], we make explicit the existence of gains to trade as follows. An allocation $\left(\ldots,x^i,\ldots\right)$ is feasible if $\sum_i x^i\leq\sum_i e^i$. Allocation $\left(\ldots,z^i,\ldots\right)$ weakly Pareto dominates allocation $\left(\ldots,x^i,\ldots\right)$ if, for every individual i, $z^i\succ^i x^i$. We impose that, if a feasible allocation involved no trade at some date-event, then a Pareto improvement would be obtained through a reallocation of consumptions if this readjustment involved real costs of the same magnitude as liquidity costs.

(R) Public Revenue. *Either (i) aggregate taxes* $g=\sum g^i$ *are strictly positive or (ii) nominal rates of interest* r *are strictly positive and every feasible allocation* $\left(\ldots,x^i,\ldots\right)$*, that coincides with the initial allocation* $\left(\ldots,e^i,\ldots\right)$ *at some date-event,*[5] *is weakly Pareto dominated by an allocation* $\left(\ldots,z^i,\ldots\right)$ *satisfying, at every date-event,*

$$\sum_i z^i_\sigma+\left(\frac{r_\sigma}{1+r_\sigma}\right)\sum_i\left(z^i_\sigma-x^i_\sigma\right)^-\leq\sum_i x^i_\sigma.$$

The hypothesis of gains to trade involves fundamentals and nominal rates of interest. Weak Pareto ordering only simplifies presentation, as the hypothesis could be equivalently stated in terms of the true Pareto ordering. Also, the requirement that trade is beneficial at all date-events is stronger than necessary, as it would suffice to require that only feasible allocations that involve no trade starting from some date-event would be weakly Pareto dominated. Though this is not necessary, it is clear that it could be assumed that the alternative allocation $\left(\ldots,z^i,\ldots\right)$ does not modify the allocation $\left(\ldots,x^i,\ldots\right)$ out of the single date-event that exhibits no trade. Finally, but importantly, postulating time additively separable preferences, one could easily provide robust examples of economies that exhibit gains to trade at all nominal rates of interest that do not exceed some upper bound.

[5] To be clearer, there is date-event σ such that $\left(\ldots,x^i_\sigma,\ldots\right)=\left(\ldots,e^i_\sigma,\ldots\right)$.

It is to be noticed that the above assumption guarantees that the intertemporal revenue from taxes and seignorage is strictly positive at every date-event. Such a condition is necessary for the existence of an equilibrium with no transfers. Indeed, starting from a date-event, the public revenue covers outstanding public liabilities, as required by

$$\sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{r_\tau}{1 + r_\tau} \right) a_\tau p_\tau \cdot \sum_i \left(x_\tau^i - e_\tau^i \right)^- + \sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{1}{1 + r_\tau} \right) a_\tau p_\tau \cdot g_\tau = a_\sigma w_\sigma .$$

As the public authority only trades in safe bonds and there are no transfers, the outstanding public liabilities at a date-event are determined at the immediately preceding date-event. If date-events τ' and τ'' are two immediate successors of date-event σ and the public revenue vanishes starting from τ', but not from τ'', there would not be any equilibrium. Under unrestricted public portfolio, existence of an equilibrium with no transfers only requires gains to trade at the initial date-event.

5.6 Existence and Indeterminacy

An equilibrium exists under assumptions on fundamentals that are not more restrictive than those needed for the existence of Walrasian equilibrium, provided that public revenue is guaranteed (assumption (R)).

Proposition 1 (Existence). *Given nominal rates of interest r, there exists an equilibrium with no transfers.*

Equilibria with vanishing transfers are those considered under the fiscal theory of the price level. With no transfers, the overall price level is typically determinate, as the intertemporal public budget constraint requires that

$$\sum_{\sigma \in \mathcal{S}} \left(\frac{r_\sigma}{1 + r_\sigma} \right) a_\sigma p_\sigma \cdot \sum_i \left(x_\sigma^i - e_\sigma^i \right)^- + \sum_{\sigma \in \mathcal{S}} \left(\frac{1}{1 + r_\sigma} \right) a_\sigma p_\sigma \cdot g_\sigma = \delta,$$

and increases in the overall price level generate a public surplus that cannot be exhausted through transfers. In addition, at a date-event, the intertemporal public budget also imposes

$$\sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{r_\tau}{1 + r_\tau} \right) a_\tau p_\tau \cdot \sum_i \left(x_\tau^i - e_\tau^i \right)^- + \sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{1}{1 + r_\tau} \right) a_\tau p_\tau \cdot g_\tau = a_\sigma w_\sigma .$$

Since the public authority only trades in safe bonds, public liabilities w_σ are determined at the preceding date-event. Dividing by a_σ, the constraint at σ is of the same form as the constraint at the initial date-event, so that the overall price level at σ is typically determinate as well. State price a_σ is then determined by present value prices at the initial date-event. This delivers a complete determination of state prices.

With no transfers, existence requires that public liabilities remain strictly positive at every date-event with a strictly positive value of public revenue. Positive public liabilities at all date-events are also required for determinateness of state prices. In addition, the fact that the public authority trades in safe bonds only (better, in a given bundle of securities only) is crucial for the conclusion. In fact, even with no transfers, a full indeterminacy of state prices, up to consistency with nominal rates of interest, would obtain if the composition of the public portfolio of securities were not restricted exogenously (that is, if no condition equivalent to (5.5) were imposed). Indeed, in this case, sequential public budget constraint (5.2) would be implied by sequential budget constraints (5.6) of individuals at equilibrium, so involving no additional restrictions. As the real allocation remains unchanged, indeterminacy of state prices has no real effects. This straightforward conclusion is stated in the following proposition.

Proposition 2 (Indeterminacy with unrestricted public portfolio). *Given nominal rates of interest r, when public portfolio is unrestricted, every equilibrium allocation remains an equilibrium for all state prices a set arbitrarily, up to consistency with nominal rates of interest r.*

Allowing for the distribution of public budget surpluses, price determination fails even when public liabilities consist of safe bonds only.

Proposition 3 (Indeterminacy). *Given nominal rates of interest r, there is $c^* > 0$ such that, for every $c \geq c^*$ and for all state prices a set arbitrarily, up to consistency with nominal rates of interest r, there exists an equilibrium with $\sum_{\sigma \in \mathcal{S}} a_\sigma \|p_\sigma\| = c$.*

That is, the overall price level, $\sum_{\sigma \in \mathcal{S}} a_\sigma \|p_\sigma\|$, is indeterminate up to a lower bound. A proportional increase in prices, in general, bears real effects because it redistributes wealth across individuals. No real effects obtain when, for instance, initial nominal wealths are proportional to transfers (that is, $\delta^i = \zeta^i \delta$).

For a given overall price level, if transfers are strictly positive, state prices exhibit degrees of purely nominal multiplicity, also when portfolio policy is pegged (that is, condition (5.5) is imposed). Indeed, without altering the real allocation and present value prices of commodities, at a date-event, intertemporal public budget constrain only imposes

$$\sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{r_\tau}{1 + r_\tau} \right) a_\tau p_\tau \cdot \sum_i \left(x_\tau^i - e_\tau^i \right)^- + \sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{1}{1 + r_\tau} \right) a_\tau p_\tau \cdot g_\tau = a_\sigma w_\sigma + \sum_{\tau \in \mathcal{S}_\sigma} a_\tau h_\tau .$$

Thus, a slight variation in state prices, up to consistency with nominal rates of interest, can be compensated by a slight variation in the intertemporal profile of transfers, so as to satisfy the requirement of a balanced intertemporal public budget at that date-event. However, as only positive transfers are allowed, large variations in state prices could not be consistent with the given overall price level.

As the overall price level increases, public budget exhibit larger and larger surpluses, to be exhausted through transfers to individuals. As such budget surpluses could be made arbitrarily large, there are in fact no restrictions on state prices at equilibrium. Though the asset market is sequentially complete, it can no longer be assumed that the allocation does not vary with arbitrarily set state prices. Some equilibrium allocations might not be preserved without large increases in the overall price level at those state prices.

5.7 Efficiency

Neither of the basic Welfare Theorems holds in a monetary economy under strictly positive nominal rates of interest: (a) equilibrium allocations, in general, fail to be Pareto efficient; (b) Pareto efficient allocations cannot, in general, be sustained as equilibrium allocations (though they could, trivially, under suitable redistributions of endowments of commodities). The Pareto inefficiency follows from the wedge driven by the cash-in-advance constraint between buying and selling prices of commodities. More importantly, one can construct robust examples of economies exhibiting Pareto-ranked equilibria at given nominal rates of interest.

To clarify our last claim, we provide a simple example without aiming at being exhaustive. There are two individuals and two physical commodities. Let the nominal rate of interest, $r > 0$, be given. Assuming a common rate of time preference across individuals, we can treat a stationary infinite horizon economy as a simple one-period economy. Individual 1's preferences are represented by $x_1^1 + (1+r)^{-1} x_2^1$ and endowments are $(0, 1)$. Individual 2's preferences are represented by $(1+r)^{-1} x_1^2 + x_2^2$ and endowments are $(1, 0)$. A symmetric allocation is represented by $0 \leq \theta \leq 1$, with consumptions $x_\theta^1 = (\theta, 1-\theta)$ and $x_\theta^2 = (1-\theta, \theta)$. The strictly positive amount of public debt is equally distributed across the two individuals. It is simple to verify that, for every $0 < \theta \leq 1$, $\left(x_\theta^1, x_\theta^2\right)$ is an equilibrium with prices π_θ proportional to $(1, 1)$. There is thus a continuum of real equilibria ranking from the no-trade to the symmetric Pareto-efficient allocation. Notice that all such equilibria involve no transfers and can be indexed by the overall price level, up to a lower bound.

The concept of constrained efficiency suitable for monetary economies is not evident. However, given nominal rates of interest, one could obtain an analogue of the two Welfare Theorems using a notion of supportability of an allocation in place of the standard notion of Pareto efficiency (Bloise, Drèze and Polemarchakis [4, Section 7]). When nominal rates of interest vanish, supportability coincides with Pareto efficiency. Otherwise, the interpretation is unclear. Still, it allows for a complete characterization of the full set of equilibria (with transfers), at given nominal rates of interest, in terms of fundamentals only. In addition, it helps in understanding what distinguishes cash-in-advance economies from economies with real intermediation costs, as those that are described by Foley [10].

5.8 Remarks

5.8.1

Various contributions over the last decade (among others, Drèze and Polemarchakis [7] and Dubey and Geanakoplos [8, 9]) have pointed out that finite time is suitable to meaningfully address issues of monetary analysis. Our current work is intended to confirm this view, as our arguments are independent for substance of the horizon being finite or infinite. Remarkably, the finite-horizon model provides a tractable disaggregate framework for a short-term analysis of, for instance, financial markets and nominal price rigidities.

5.8.2

Throughout our analysis, we have maintained the assumption of a sequentially complete asset market. This has allowed for a focus only on balances needed for transaction purposes. A sequentially incomplete asset market would enrich our analysis in a number of ways and, in particular, it would make the variability of inflation rates of real allocative relevance.

5.8.3

Our analysis points at a limited relevance of the fiscal theory of the price level (Woodford [17, 18] and Cochrane [6]). Differently from the framework of that theory, we only assume that eventual public budget surpluses are distributed to individuals through transfers. This seems innocuous and, yet, dramatically changes the conclusions on the determinacy of prices.

Proofs

Proof of lemma 1

Solvency constraints imply that

$$\sum_{\tau \in \mathcal{S}_\sigma} a_\tau \left(h^i_\tau + \left(\frac{1}{1+r_\tau} \right) p_\tau \cdot \left(e^i_\tau - g^i_\tau \right) \right)$$

takes finite value at every non-initial date-event and, hence, at every date-event. By assumptions (T) and (F),

$$\sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{1}{1+r_\tau} \right) a_\tau p_\tau \cdot \left(e^i_\tau - g^i_\tau \right)$$

is finite. Hence, by assumptions (M) and (F), the claim easily follows.

Proof of lemma 2

The argument is standard. Suppose that a plan $\left(x^i, m^i, w^i\right)$ satisfies sequential budget, liquidity and solvency constraints, given initial nominal claims. Multiplication of the sequential budget constraints by a_σ and summation over $\mathcal{S}^t$ yield

$$\sum_{\sigma \in \mathcal{S}_{t+1}} a_\sigma w^i_\sigma + \sum_{\sigma \in \mathcal{S}^t} \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m^i_\sigma + \sum_{\sigma \in \mathcal{S}^t} a_\sigma p_\sigma \cdot x^i_\sigma \leq$$

$$\delta^i + \sum_{\sigma \in \mathcal{S}^t} a_\sigma h^i_\sigma + \sum_{\sigma \in \mathcal{S}^t} a_\sigma p_\sigma \cdot e^i_\sigma - \sum_{\sigma \in \mathcal{S}^t} \left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot g^i_\sigma.$$

The solvency constraint at every date-event, then, implies

$$\sum_{\sigma \in \mathcal{S}^t} \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m^i_\sigma + \sum_{\sigma \in \mathcal{S}^t} a_\sigma p_\sigma \cdot x^i_\sigma - \sum_{\sigma \in \mathcal{S}^t} \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot e^i_\sigma \leq$$

$$\delta^i + \sum_{\sigma \in \mathcal{S}} a_\sigma h^i_\sigma + \sum_{\sigma \in \mathcal{S}} \left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot e^i_\sigma - \sum_{\sigma \in \mathcal{S}} \left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot g^i_\sigma.$$

Since the left-hand side is bounded, the first term is non-decreasing and the other two terms converge, taking the limit as $t \to \infty$ implies

$$\sum_{\sigma \in \mathcal{S}} \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m^i_\sigma + \sum_{\sigma \in \mathcal{S}} a_\sigma p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right) \leq$$

$$\delta^i + \sum_{\sigma \in \mathcal{S}} a_\sigma h^i_\sigma - \sum_{\sigma \in \mathcal{S}} \left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot g^i_\sigma.$$

Therefore, $\left(x^i, m^i\right)$ satisfy the intertemporal budget constraint and sequential liquidity constraints.

Conversely, suppose that a plan $\left(x^i, m^i\right)$ satisfies the intertemporal budget constraint and sequential liquidity constraints and define w^i, at all non-initial date-events, by

$$a_\sigma w^i_\sigma = \sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{r_\tau}{1+r_\tau}\right) a_\tau m^i_\tau + \sum_{\tau \in \mathcal{S}_\sigma} a_\tau p_\tau \cdot \left(x^i_\tau - e^i_\tau\right)$$

$$+ \sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{1}{1+r_\tau}\right) a_\tau p_\tau \cdot g^i_\tau - \sum_{\tau \in \mathcal{S}_\sigma} a_\tau h^i_\tau.$$

Solvency constraints are satisfied, since liquidity constraints imply that

$$-a_\sigma^{-1} \sum_{\tau \in \mathcal{S}} \left(\frac{1}{1+r_\tau}\right) a_\tau p_\tau \cdot e^i_\tau \leq$$

$$-a_\sigma^{-1} \sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{1}{1+r_\tau}\right) a_\tau p_\tau \cdot \left(x^i_\tau - e^i_\tau\right)^- + a_\sigma^{-1} \sum_{\tau \in \mathcal{S}_\sigma} a_\tau p_\tau \cdot \left(x^i_\tau - e^i_\tau\right)^+ \leq$$

$$w^i_\sigma + a_\sigma^{-1} \sum_{\tau \in \mathcal{S}_\sigma} a_\tau h^i_\tau - a_\sigma^{-1} \sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{1}{1+r_\tau}\right) a_\tau p^i_\tau \cdot g^i_\tau.$$

To see that sequential budget constraints are satisfied as well, observe that, at every non-initial date-event, the definition of w^i implies that

$$\left(\frac{r_\sigma}{1+r_\sigma}\right) m^i_\sigma + a^{-1}_\sigma \sum_{\tau\in\sigma_+} a_\tau w^i_\tau + p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right) = w^i_\sigma + h^i_\sigma - \left(\frac{1}{1+r_\sigma}\right) p_\sigma \cdot g^i_\sigma.$$

At the initial date-event, the intertemporal budget constraint and the definition of w^i imply that

$$\left(\frac{r_\phi}{1+r_\phi}\right) m^i_\phi + \sum_{\sigma\in\phi^+} a_\sigma w^i_\sigma + p_\phi \cdot \left(x^i_\phi - e^i_\phi\right) \leq \delta^i + h^i_\phi - \left(\frac{1}{1+r_\phi}\right) p_\phi \cdot g^i_\phi.$$

At an optimal plan, the intertemporal budget constraint must hold with equality since preferences are strictly monotone. Moreover, it is clear that the liquidity constraint is non-binding only if the nominal rate of interest is zero.

Concerning transversality, a plan satisfies solvency constraints only if

$$\liminf \sum_{\sigma\in\mathcal{S}_t} a_\sigma w^i_\sigma \geq 0.$$

It, then, suffices to show that a plan is maximal only if

$$\limsup \sum_{\sigma\in\mathcal{S}_t} a_\sigma w^i_\sigma \leq 0.$$

If not, then, for infinitely many dates, n, and some $\epsilon > 0$,

$$\epsilon + \sum_{\sigma\in\mathcal{S}^n} \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m^i_\sigma + \sum_{\sigma\in\mathcal{S}^n} a_\sigma p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right) \leq$$
$$\delta^i + \sum_{\sigma\in\mathcal{S}^n} a_\sigma h^i_\sigma - \sum_{\sigma\in\mathcal{S}^n} \left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot g^i_\sigma.$$

From the limit, since all series must converge, it follows that

$$\sum_{\sigma\in\mathcal{S}} \left(\frac{r_\sigma}{1+r_\sigma}\right) a_\sigma m^i_\sigma + \sum_{\sigma\in\mathcal{S}} a_\sigma p_\sigma \cdot \left(x^i_\sigma - e^i_\sigma\right) <$$
$$\delta^i + \sum_{\sigma\in\mathcal{S}} a_\sigma h^i_\sigma - \sum_{\sigma\in\mathcal{S}} \left(\frac{1}{1+r_\sigma}\right) a_\sigma p_\sigma \cdot g^i_\sigma,$$

which violates optimality.

Proof of propositions 1-3

The proof is organized as follows. First (I), we introduce a notion of abstract equilibrium, which allows for the determination of present value prices of commodities independently of state prices. Second (II), we show that an abstract equilibrium exists in every truncated economy. Third (III), we prove that the limit of truncated equilibria is an abstract equilibrium of the infinite-horizon economy. Fourth (IV), we show that every abstract equilibrium corresponds to an equilibrium with no transfers for some state prices and, if the overall price level is high enough, to an equilibrium with transfers for arbitrarily set state prices.

(I) Abstract equilibrium

Let X^i be the consumption space of individual i, the positive cone of $\ell_\infty(\mathcal{S} \times \mathcal{N})$, and Π the space of normalized present value prices of commodities, the subset of the positive cone of $\ell_1(\mathcal{S} \times \mathcal{N})$ satisfying the normalization $\|\pi\|_1 = 1$. For (π, x) in $\ell_1(\mathcal{S} \times \mathcal{N}) \times \ell_\infty(\mathcal{S} \times \mathcal{N})$, $\pi \cdot x = \sum_{\sigma \in \mathcal{S}} \pi_\sigma \cdot x_\sigma$ denotes the duality operation.[6]

An abstract equilibrium consists of present value prices of commodities, π, an allocation, $(\ldots, x^i, \ldots)$, an index for (the reciprocal of) the overall price level, $\mu \geq 0$, and an aggregate transfer, $\beta \geq -\mu\delta$, such that:

(a) market clearing is achieved,

$$\sum_i x^i - \sum_i e^i = 0;$$

(b) for every individual,

$$z^i \succ^i x^i \text{ implies } \pi \cdot z^i + \left(\frac{r}{1+r}\right) \pi \cdot \left(z^i - e^i\right)^- > \pi \cdot x^i + \left(\frac{r}{1+r}\right) \pi \cdot \left(x^i - e^i\right)^-$$

and

$$\pi \cdot \left(x^i - e^i\right) + \left(\frac{r}{1+r}\right) \pi \cdot \left(x^i - e^i\right)^- = \mu\delta^i + \zeta^i\beta - \left(\frac{1}{1+r}\right) \pi \cdot g^i.$$

To offset the redundancy stemming from the choice of the unit of account, we add the normalization $\pi \in \Pi$. Notice that, in an abstract equilibrium, $\mu = 0$ is allowed.

[6] Concerning notation, for a countable set $\mathcal{A}$, $\ell(\mathcal{A})$, $\ell_\infty(\mathcal{A})$ and $\ell_1(\mathcal{A})$ denote, respectively, the vector space of all real-valued maps, bounded real-valued maps and summable real-valued maps on $\mathcal{A}$, where summable means that $\|x\|_1 = \sum_{\alpha \in \mathcal{A}} |x_\alpha|$ is finite. For (x, z) in $\ell(\mathcal{S}) \times \ell(\mathcal{S} \times \mathcal{N})$, $xz = zx$ is the element of $\ell(\mathcal{S} \times \mathcal{N})$ obtained by point-wise product, $(\ldots, x_\sigma(\ldots, z_{\sigma\nu}, \ldots), \ldots) = (\ldots, (\ldots, z_{\sigma\nu}, \ldots) x_\sigma, \ldots)$. Moreover, we use

$$\left(\frac{1}{1+r}\right) = \left(\ldots, \left(\frac{1}{1+r_\sigma}\right), \ldots\right) \quad \text{and} \quad \left(\frac{r}{1+r}\right) = \left(\ldots, \left(\frac{r_\sigma}{1+r_\sigma}\right), \ldots\right)$$

for notational convenience. Aliprantis and Border [1] provide a useful treatment of infinite-dimensional analysis.

(II) Truncations

Suppose that all vector spaces are of finite dimension, which corresponds to a truncated economy. We show that, given any $\mu \geq 0$ small enough, an abstract equilibrium exists for some $\beta \geq -\mu\delta$; alternatively, setting $\beta = 0$, an abstract equilibrium exists for some $\mu \geq 0$.

Choose any $\mu \geq 0$ small enough so that, for every individual i,

$$0 < \epsilon \leq \mu \left(\delta^i - \zeta^i \delta\right) + \left(\frac{1}{1+r}\right) \pi \cdot \left(e^i - g^i\right), \tag{5.14}$$

for all $\pi \in \Pi$. (This can be done as the net endowment is interior and nominal rates of interest are bounded.) Consider the space of all

$$f = \left(\left(\ldots, x^i, \ldots\right), \pi, \beta\right) \in \cdots \times X^i \times \cdots \times \Pi \times B = F,$$

where X^i is the consumption space of individual i, Π is the space of normalized present value prices and $B = \{\beta \in \mathbb{R} : \beta \geq -\mu\delta\}$. A correspondence $\hat{f} \mapsto \bar{f}$ is defined by:

(a) $\bar{x}^i$ is an optimal choice subject to

$$\hat{\pi} \cdot \left(x^i - e^i\right) + \left(\frac{r}{1+r}\right) \hat{\pi} \cdot \left(x^i - e^i\right)^- \leq \mu\delta^i + \zeta^i \hat{\beta} - \left(\frac{1}{1+r}\right) \hat{\pi} \cdot g^i;$$

(b) $\bar{\beta}$ solves

$$\left(\frac{r}{1+r}\right) \hat{\pi} \cdot \sum_i \left(\hat{x}^i - e^i\right)^- + \left(\frac{1}{1+r}\right) \hat{\pi} \cdot g = \mu\delta + \beta;$$

(c) $\bar{\pi}$ maximizes

$$\pi \cdot \sum_i \left(\hat{x}^i - e^i\right).$$

A fixed point exists and it can be shown to be an abstract equilibrium of the truncated economy (Drèze and Polemarchakis [7]). Therefore, in a truncated economy, an abstract equilibrium exists for all arbitrarily chosen $\mu \geq 0$ small enough.

Alternatively, set $\beta = 0$. Notice that, for every $\mu \geq 0$,

$$0 < \epsilon \leq \mu\delta^i + \left(\frac{1}{1+r}\right) \pi \cdot \left(e^i - g^i\right). \tag{5.15}$$

Consider the space of all

$$f = \left(\left(\ldots, x^i, \ldots\right), \pi, \beta\right) \in \cdots \times X^i \times \cdots \times \Pi \times M = F,$$

where X^i is the consumption space of individual i, Π is the space of normalized present value prices and $M = \{\mu \in \mathbb{R} : \mu \geq 0\}$. A correspondence $\hat{f} \mapsto \bar{f}$ is defined by:

(a) $\bar{x}^i$ is an optimal choice subject to

$$\hat{\pi} \cdot \left(x^i - e^i\right) + \left(\frac{r}{1+r}\right) \hat{\pi} \cdot \left(x^i - e^i\right)^- \leq \mu\delta^i - \left(\frac{1}{1+r}\right) \hat{\pi} \cdot g^i;$$

(b) $\bar{\mu}$ solves

$$\left(\frac{r}{1+r}\right) \hat{\pi} \cdot \sum_i \left(\hat{x}^i - e^i\right)^- + \left(\frac{1}{1+r}\right) \hat{\pi} \cdot g = \mu\delta;$$

(c) $\bar{\pi}$ maximizes

$$\pi \cdot \sum_i \left(\hat{x}^i - e^i\right).$$

A fixed point exists and it can be shown to be an abstract equilibrium of the truncated economy (Drèze and Polemarchakis [7]). Therefore, in a truncated economy, an abstract equilibrium exists with no aggregate transfer, $\beta = 0$.

(III) Limit

We now make truncation explicit. For an element x of $\ell\left(\mathcal{S} \times \mathcal{N}\right)$, let $x\chi_t$ denote its truncation at t. That is, $\left(x\chi_t\right)_\sigma = x_\sigma$, if $0 \leq t_\sigma \leq t$, and $\left(x\chi_t\right)_\sigma = 0$, otherwise. A t-truncated economy is constructed as follows: preferences on the consumption space, X^i, the positive cone of $\ell_\infty\left(\mathcal{S} \times \mathcal{N}\right)$, are recovered using $x^i \succeq^{it} z^i$ if and only if $x^i\chi_t + \left(e^i - e^i\chi_t\right) \succeq^i z^i\chi_t + \left(e^i - e^i\chi_t\right)$; truncated present value prices of commodities are elements of $\Pi^t = \{\pi \in \Pi : \pi\chi_t = \pi\}$.

Consider a sequence of abstract equilibria of t-truncated economies: for every t, the allocation is $\left(\ldots, x^{it}, \ldots\right)$, present value prices of commodities are π^t, the aggregate transfer is β^t and the index for the overall price level is μ^t. Along such a sequence of truncated equilibrium, one might assume that either $\mu^t = \mu \geq 0$ is constant or $\beta^t = \beta = 0$ is constant. To simplify, write $\alpha^{it} = \mu^t\delta^i + \zeta^i\beta^t \geq 0$.

Letting

$$\varphi^t = \left(\ldots, \varphi^t_\sigma, \ldots\right) = \left(\ldots, \left(\frac{1}{1+r_\sigma}\right) \pi^t_\sigma, \ldots\right),$$

π^t and φ^t can be viewed as elements of $ba\left(\mathcal{S} \times \mathcal{N}\right)$, the norm dual of $\ell_\infty\left(\mathcal{S} \times \mathcal{N}\right)$ consisting of all finitely additive set functions on $\mathcal{S} \times \mathcal{N}$ and endowed with the norm $\|\cdot\|_{ba}$ (the norm of total variation). Let $\sigma\left(ba, \ell_\infty\right)$ denote the weak* topology of $ba\left(\mathcal{S} \times \mathcal{N}\right)$. Since

$$\left\|\varphi^t\right\|_1 = \left\|\varphi^t\right\|_{ba} \leq \left\|\pi^t\right\|_{ba} = \left\|\pi^t\right\|_1 = 1$$

and since, by Alaoglu Theorem, the unit sphere in $ba\left(\mathcal{S} \times \mathcal{N}\right)$ is $\sigma\left(ba, \ell_\infty\right)$ compact, without loss of generality, $\{\pi^t\}$ and $\{\varphi^t\}$ converge to π and φ, respectively, in the $\sigma\left(ba, \ell_\infty\right)$ topology. Moreover, both π and φ, as well as $\pi - \varphi$, are positive elements of $ba\left(\mathcal{S} \times \mathcal{N}\right)$ and $0 < \|\varphi\|_{ba} \leq \|\pi\|_{ba} = 1$.

By Tychonov Theorem, without loss of generality, every $\left\{x^{it}\right\}$ converges to x^i in the product topology. Since the product and the Mackey topology coincide on bounded subsets of $\ell_\infty(\mathcal{S} \times \mathcal{N})$, it follows that every $\{x_{it}\}$ converges to x^i in the Mackey topology.

As $\{\mu^t\}$ and $\{\beta^t\}$ can be assumed to be bounded, without loss of generality, they converge to μ and β, respectively. Defining $\alpha^i = \mu\delta^i + \zeta^i\beta$, it follows that every α^{it} converges to α^i.

We now show that the limit of abstract equilibria in the truncated economies is an abstract equilibrium of the economy over an infinite horizon. The proof, which is presented in a sequence of steps (1)-(5), uses standard arguments.

1

Decomposition. Since π (φ) is a positive linear functional, it follows from the Yosida-Hewitt Theorem that there is a unique decomposition $\pi = \pi_f + \pi_b$ ($\varphi = \varphi_f + \varphi_b$), where π_f (φ_f) is a positive functional in $\ell_1(\mathcal{S} \times \mathcal{N})$, the Mackey-topology dual of $\ell_\infty(\mathcal{S} \times \mathcal{N})$, and π_b (φ_b) is a positive finitely additive measure (a pure charge) vanishing on all vectors having only a finite number of non-zero components.

2

$z^i \succeq^i x^i$ *implies*

$$\varphi \cdot g^i + \pi \cdot \left(z^i - e^i\right)^+ \geq \alpha^i + \varphi \cdot \left(z^i - e^i\right)^- .$$

For a strictly positive real number, λ, $z^i + \lambda e^i \succ^{it} x^{it}$ for all t large enough, which implies that

$$\varphi^t \cdot g^i + \pi^t \cdot \left(z^i - (1-\lambda)\, e^i\right)^+ \geq \alpha^{it} + \varphi^t \cdot \left(z^i - (1-\lambda)\, e^i\right)^- .$$

Taking the limit, one obtains

$$\varphi \cdot g^i + \pi \cdot \left(z^i - (1-\lambda)\, e^i\right)^+ \geq \alpha^i + \varphi \cdot \left(z^i - (1-\lambda)\, e^i\right)^- .$$

As lattice operations are continuous in the norm topology and π and φ are norm-continuous linear functionals, letting λ go to zero, the claim is proven.

3

$z^i \succ^i x^i$ *implies*

$$\varphi \cdot g^i + \pi \cdot \left(z^i - e^i\right)^+ > \alpha^i + \varphi \cdot \left(z^i - e^i\right)^- .$$

Continuity of preferences implies that $\lambda z^i \succ^i x^i$ for some $0 < \lambda < 1$. Since

$$\varphi \cdot g^i + \lambda\pi \cdot \left(z^i - e^i\right) + (\pi - \varphi) \cdot \left(\lambda z^i - e^i\right)^- \geq \alpha^i + (1-\lambda)\, \pi \cdot e^i$$

and

$$\lambda \left(z^i - e^i\right)^- + (1-\lambda)\, e^i \geq \left(\lambda z^i - e^i\right)^-,$$

one obtains

$$\begin{aligned}\varphi \cdot g^i + \pi \cdot \left(z^i - e^i\right)^+ &\geq \alpha^i + \varphi \cdot \left(z^i - e^i\right)^- + \left(\frac{1-\lambda}{\lambda}\right)\left(\varphi \cdot \left(e^i - g^i\right) + \alpha^i\right) \\ &\geq \alpha^i + \varphi \cdot \left(z^i - e^i\right)^- + \left(\frac{1-\lambda}{\lambda}\right)\epsilon,\end{aligned}$$

where the last inequality follows from solvency conditions (5.14)-(5.15).

4

$\pi_b = \varphi_b = 0$ *and*

$$\varphi \cdot g^i + \pi \cdot \left(x^i - e^i\right)^+ = \alpha^i + \varphi \cdot \left(x^i - e^i\right)^-.$$

For a vector u in $\ell\left(\mathcal{S} \times \mathcal{N}\right)$,

$$\varphi^t \cdot u = \pi^t \cdot \left(\left(\frac{1}{1+r}\right) u\right)$$

holds at every t and, hence, in the limit,

$$\varphi \cdot u = \pi \cdot \left(\left(\frac{1}{1+r}\right) u\right).$$

Using truncations, one can show that

$$\varphi_b \cdot u = \pi_b \cdot \left(\left(\frac{1}{1+r}\right) u\right).$$

It follows that $\pi_b = 0$ if and only if $\varphi_b = 0$.

Suppose that $\pi_b > 0$, so that, by interior net endowments, $\varphi_b \cdot \left(e^i - g^i\right) = \xi > 0$. Since $x^i\chi_t + \lambda e^i \chi_t \succ^i x^i$ for all t large enough and all strictly positive real numbers, λ,

$$\begin{aligned}\varphi \cdot g^i + \pi \cdot \left(x^i - e^i\right)^+ \chi_t + \lambda \pi \cdot e^i \chi_t &\geq \\ \varphi \cdot g^i + \pi \cdot \left(x^i \chi_t + \lambda e^i \chi_t - e^i\right)^+ &\geq \\ \alpha^i + \varphi \cdot \left(x^i \chi_t + \lambda e^i \chi_t - e^i\right)^- &\geq \\ \alpha^i + \varphi \cdot \left(x^i - e^i\right)^- \chi_t - \lambda \varphi \cdot e^i \chi_t + \varphi \cdot \left(e^i - e^i \chi_t\right)&.\end{aligned}$$

In the limit, one obtains

$$\begin{aligned}\varphi_f \cdot g^i + \pi_f \cdot \left(x^i - e^i\right)^+ + \lambda\left(\pi_f - \varphi_f\right) \cdot e^i &\geq \\ \alpha^i + \varphi_f \cdot \left(x^i - e^i\right)^- + \varphi_b \cdot \left(e^i - g^i\right) &\geq \\ \alpha^i + \varphi_f \cdot \left(x^i - e^i\right)^- + \xi&.\end{aligned}$$

Thus,

$$\varphi_f \cdot g^i + \pi_f \cdot \left(x^i - e^i\right)^+ \geq \alpha^i + \varphi_f \cdot \left(x^i - e^i\right)^- + \xi.$$

To prove equality, notice that, for all $0 \leq s \leq t$,

$$\varphi^t \cdot g^i + \pi^t \cdot \left(x^{it} - e^i\right)^+ \chi_s \leq \alpha^i + \varphi^t \cdot \left(x^{it} - e^i\right)^- \chi_s + \varphi^t \cdot \left(e^i - e^i \chi_s\right).$$

Therefore, in the limit,

$$\varphi_f \cdot g^i + \pi_f \cdot \left(x^i - e^i\right)^+ \leq \alpha^i + \varphi_f \cdot \left(x^i - e^i\right)^- + \xi.$$

Summing over individuals,

$$\varphi_f \cdot g + (\pi_f - \varphi_f) \cdot \sum_i \left(x^i - e^i\right)^- > \sum_i \alpha^i.$$

Observe that, for all $0 \leq s \leq t$,

$$\varphi^t \cdot g \chi_s + \left(\pi^t - \varphi^t\right) \cdot \left(\sum_i \left(x^{it} - e^i\right)^-\right) \chi_s \leq \sum_i \alpha^{it}.$$

In the limit,

$$\varphi_f \cdot g + (\pi_f - \varphi_f) \cdot \sum_i \left(x^i - e^i\right)^- \leq \sum_i \alpha^i,$$

a contradiction to the previous reverse strict inequality.

5

Limit is an abstract equilibrium. By point-wise limits, one obtains

$$\varphi = \left(\frac{1}{1+r}\right) \pi,$$

thus proving the claim.

(IV) Equilibrium

In every abstract equilibrium, it is clear that present value prices π are strictly positive. We show that, at every date-event,

$$\left(\frac{r_\sigma}{1+r_\sigma}\right) \pi_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- + \left(\frac{1}{1+r_\sigma}\right) \pi_\sigma \cdot g_\sigma > 0. \tag{5.16}$$

Indeed, if $g_\sigma = 0$, then $r_\sigma > 0$ by our assumption (R). If public revenue vanishes at some date-event, it follows that the abstract equilibrium allocation $\left(\ldots, x^i, \ldots\right)$ coincides with the initial allocation $\left(\ldots, e^i, \ldots\right)$ at that date-event. By the condition on trade at equilibrium (R), there exists an allocation $\left(\ldots, z^i, \ldots\right)$ that weakly Pareto dominates allocation $\left(\ldots, x^i, \ldots\right)$ and satisfies

$$\sum_i z^i + \left(\frac{r}{1+r}\right) \sum_i \left(z^i - x^i\right)^- \leq \sum_i x^i.$$

Since $\left(z^i - e^i\right)^- \leq \left(z^i - x^i\right)^- + \left(x^i - e^i\right)^-$, it follows that

$$\sum_i z^i + \left(\frac{r}{1+r}\right) \sum_i \left(\left(z^i - e^i\right)^- - \left(x^i - e^i\right)^-\right) \leq \sum_i x^i.$$

By the optimality of consumption plans, for every individual,

$$\pi \cdot z^i + \left(\frac{r}{1+r}\right) \pi \cdot \left(z^i - e^i\right)^- > \pi \cdot x^i + \left(\frac{r}{1+r}\right) \pi \cdot \left(x^i - e^i\right)^- .$$

Thus,

$$\pi \cdot \left(z^i - x^i + \left(\frac{r}{1+r}\right) \left(\left(z^i - e^i\right)^- - \left(x^i - e^i\right)^-\right)\right) > 0,$$

which, summing over individuals, implies

$$\pi \cdot \left(\sum_i z^i - \sum_i x^i + \left(\frac{r}{1+r}\right) \sum_i \left(\left(z^i - e^i\right)^- - \left(x^i - e^i\right)^-\right)\right) > 0,$$

a contradiction.

Letting

$$u_\sigma = \sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{r_\tau}{1+r_\tau}\right) \pi_\tau \cdot \sum_i \left(x^i_\tau - e^i_\tau\right)^- + \sum_{\tau \in \mathcal{S}_\sigma} \left(\frac{1}{1+r_\tau}\right) \pi_\tau \cdot g_\tau,$$

the above argument shows that, in every abstract equilibrium, u is a strictly positive element of $\ell\left(\mathcal{S}\right)$ with $u_\phi = \mu\delta + \beta > 0$. In particular, this proves that, if there is no aggregate transfer, $\beta = 0$, then $\mu > 0$ in every abstract equilibrium.

To obtain an equilibrium from an abstract equilibrium with $\mu > 0$, one needs only to show the existence of state prices a, consistent with nominal rates of interest r, public liabilities w and transfers h that satisfy, at every date-event σ,

$$\frac{1}{\mu}\frac{u_\sigma}{a_\sigma} = w_\sigma + \frac{1}{a_\sigma} \sum_{\tau \in \mathcal{S}_\sigma} a_\tau h_\tau. \tag{5.17}$$

and

$$w_{\tau'} = w_{\tau''} \text{ for all } \left(\tau', \tau''\right) \in \sigma_+ \times \sigma_+. \tag{5.18}$$

Prices are then obtained by

$$\left(\ldots, \left(\ldots, p_{\sigma\nu}, \ldots\right), \ldots\right) = \frac{1}{\mu}\left(\ldots, \left(\ldots, \frac{\pi_{\sigma\nu}}{a_\sigma}, \ldots\right), \ldots\right).$$

Condition (5.17) ensures that the asset market clears, while condition (5.18) guarantees that public liabilities consist of safe bonds only.

Given any abstract equilibrium with $\beta = 0$, an equilibrium with no transfers, $h = 0$, corresponds to state prices a that, subject to no arbitrage (restrictions (5.1)), solve, at every date-event σ,

$$\frac{a_\tau}{\sum_{\tau \in \sigma_+} a_\tau} = \frac{u_\tau}{\sum_{\tau \in \sigma_+} u_\tau} \text{ for all } \tau \in \sigma_+.$$

This proves proposition 1.

To prove proposition 3, observe that there is an abstract equilibrium for every $\mu > 0$ small enough, with associated aggregate transfers $\beta \geq -\mu\delta$. Suppose that, as μ vanishes, there is an abstract equilibrium with

$$\left(\frac{r_\phi}{1+r_\phi}\right) \pi_\phi \cdot \left(x^i_\phi - e^i_\phi\right)^- + \left(\frac{1}{1+r_\phi}\right) \pi_\phi g_\phi \leq \mu\delta.$$

One can show that the limit is also an abstract equilibrium, which contradicts the strict positivity of public revenue established by condition (5.16). Hence, for every $\mu > 0$ small enough, there is an abstract equilibrium with

$$\left(\frac{r_\phi}{1+r_\phi}\right) \pi_\phi \cdot \left(x^i_\phi - e^i_\phi\right)^- + \left(\frac{1}{1+r_\phi}\right) \pi_\phi g_\phi > \mu\delta, \tag{5.19}$$

that is, such that the public revenue at the initial date-event exceeds the initial public liability. Consider any abstract equilibrium such that (5.19) is satisfied and set state prices a arbitrarily, subject to no arbitrage (restrictions (5.1)). Transfers h are obtained, at the initial date-event ϕ, by

$$\left(\frac{r_\phi}{1+r_\phi}\right) \pi_\phi \cdot \sum_i \left(x^i_\phi - e^i_\phi\right)^- + \left(\frac{1}{1+r_\phi}\right) \pi_\phi \cdot g_\phi - \mu\delta = \mu h_\phi \geq 0;$$

at every non-initial date-event σ, by

$$\left(\frac{r_\sigma}{1+r_\sigma}\right) \pi_\sigma \cdot \sum_i \left(x^i_\sigma - e^i_\sigma\right)^- + \left(\frac{1}{1+r_\sigma}\right) \pi_\sigma \cdot g_\sigma = \mu a_\sigma h_\sigma \geq 0.$$

This construction trivially fulfills conditions (5.17)-(5.18) with public liabilities w vanishing at all non-initial date-event, that is, with $w_\sigma = 0$ for every non-initial date-event σ. The argument proves proposition 3.

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6

Do the Wealthy Risk More Money? An Experimental Comparison*

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Summary. Are poor people more or less likely to take money risks than wealthy folks? We find that risk attraction is more prevalent among the wealthy when the amounts of money at risk are small (not surprising, since ten dollars is a smaller amount for a wealthy person than for a poor one), but, interestingly, for the larger amounts of money at risk the fraction of the nonwealthy displaying risk attraction actually exceeds that of the wealthy. We also replicate our previous finding that many people display risk attraction for small money amounts, but risk aversion for large ones.

Key words: Risk attraction, Risk aversion, Wealth, Experiments.

JEL Classification Numbers: C91, D81.

6.1 Introduction

Are poor people more or less likely to display attraction to pure money risks than wealthy folks? We experimentally address the dependence of risk attitudes (risk aversion or attraction) on wealth by conducting the same experiment on two groups of participants, the Nonwealthy and the Wealthy.

Because we are interested in the dependence on wealth of risk attitude, rather that the degree of risk aversion, our participants are required to choose between alternatives with the same expected money value: all risk averse individuals will then choose the safe alternative, no matter what their degree of risk aversion is.[3] Thus,

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[3] See footnote 10 below for a comment on risk neutrality in the context of our experiments.

we do not directly address the related, and often-studied, issue of the dependence of absolute or relative risk aversion on wealth.[4]

In the experiment, participants (subjects) were told that they would be randomly assigned, without replacement, to one of seven money amounts. But they had a 20% chance of losing the amount, and could buy an actuarially fair insurance against this loss. Participants were asked to decide, before knowing the amount of money they would be assigned, whether to insure or not each of the seven possible amounts. If the participant chose not to insure a given money amount, then we say that he displayed risk attraction for that amount. If, on the contrary, he (or she) chose to insure, then we say that he displayed risk aversion for that amount.

In a nutshell, we found that risk attraction was more prevalent among the Wealthy when the amounts of money at risk were small; but this pattern does not carry over to large money amounts where, if anything, risk attraction was more prevalent among Nonwealthy.

Replicating the feature evidenced in Bosch-Domènech and Silvestre (1999, 2005), we also found that a large majority of participants display what we call the *standard pattern*: whenever risk attraction is displayed in a choice involving a given amount of money, risk attraction is also displayed for any smaller amount of money. We can then define a participant's highest risked amount (HRA) as the highest money amount that she or he fails to insure (we set at zero the HRA of a participant who insures all amounts). In our experiments, the bottom 86% of the Wealthy distribution have a higher HRA than the bottom 86% of the Nonwealthy distribution, indicating that risk attraction is more prevalent among Wealthy than among Nonwealthy. But the top 14% of the Nonwealthy distribution have a higher HRA than the top 14% of the Wealthy distribution, i.e., the very risk-attracted Nonwealthy (relative to their fellow Nonwealthy) risk more that the very-risk attracted Wealthy.

Given our previous results showing that many people display risk attraction for small money amounts, but risk aversion for large ones, the finding that Wealthy are more likely to display risk attraction for small money amounts is not surprising: ten dollars represent a smaller sum for a wealthy person than for a poor one. But Nonwealthy's higher likelihood of displaying risk attraction when the amounts of money at stake are large is noteworthy.

6.2 The experiment

We run the experiment with two groups of Catalan participants, all in their last year of *batxillerat*, which is the university-bound track in high school. The two groups have the same age, identical formal education, and involve similar proportions of males and females.[5]

[4] The large literature on this issue starts with the pioneering work of Kenneth Arrow (1965, 1970) and John Pratt (1964).

[5] According to Luigi Guiso and Monica Paiella (2001, p. 9): "risk averse are younger and less educated; they are less likely to be male..." Empirical research on wealth and risk has

The first group includes students of a public high school in a low-income neighborhood in Barcelona. The second group includes students attending a high-tuition private school in a plush area in the same city. We will call these groups Nonwealthy and Wealthy, respectively. In Spain, public schools are free and, in large cities, attract mostly students from the neighborhood. A public school in a low-income neighborhood is unlikely to receive any applications from students living in well-to-do neighborhoods. Therefore, by choosing participants among the students in these two schools we were reasonably certain to observe children from families with middle to low incomes in one place and children from high–income families in the other. A questionnaire about family and social background, which the participants in the experiment had to answer, reveals that this assumption appears to be correct. In Table 1 we report their answers to the question about their parents' jobs, showing that the ratio of low-paid jobs over high-paid jobs is clearly larger among the Nonwealthy group.

Table 1. Distribution of parents' jobs in the two groups of Nonwealthy and Wealthy participants, in percentage of answers. (Out of 42 possible answers for each group, we received 39 from Nonwealthy and 28 from Wealthy).

Parents' jobs	Nonwealthy	Wealthy
Housewife	18%	0%
Blue collar	18%	0%
White collar	51%	21%
Professional	8%	43%
Small business owner	15%	21%
Business executive	0%	14%

While from the answers to the questionnaire we cannot ascertain the degree of wealth dispersion within the two groups, it appears unlikely that the highest levels of wealth in the Nonwealthy group could be above the lowest in the Wealthy group. We therefore assume that Wealthy and Nonwealthy are two groups clearly separated by their wealth levels. Needless to say, characteristics other than wealth differences can be a factor in the experimental results.[6] Yet, our participants share those character-

to wrestle to separate the effects of different types of wealth, in particular, wealth measured in human capital and wealth measured in net assets, two types of wealth that often yield opposite effects on risk taking (see Martin Halek and Joseph Eisenhauer, 2001, p. 13 and 22). We have no such problem in our experiment, since we can safely assume that participants have similar amounts of human capital.

[6] We know from the responses to our questionnaire that 51% of parents' jobs among Nonwealthy are "white collar," while only 20% are of this type among Wealthy. Could it be that white collars, who tend to receive their salary on a regular basis, are less risk-takers than business executives and business owners? And, if so, could it be that children in these families have been socialized to become less risk-takers? Or, on the contrary, is it the case that professionals (many of whom, in Spain, could be civil servants with secure jobs) are less risk-taking and, representing 43% of Wealthy parents but only 8% of Non-Wealthy parents,

istics that are usually singled out as influencing risk attitudes, such as religion, race, employment, marital status, age, or education.

We performed the experiment with each group in a single session (no preliminary pilot sessions) using twenty-one participants per group who were chosen randomly among the male and female volunteers. We tried to maintain a similar proportion of sexes in both experiments (the female/male ratio was 10/11 in Wealthy and 9/12 in Nonwealthy). Participants were told that they would be randomly assigned, without replacement, to one of seven money amounts, denominated in the (former) Spanish currency, pesetas: 500, 1000, 2000, 5000, 7500, 10000 or 15000 (i.e., approximately, in PPP, from US$ 3 to US$ 100).[7] But participants had a 20% chance of losing the amount, and could buy an actuarially fair insurance against this loss.[8] Participants were asked to decide, before knowing to which group they would belong, whether for each of the seven possible amounts to insure or not to insure it.

To record their decisions, as in other similar experiments that we have run (Bosch-Domènech and Silvestre 1999, 2004), participants were given a seven-page booklet, one page for each possible money amount. Every page had five boxes arranged vertically. The amount of money was printed in the first box and the insurance premium in the second one, with the statement that the premium was exactly 20% of the amount. The third box contained two check cells, one for insuring the amount, and another one for not insuring it. Below a separating horizontal line, two more boxes were later used to record first whether the money amount was lost and, second, the take-home sum. In order to facilitate decisions, a matrix on the back of the page showed all the payoffs. The information was given to the participants as written instructions (available on request), which were read aloud by the experimenter. The experiment began after all questions were answered in private.

Once all participants had registered their decisions (under no time constraint: nobody used more than 15 minutes), their pages were collected. Participants were then called one by one to an office with an urn that initially contained twenty one pieces

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have socialized a larger proportion of Wealthy respondents to being less risk-taking? Since the questionnaire was answered anonymously, we cannot associate observed behavior to parents jobs and, consequently, we cannot even try to answer these questions.

[7] 15000 pesetas is a large amount of money for Catalan high school students. In the questionnaire mentioned above, we also asked participants to compare this amount of money with their monthly income. For Nonwealthy it represented an average of 175% (16 answers out of 21), while for Wealthy the average was lower and equal to 113% (12 answers out of 21).

[8] We avoided extreme probabilities: 0.2 seems to be above the range that tends to be overweighted (Malcolm Preston and Philip Baratta, 1948) and below the range that tends to be underweighted (0.3 to 0.8 according to Michele Cohen *et al.*, 1985). Also, one observes in Steven Kachelmeier and Mohamed Shehata (1992) that, at a 0.8 probability of winning, participants tend to be risk neutral, which is not the case for lower probabilities. One could be more confident, then, that the choice of a probability of winning of 0.8 may not bias, by itself, the degree of risk aversion. But Amos Tversky and Daniel Kahneman (1992) suggest that there is overweighting at 0.2 and underweighting at 0.8 (Figure 3.3), whereas the earlier Figure 2.4 in Kahneman and Tversky (1979) suggests no overweighting at 0.2. At the other extreme, William Harbaugh *et al.*, (2002) claim to observe overweighting at 0.8.

of paper: each piece indicated one money amount, and each of the seven amounts occurred three times. A piece of paper was randomly drawn without replacement: the experimenter and the participant then checked whether the participant had insured that particular amount. If she had, then the premium was subtracted from the money amount to obtain the take-home sum. If she had not, then a number from one to five was randomly drawn from another urn. If the number one was drawn, then the participant would take nothing home. Otherwise, she would take home the money amount. The participant was then paid and dismissed, and the next participant was escorted into the office.

The following element of the experiment was not included in the written instructions. As described above, we asked participants to consider all seven possible money amounts with the intention of obtaining a larger data set. We wanted to check whether this procedure tends to elicit the same choices as when participants make only one choice, as reported by Chris Starmer and Robert Sugden (1991). Consequently, we allowed each participant to reconsider his or her reported decision after his or her money amount was selected. Of the forty-two participants involved, only one, labeled JN, changed his mind (from non-insurance to insurance). This observation seems insufficient to negate the overall reliability of hypothetical decisions as accurate descriptions of real choices, but it does exemplify a higher likelihood of risk aversion in played games (Robin Hogarth and Hillel Einhorn, 1990).

The experimental data are presented in Tables A1 and A2 in the Appendix.

6.3 Stylized facts

From the results in Tables A1 and A2, we can construct Table 2, which supports the following result:

Result 1. *On average, Wealthy participants are more likely to risk (decline to insure) small money amounts, but our experimental outcomes suggest that Nonwealthy participants may be more likely to risk large money amounts.*[9]

In particular:

- For any amount lower than or equal to 5000 (about US$30), the number of Wealthy participants who risk that amount exceeds that of Nonwealthy participants.
- And for any amount larger than 5000, the number of Nonwealthy participants who risk that amount exceeds that of Wealthy participants.

[9] Note the contrast with the empirical data reported by Guiso and Paiella (2001, p. 9): "... the risk-averse are significantly less wealthy than the risk lovers or neutrals." But notice that the authors characterize each individual by one single measure of risk aversion, while we observe that individuals may have different attitudes towards risk depending on the income at risk. Also, their statement should be qualified by their own conclusion (p. 31) that there is limited empirical evidence on the sign of the relationship between risk attitude and wealth. But see Bas Donkers *et al.*, (2001) p. 182, who observe that risk aversion appears to decrease with income using data from a questionnaire on hypothetical risks.

A preliminary conclusion would be, therefore, that Wealthy participants are more likely to display risk aversion than less wealthy participants for large enough amounts of money at stake, but more likely to display risk attraction when the amounts of money are small.[10, 11]

Table 2. Number and fraction of participants in the Wealthy and Nonwealthy groups that risk the various money amounts.

	Amount of Money						
	500	1000	2000	5000	7500	10000	15000
Number of Nonwealthy Participants Who Risk the Amount	11	7	3	4	5	2	4
Fraction of Nonwealthy Participants Who Risk the Amount	0.52	0.33	0.14	0.19	0.24	0.10	0.19
Number of Wealthy Participants Who Risk the Amount	18	11	6	5	1	1	2
Fraction of Wealthy Participants Who Risk the Amount	0.86	0.52	0.29	0.24	0.05	0.05	0.10

We say that an individual follows the standard pattern if, whenever she displays risk attraction in a choice involving a given amount of money, she also displays risk attraction for any smaller amount of money. The inspection of Tables A.1 and A.2 yields the following result.

Result 2. *A large proportion of participants* (18/21= 86%) *follow the standard pattern in either group.*

6.4 The distribution of the degree of risk attraction

Within the standard pattern, we can rank a participant's risk attraction by the highest amount that she fails to insure: we call it *highest risked amount*, or HRA. We set at zero the HRA of a participant who insures all amounts.

If we disregard the participants who violate the standard pattern, then our experimental observations, complemented by linear interpolation, generate a distribution

[10] A risk-neutral participant could choose either the certain or the uncertain prospect, his choice being random. But the likelihood that the results of the experiment consist of random variation is statistically undistinguishable from zero.

[11] While our experimental data show women being less risk averse than men for very small amounts (500 or 1000) and more risk averse for larger amounts, this effect is dominated by the effect of wealth.

of HRA for each of the two groups. Figure 1 shows the corresponding CDF's (the "types of behavior" are discussed in Section 7 below), and Table 3 gives some of the statistics.

Table 3. Percentile distribution of the highest amount of money that participants risked (with linear interpolation) in Nonwealthy and Wealthy groups. Higher HRA's per percentile group appear in bold.

Percentile (Lowest HRA = Lowest Risk Attraction = Highest Risk Aversion)	HRA in Nonwealthy Group (Standard pattern)	HRA in Wealthy Group (Standard pattern)
17% lowest HRA (Bottom 17%)	0	**0**
25% (Bottom quartile)	0	**125**
44% (Bottom 44%)	0	**417**
50% (Median)	125	**500**
75% (Top quartile)	750	**1250**
$85.\overset{\frown}{5}\%$ (Top $14.\overset{\frown}{4}\%$)	2600	2600
90% (Top 10%)	**5500**	3800
95% (Top 5%)	**8250**	5500
98% (Top 2%)	**12300**	8200

We observe that

- The bottom 17% of the Wealthy distribution, and the bottom 44 % of the Nonwealthy distribution, insure all risks.
- The bottom 86% of the Wealthy distribution have a (weakly) higher HRA than the bottom 86% of the Nonwealthy distribution.
- But the top 14% of the Nonwealthy distribution have a higher HRA than the top 14% of the Wealthy distribution, i.e., the very risk-attracted Nonwealthy (relative to their fellow Nonwealthy) risk more that the very risk-attracted Wealthy.

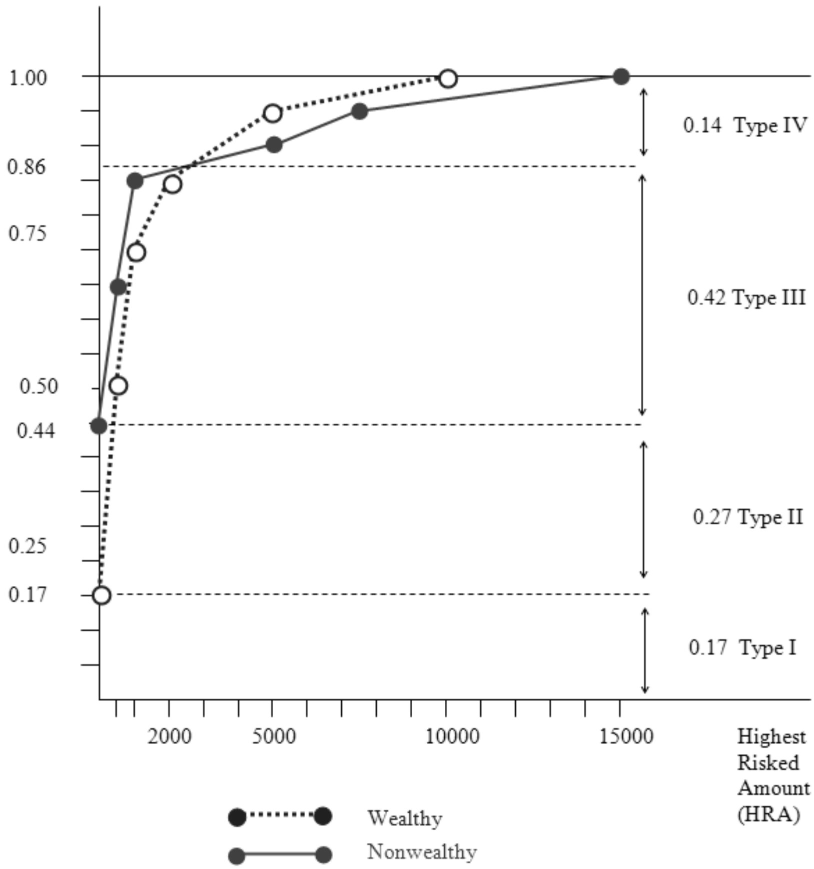

Fig. 1. Cumulative distributions (standard pattern only) and suggested types of behavior.

6.5 Statistical model

To tighten up the previous observations, we consider the logit regression model with random intercept (to allow for the heterogeneity of individual tastes represented by u_i),

$$\ln \frac{p_{ij}}{1-p_{ij}} = \alpha + u_i + bz_j,$$

$i \in \{1, \ldots I\}$, where I is the number of participants,
$j \in \{1, \ldots, 7\}$, the seven levels of money,
$(z_1, z_2, z_3, z_4, z_5, z_6, z_7) =$ (0.5, 1, 2, 5, 7.5, 10, 15).

The variable p_{ij} is the probability that participant i chooses to insure (and thus displays risk aversion) when the amount of money at stake is z_j in thousands of pesetas (so as to avoid too many decimals in the estimates of the regression coefficients).

The individual effect u_i allows for heterogeneous individual tastes, assumed to be normally distributed with mean zero and standard deviation σ_u so that $(\alpha + u_i)$ is the random intercept.

The results of the maximum likelihood estimation of this equation for the Nonwealthy and Wealthy groups, estimated separately (147 observations in each estimation)[12], are reported in Tables 4 and 5.

Table 4. Results of the ML estimation with the Nonwealthy data.

v	Coef.	Std. Err.	t	$P>\|t\|$	95% Conf. Int.	
z	0.1390	0.0564	2.464	0.014	0.2846	0.2496
Constant	1.0490	0.6059	1.731	0.083	-0.1385	2.2365
ln σ^2	1.2996	0.5604	2.319	0.020	0.2012	2.3980
σ	1.9152	0.5366			1.1058	3.3168
ρ	0.7858	0.0943			0.5501	0.9166

Table 5. Results of the ML estimation with the Wealthy data.

v	Coef.	Std. Err.	t	$P>\|t\|$	95% Conf. Int.	
z	0.4503	0.0988	4.555	0.000	0.2564	0.6439
Constant	-0.7903	0.5248	-1.506	0.132	-1.8190	0.2384
ln σ^2	0.9916	0.6110	1.623	0.105	-0.2058	2.1892
σ	1.6418	0.5016			0.9021	2.9880
ρ	0.7294	0.1206			0.4487	0.8992

If, for each group, we plot the probability of insuring with respect to the money amounts (divided by 1000), we obtain Figure 2.

Notice first that all estimates are significant. Second, the graph seems to indicate that the two curves describe different behavior. To verify that the intercepts are statistically different, we computed a t-test of the equality of the two groups' intercepts (we used the fact that the two samples are independent) obtaining t = 2.73, a value that rejects the null hypothesis of equality of intercepts (p-value = 0.006 for a two-sided test).

On the basis of the estimated regression model for each group, we want now to verify that our assumption of heterogeneity of individual decisions is appropriate. For this we run a χ^2 test of the null hypothesis ρ =0. The hypothesis indicates that there is no intraclass correlation of the individual decisions. But the test rejects the hypothesis for both groups with a p-value close to 0. Therefore, the individual effect

[12] We did an estimation of the joint data (294 observations) including a dummy group-variable. The joint estimation uses more data but forces a common intercept and slope. The estimation was not significant for some variables. The discussion below will help to understand why.

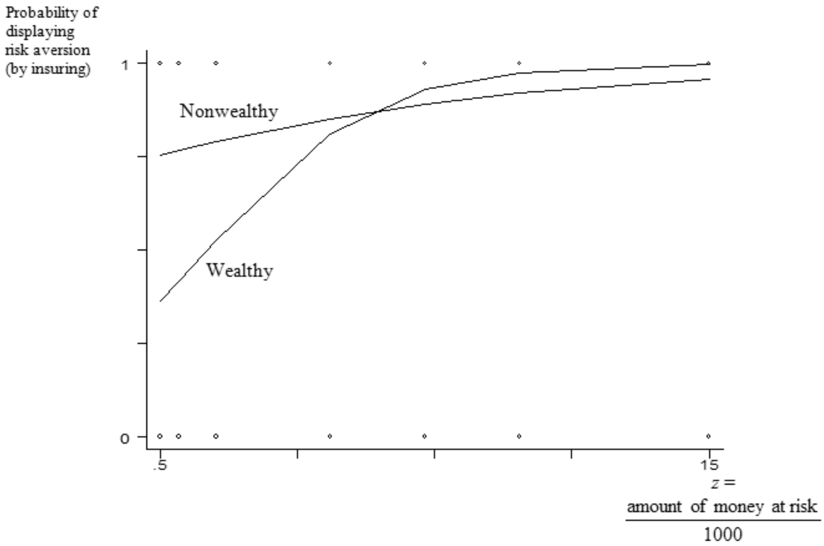

Fig. 2. Estimated functional relations between the amount of money at risk and the probability of displaying risk aversion (by insuring it) in Wealthy and Non-Wealthy participants.

is highly significant, as previous empirical analysis of risk had noticed, and can be confirmed by looking at the 95% confidence interval of σ_u.

The magnitudes of the parameter estimates show that, for the Nonwealthy, the odds of insuring increase 15% when the money at risk increases by 1000 pesetas (about US$7), while for Wealthy, the odds of insuring increase by as much as 57% when the amount increases by 1000 pesetas.[13] Note also that for the Nonwealthy, the probability of insuring is high when the amount of money at risk is close to zero, namely, 74%. For Wealthy, the limit of the probability of insuring an amount that tends to zero is lower than for Nonwealthy and equal to 31%.[14]

The statistical analysis confirms the previous observation that Nonwealthy insure small incomes at risk more than Wealthy.[15] The probability of insuring high amounts is close to one for both the Nonwealthy and the Wealthy, with the Wealthy insuring somewhat more frequently than the Nonwealthy. This analysis supports the preliminary conclusions stated as Result 1 in Section 3 above.

[13] By the classical transformation of the regression coefficients we obtain the percentage change on the dependent variable, namely 100[exp (.139) - 1] = 15%,
100[exp (.45028) – 1] = 57%.

[14] Similarly, exp (1.049)/(1+ exp (1.049)) = 0.74, and exp (-.7903)/(1+ exp (-.7903)) = 0.31.

[15] Of course, we cannot rule out that the monetary rewards were too low for the Wealthy to completely dominate nonmonetary influences.

6.6 Risk attitudes and the amount of money at risk

We showed in Bosch-Domènech and Silvestre (1999, 2005) that experimental participants become more likely to display risk aversion as the amount of the money at risk increases. All the results of the experiments reported here confirm that the probability of insuring increases with the amount of money at risk.

In particular, if we add together the data from Wealthy and Nonwealthy (294 observations) and we run the same regression model as above with z_j as the independent variable and log of the odds as the dependent variable, we obtain the results in Table 6.

Table 6. Results of the joint ML estimation with the Nonwealthy and Wealthy data.

v	Coef.	Std. Err.	t	$P>\|t\|$	95% Conf. Int.	
z	0.2645	0.0482	5.485	0.000	0.1700	0.3591
Constant	0.1416	0.3648	0.388	0.698	-0.5735	0.8567
ln σ^2	1.0596	0.4159	2.548	0.011	0.2446	1.8747
σ	1.6986	0.3531			1.1300	2.5531
ρ	0.7426	0.0794			0.5608	0.8669

Observe that there is a significant effect of the independent variable on the probability of insuring. Moreover, as shown in the previous estimations, there is an individual variation on the propensity to insure. This is captured by a random individual effect which is also significant (the hypothesis that individual correlation is zero being rejected by a χ^2 test with p-value = 0.0000). More important, the odds of insuring increase by 30% for increases of 1000 pesetas in the amount of money at risk. Note also that the overall probability of insuring is high for very small amounts of money, at approximately 53%.

The regression clearly supports Result 2 in Section 3 above. This result, also observed in Bosch-Domènech and Silvestre (1999, 2005), agrees with the empirical evidence reported by Roel Beetsma and Peter Schotman (2001) who claim, p. 847, that "the required minimum probability of winning in a lottery [...] rises from a 53% for a stake of f1,000 to 73% for a stake of f8,000". In other words, as the income at risk increases, so increases the aversion to risk. Similar results have been observed by Charles Holt and Susan Laury (2002). Strangely, Kachelmeier and Shehata (1992) observe that at an 80% probability of winning, the average risks attitudes are similar (risk neutrality in both cases) for two groups that risk incomes that are different by a factor of ten. In our experiments, the difference from the lowest to the largest income was a factor of thirty, and the slope of the estimated function of probability of insuring with respect to income was never flat.

6.7 Individual behavior at different wealth levels, and preferences

We ask the hypothetical question: if a Nonwealthy participant were wealthy, what would be her HRA? Assume that the distribution of HRA in either wealth category is invariant. Assume moreover that, when moving across wealth categories, a participant's position in the distribution of the HRA does not change, i.e., a participant who has the median HRA when non-wealthy also has the median HRA when wealthy. Similarly, a participant who, when non-wealthy, has a HRA in the 75% percentile of the distribution of non-wealthy HRA also is in the 75% percentile of the wealthy HRA when wealthy, and so on. Under these assumptions, we can use Figure 1 and Table 3 to identify the four types of behavior of Table 7.

Table 7. Types of behavior and their percentages.

Type	Percentage	Description
I	17%	Avoids all fair risks at all wealth levels
II	27%	Avoids all fair risks when nonwealthy; Takes very small fair risks when wealthy
III	42%	Takes medium fair risks when nonwealthy; Takes larger, but not very large, fair risks when wealthy
IV	14%	Takes a relatively large fair risk when nonwealthy; Takes a lower fair risk when wealthy

The various types of behavior have different implications for preferences. A participant in our experiments, with initial wealth w, has to choose between the risky prospect of a money gain of z with known probability p and the certain money gain of pz. The risky prospect induces the contingent final money balances $(x_1, x_2) = (w, w + z)$, with x_1 occurring with probability p and x_2 with probability $1 - p$, whereas the certain gain induces $(x_1, x_2) = (w + pz, w + pz)$.[16]

The person's choice displays *risk attraction* (resp. *aversion*) if she chooses the risky prospect (resp. the certain money gain pz). Our experiments have studied the dependence of risk attitudes on (i) the person's initial wealth, and (ii) whether the amount at risk z is large or small.

A person's attitude is *wealth-dependent* if she prefers the risky prospect at some wealth level, but prefers its certain expected money value at a different level of wealth.[17] On the other hand, we say that a person's risk attitude is *amount-dependent*

[16] Equivalently, we could write $x_1 = w + z$, occurring with probability p, and $x_2 = w$, occurring with probability $1 - p$.

[17] As noted in the introduction, the Arrow–Pratt literature considers a related issue within the canonical expected utility model: it assumes risk aversion and considers the acceptance or rejection of actuarially favorable risks of various sizes depending on the initial wealth.

if she displays risk attraction when the amounts at risk are small, but aversion for large ones, *at all levels of wealth* (or, at least, for a wide interval of wealth values). Our experimental results (including the ones reported here, the ones in Bosch-Domènech and Silvestre, 1999, 2005, and various classroom experiments) have convinced us that this is a highly realistic feature, well represented in real-life populations.

We say that preferences agree with the *canonical expected utility model* if (in the case of two possible monetary outcomes) they can be represented by a utility function of the form $U : \mathbb{R}^2_+ \times [0,1] \to \mathbb{R} : U(x_1, x_2, p) = (1-p)u(x_1) + pu(x_2)$, for some function $u : \mathbb{R}_+ \to \mathbb{R}$, which is called the von Neumann-Morgenstern (or the vNM) utility function (p is the probability of the state where x_2 occurs). Note that, in our definition, the arguments of the utility function are final wealth levels, rather than changes of wealth relative to a wealth reference level. This notion can be generalized to that of *single-self preferences*, where the utility function $U(x_1, x_2, p)$still has as arguments final wealth levels, but is not necessarily of the form $(1-p)u(x_1) + pu(x_2)$.[18]

Wealth-dependent attitudes are in principle compatible with the canonical expected utility model, witness Milton Friedman and Leonard Savage (1948). All it takes is a vNM utility function that is concave in part of its domain (that of wealth levels at which the person displays risk aversion for small risks), and convex in some other parts (attraction). But amount-dependent attitudes would require, in the canonical expected utility model, the vNM utility function u to be locally convex everywhere, implying convexity on its domain, which would contradict the aversion to large risks.

It follows that, while Type I is consistent with the canonical expected utility model (displaying risk aversion, i.e., with a strictly concave vNM utility function), types Types II-IV do not, because they display amount-dependent attitudes.

Yet they may be consistent with single-self preferences. Recall that a Type III decision maker is willing to take small risks, but not large ones, at all levels of wealth, "small" being understood relative to her wealth. The interesting Type IV reverses Type III: the decision maker is willing to take larger risks when Nonwealthy than when Wealthy.[19] A hypothetical example of preferences consistent with Type III can be constructed as follows. Define $U : \mathbb{R}^2_+ \times [0,1] \to \mathbb{R}$:

[18] Machina (1982) emphasizes the distinction between these two notions, albeit without using our terms.

[19] It includes the extreme case of taking all risks when Nonwealthy, as one of our Nonwealthy participants did.

$$U(x_1,x_2,p)=\begin{cases}\dfrac{\left[(1-p)+p\alpha^{1-\rho^-}\right]^{\frac{1}{1-\rho^-}}}{\left[(1-p)+p\alpha^{1-\rho^+}\right]^{\frac{1}{1-\rho^+}}}\left[(1-p)x_1^{1-\rho^+}+px_2^{1-\rho^+}\right]^{\frac{1}{1-\rho^+}}, \text{if } x_2>\alpha x_1\\ \left[(1-p)x_1^{1-\rho^-}+px_2^{1-\rho^-}\right]^{\frac{1}{1-\rho^-}}, \text{if } x_2\in[\beta x_1,\alpha x_1]\\ \dfrac{\left[(1-p)+p\beta^{1-\rho^-}\right]^{\frac{1}{1-\rho^-}}}{\left[(1-p)+p\beta^{1-\rho^+}\right]^{\frac{1}{1-\rho^+}}}\left[(1-p)x_1^{1-\rho^+}+px_2^{1-\rho^+}\right]^{\frac{1}{1-\rho^+}}, \text{if } x_2<\beta x_1\end{cases},$$

where $\rho^- < 0$, $\rho^+ \in (0,1)$, $0 < \beta < 1 < \alpha$ and p (resp. 1-p) is the probability of the state where x_2 (resp. x_1) occurs. It can be checked that $U(x,x,p) = x$, that $\frac{\partial U}{\partial x_1} > 0$ if 1-$p > 0$ (zero if 1-p =0), and that $\frac{\partial U}{\partial x_2} > 0$ if $p > 0$ (zero if $p = 0$).[20] In principle, α and β could be functions of the probabilities, restricted to satisfy some natural conditions, but here for simplicity we take α and β to be constants, with $\beta = 1/\alpha$. Tedious algebra shows that $sgn\frac{\partial U}{\partial p} = sgn(x_2 - x_1)$, i.e., utility is increasing in the probability of the better outcome. Figure 3 displays the contour lines of U in the space of contingent money balances for $p = 0.5$, $\rho^- = -0.5$, $\rho^+ = 0.5$, $\alpha = 2$, $\beta = 1/2$, i.e., the rays along which the kinks occur are given by $x_2 = 2\,x_1$ and $x_2 = 0.5\,x_1$.

One can compute that, if $x_2 = 2.67242x_1$ (or $x_2 = [2.67242]^{-1}x_1 = 0.374193x_1$), then the decision maker is indifferent between a final balance of x_2 with probability 0.5 and of x_1 with probability 0.5, and a final balance of 0.5 $(x_1 + x_2)$. For instance, she is indifferent between a final money balance of 600 for sure (point C in Figure 3) or a final balance of 873.24 with probability 0.5 and of 326.76 with probability 0.5 (point B or B'). But she prefers any point in the segment (C, B) to point C, displaying risk attraction in theses choices, while she prefers point C to any point in the segment (B, D], thus displaying risk aversion in those choices.

If her wealth were w= 326.76, then she would be indifferent between a gain of z = 546.48 with probability p = 0.5 (and no gain with probability 0.5) and a certain gain of pz = 273.24. which yields a certain balance of $w + pz$ = 600. But she would choose the risky alternative for $z \in (0, 546.48)$ (i.e., for small amounts of money at stake), and the safe alternative for z >546.48 (i.e., for large amounts of money at stake), therefore displaying an amount effect. It is also clear that the supremum amount of money at stake for which she would choose the risky alternative is increasing in w, thus displaying a wealth effect of Type III in Table 7.

6.8 Relation to the literature

Experimental work has always shown interest in socio-demographic characteristics of participants, like sex or age, and many experiments deal with other non-demographic or cultural effects on behavior.[21] All this has influenced the experimental research on risk attitudes, resulting in experiments that relate risk-taking to age

[20] When the partial derivatives are defined. Clearly, U is differentiable everywhere in the interior of the quadrant except along the rays $x_2 = \alpha x_1$ and $x_2 = \beta x_1$. It is in particular differentiable along the certainty line.

[21] From Alvin E. Roth *et al.* (1991) to Joseph Henrich *et al.*, (2001).

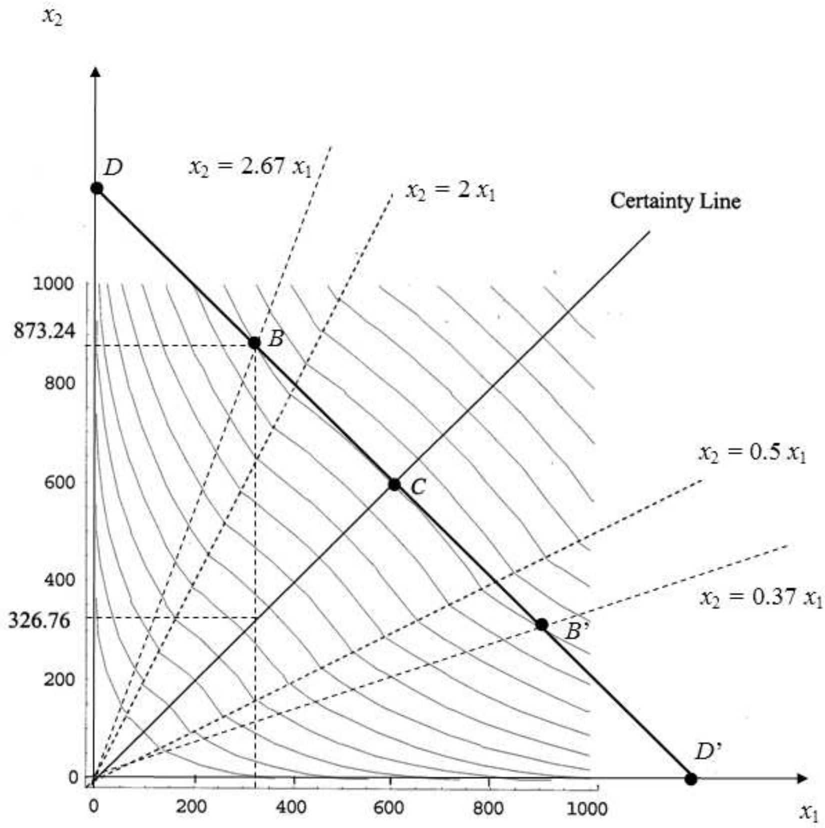

Fig. 3. Contour lines of *U* ($p = 0.5$).

(Harbaugh *et al.*, 2002), to sex (Renate Schubert *et al.*, 1999, Catherine Eckel and Philip Grossman, forthcoming), and to non-demographic factors, like the effects of the experimental medium (i.e., the lab or the internet, see Tal Shavit *et al.*, 2001), or the frequency of evaluation (Uri Gneezy and Jan Potters, 1997).[22] Yet, surprisingly, economists have not shown much interest in the effect of differences in personal or family wealth on experimental behavior.[23] This is particularly odd -even considering

[22] There is also a growing field of evidence about risk attitudes from natural experiments, mostly television games or racetrack betting. See, e.g., Beetsma and Schotman (2001), Bruno Jullien and Bernard Salanié (2000), and the references mentioned there.

[23] There is a variety of experiments that link so-called wealth with different behavior. But what is called wealth in these experiments is not what we mean here. It is either the endowed income provided by the experimenter as, for instance, in Olivier Armantier (forthcoming), or the accumulated earnings of participants as they keep participating in an experiment, as in Kachelmeier and Shehata (1992). Holt and Laury (2002, p. 11) report, almost as an afterthought, that "income seems to have a mildly negative effect on risk aversion."

the difficulty of finding the relevant information- concerning, as it does, economists, and the oversight is even more striking when it refers to the study of risk aversion, because of the long-standing awareness that risk aversion may vary with wealth and that this relation "is of the greatest importance for prediction of economic reactions in the presence of uncertainty" (Arrow, 1965). But the fact is that differences in personal wealth among participants from the same culture do not appear to have ever been controlled in the lab, or used as a treatment to explain behavior.[24]

Field studies by development economists and anthropologists provide some information on whether wealthy people are more or less likely to exhibit attraction to money risks.[25] Frank Cancian (1972) reports on a variety of studies, including his own in an area of Chiapas, that relate the degree of risk taking (measured by an index of the speed or depth in the adoption of various innovations in the production or marketing of corn) to the person's position in a four-tier wealth classification: low, low middle, high middle and high. The main observation is that the relation is increasing except for the middle-high group, i.e., low and middle high take fewer risks than middle low, which in turn take fewer risks than high. The findings do not directly address the issue of risk attraction, but suggest that it is more likely to be found in the low-middle and the high groups than in the low or high middle.

John Dillon and Pasquale Scandizzo (1978) studied the risk attitudes of two groups of subsistence farmers in the Brazilian Sertão, namely small owners and shareholders. The two groups showed different socioeconomic characteristics: in particular, small owners were wealthier, with an average income which was 140% that of the sharecroppers. Their risk attitudes were elicited by two sets of hypothetical questions, one of which involved a potential fall below subsistence. Even though risk aversion was more common in either group, a non-negligible fraction displayed risk attraction, and this fraction was substantially higher for the (relatively) poor shareholders than for the small owners. These observations agree with the conclusion of Hans Binswanger's (1980) study in rural India that tenant farmers are more risk-attracted than landowners.

Lawrence Kuznar (2001) adopts a similar method in his study of Andean pastoralists, and his questions are targeted to elicit the probability premium for given lotteries involving hypothetical herds of goats, sheep or cows. (The probability premium is the excess of winning probability over fair odds that makes the individual indifferent between a certain amount and a symmetric lottery centered in that amount: it is an index of risk aversion, with negative values corresponding to risk attraction). He finds that these premia are the lowest for the poorest herders (with one instance

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But, as will be described below, anthropologists and development economists do appear to be interested in the effect of wealth on risk behavior.

[24] However, there is a literature of experiments in the field that uses wealth and wealth differences as parametric factors for explaining cooperative behavior. See, e.g., Juan-Camilo Cárdenas (2003) and the references he provides

[25] Here we focus on money, rather than lifestyle, risks. The conventional wisdom is that a low income tends to increase lifestyle risky behavior, such as smoking, unprotected sex or excessive drinking, in particular for those behaviors that do not require purchases, such as seat-belt use. See Thomas Dee and William Evans (2001) and Phillip Levine (2001).

of risk attraction), highest for herders with mean wealth, and relatively low for the wealthiest herders.

Henrich and Richard McElreath (2002) report on several experiments, with real payoffs, involving four groups of participants: they find widespread risk attraction. A first experiment, involving Huinca and Mapuche participants in Southern Chile, elicits the certainty equivalent of a fifty-fifty lottery of 2000 pesos (about $30) or nothing. A whopping 80% of the Mapuche show certainty equivalents above the expected value of 1000 pesos, evidencing risk attraction, whereas among the Huinca only 16% display risk attraction. It is interesting to note that the more risk-loving Mapuches are considered (both by Mapuche and by Huincas) to be poorer and of lower social status.

They also perform an experiment with three groups of participants: Mapuche, Sangu (Tanzanian agro-pastoralists) and UCLA undergraduates. They make binary choices between a certain gain of (the equivalent to) $15 and various actuarially fair lotteries of increasing variance. In all three groups, the lottery is preferred by more than 70% of the participants when the odds of winning are 50% (i.e., the lottery gives $30 with probability 50%), evidencing pervasive risk attraction. When the lottery becomes riskier (say, 20% chance of winning $75, or 5% chance of winning $300), then only 20% of the UCLA undergraduates take the risk, versus at least 65% for the Mapuche and the Sengu. Thus, strong risk attraction seems to be more prevalent among Mapuche and Sangu than among the comparatively better-off UCLA undergraduates.

Finally, empirical work on risk based on surveys taken in developed countries seems to indicate that some socio-economic variables, like earnings, age, sex, employment experience and wealth, have a bearing on risk attitudes, but that these variables can only explain a small amount of the variability in attitudes towards risks, reflecting genuine differences in tastes (see Guiso and Paiella, 2001).[26] In particular, there seems to be limited empirical evidence of the sign of the relationship between risk attitudes and wealth.[27] This enhances the need for further experiments.

6.9 Conclusions

A fraction of our participants display risk aversion for all amounts of money at risk, but many do not, displaying risk attraction for small amounts of money, and risk aversion for higher amounts. We compare the likelihood of displaying risk attraction for various amounts of money between the Wealthy and Nonwealthy groups.

[26] Willem Saris, in a personal communication, confirms the heterogeneity of risk attitudes observed in a national household survey conducted in Holland for a private investment company. Unfortunately the study that resulted from this survey is private information and cannot be quoted. See also Robert Barsky *et al.*, (1997) who confirm the heterogeneity of risk preferences.

[27] Halek and Eisenhauer (2001) find a parabolic relation between relative risk aversion and wealth, whereas Bas Donkers *et al.* (2001) a negative relation between income and risk aversion.

In our experiment we observe a higher frequency of risk attraction for large amounts of money at risk in the Nonwealthy groups than in the Wealthy group: Nonwealthy participants at the higher end of the risk-attraction scale (relative to their fellow Nonwealthy) risk higher amounts, in absolute value and thus, a fortiori, as a fraction of their wealth, than the corresponding risk-attracted Wealthy. On the other hand, the Nonwealthy are more likely to display risk aversion for small amounts of money at stake, in agreement with the intuition that a given amount of dollars may be seen as "small" by a wealthy person, yet large by a poor one.

We view our work as a first exploration of an important issue. From the empirical viewpoint, because our result is admittedly based on only a few observations, particularly on risk attraction for large amounts of money at stake, its robustness should be tested by further experiments with larger samples. Conceptually, our findings that Wealthy participants take, as a group, more risks when the stakes are low, but fewer when high, suggest a complex relationship between wealth and risk attitudes towards money that invites further analysis.

Appendix. Experimental Data

Table A1. Nonwealthy participants. A letter y indicates insuring (thus displaying risk aversion), while a letter n indicates not insuring (thus displaying risk attraction). Capitals indicate the actual decision implemented. Participant *JN* changed his mind from not insuring to insuring when confronted with the real choice. In this table, as in a similar table below, participants have been ordered to help reading the table.

	Amount of Money						
	500	1000	2000	5000	7500	10000	15000
Participant *AN*	Y	y	y	y	y	y	y
Participant *BN*	y	y	Y	y	y	y	y
Participant *CN*	Y	y	y	y	y	y	y
Participant *DN*	y	Y	y	y	y	y	y
Participant *EN*	y	y	y	y	y	y	Y
Participant *FN*	y	y	Y	y	y	y	y
Participant *GN*	y	y	Y	y	y	y	y
Participant *HN*	y	y	y	y	y	Y	y
Participant *IN*	N	y	y	y	y	y	y
Participant *JN*	n	y	y	y	y	n, Y	y
Participant *KN*	n	y	y	y	Y	y	y
Participant *LN*	n	Y	y	y	y	y	y
Participant *MN*	n	n	y	y	y	y	Y
Participant *NN*	n	n	y	Y	y	y	y
Participant *ON*	n	n	y	Y	y	y	y
Participant *PN*	n	n	n	N	y	y	y
Participant *QN*	n	n	n	n	N	y	y
Participant *RN*	n	n	n	n	n	n	N
Participant *SN*	y	y	y	n	N	y	n
Participant *TN*	y	Y	y	Y	n	n	n
Participant *UN*	n	n	y	y	n	Y	n

Table A2. Wealthy participants. A letter y indicates insuring (thus displaying risk aversion), while a letter n indicates not insuring (thus displaying risk attraction). Capitals indicate the actual decision implemented.

	Amount of Money						
	500	1000	2000	5000	7500	10000	15000
Participant *AW*	y	y	y	y	y	y	Y
Participant *BW*	Y	y	y	y	y	y	y
Participant *CW*	y	y	Y	y	y	y	y
Participant *DW*	n	y	y	y	y	Y	y
Participant *EW*	n	y	Y	y	y	y	y
Participant *FW*	n	y	Y	y	y	y	y
Participant *GW*	n	y	y	Y	y	y	y
Participant *HW*	n	y	y	y	y	y	Y
Participant *IW*	n	y	y	Y	y	y	y
Participant *JW*	n	n	y	y	Y	y	y
Participant *KW*	n	n	y	y	y	Y	y
Participant *LW*	n	n	y	y	y	y	Y
Participant *MW*	n	n	y	y	Y	y	y
Participant *NW*	n	n	n	y	Y	y	y
Participant *OW*	n	N	n	y	y	y	y
Participant *PW*	n	N	n	n	y	y	y
Participant *QW*	n	n	n	n	y	Y	y
Participant *RW*	n	n	n	N	n	n	y
Participant *SW*	N	y	y	y	y	y	n
Participant *TW*	N	n	y	n	y	y	y
Participant *UW*	n	N	n	n	y	y	n

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7

Are Incomplete Markets Able to Achieve Minimal Efficiency?*

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Summary. We consider economies with incomplete markets, one good per state, two periods, $t = 0, 1$, private ownership of initial endowments, a single firm, and no assets other than shares in this firm. In Dierker, Dierker, Grodal (2002), we give an example of such an economy in which all market equilibria are constrained inefficient. In this paper, we weaken the concept of constrained efficiency by taking away the planner's right to determine consumers' investments. An allocation is called minimally constrained efficient if a planner, who can only determine the production plan and the distribution of consumption at $t = 0$, cannot find a Pareto improvement. We present an example with arbitrarily small income effects in which no market equilibrium is minimally constrained efficient.

Key words: Incomplete markets with production, Constrained efficiency, Drèze equilibria.

JEL Classification Numbers: D2, D52, D61, G1.

7.1 Introduction

We consider finance economies with production. More precisely, we assume incomplete markets, one good per state, private ownership of initial endowments, production, and two time periods. Due to the incompleteness of markets, shareholders typically disagree about which production decision their firm should take. Drèze (1974) presents a way of resolving the conflict among shareholders by introducing an equilibrium concept that is based on Pareto comparisons with the aim of achieving constrained efficiency. We restrict ourselves to economies with one good per state in order to rule out price effects, which are a well-known cause of constrained inefficiency (cf. Geanakoplos et al., 1990).

In this paper, we show that the market in such economies may not be able to achieve an allocation that satisfies minimal efficiency requirements as soon as the

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quasilinear framework is left. This phenomenon is illustrated in economies with only one firm.

The firm has constant returns to scale and makes zero profit. Its state dependent output at $t = 1$ is sold on the asset market in exchange for the corresponding input. When the firm proposes a production ray, consumers choose their optimal investments and this determines their consumption in all states. The firm adjusts its production level to the market clearing scale. The resulting allocation is called a market equilibrium. The set of all allocations the market can achieve consists of all market equilibria corresponding to some production decision of the firm.

A Drèze equilibrium is a market equilibrium with the following property: The (new) shareholders of the firm meet at $t = 0$ after they have chosen their shares optimally. If these shares are held fixed, there is no other production plan such that the shareholders of the firm can achieve a Pareto improvement by adopting that production plan and by making sidepayments at time $t = 0$ to reach unanimity.[3]

Constrained efficiency means that a hypothetical planner cannot find a Pareto improvement by simultaneously choosing the production plan, the shares, and each individual's consumption at $t = 0$. Note that a constrained efficient market equilibrium is a Drèze equilibrium.

An example of an economy with a unique, but constrained inefficient Drèze equilibrium is presented in Dierker, Dierker, and Grodal, henceforth DDG (2002). This example is driven by the existence of a consumer whose preferences exhibit strong income effects. If there are no income effects, that is to say, if all consumers have quasilinear utility functions, then at least one constrained efficient Drèze equilibrium exists, since the social surplus is well defined and is maximized at a Drèze equilibrium.

Since the planner, who can implement constrained efficient allocations, is more powerful than the market, we reduce the planner's power substantially and explore whether the planner can still outperform the market. We introduce the following very weak version of constrained efficiency, in which tomorrow's consumption can only be affected by the planner through the choice of the production plan. After the planner has chosen a production plan with input normalized to -1, consumers choose their optimal investments subject to their budget constraints. The firm adjusts production to the market clearing scale. The planner, who is no longer allowed to alter individual consumption at $t = 1$, can only distribute the resources remaining at $t = 0$ after subtracting the input. An allocation is called minimally constrained efficient if the planner, who is subject to these constraints, cannot find a Pareto improvement. It turns out that the example in DDG (2002) is minimally constrained efficient.

However, there are economies without any minimally constrained efficient allocations. To show this, we start with a quasilinear economy with three Drèze equilibria. Two of them are surplus maxima and the third is a surplus minimum. Then we perturb the quasilinear utility functions of the example by adding a small term to the utility at $t = 0$. The perturbation does not affect the way in which future consumption

[3] For an extensive treatment of Drèze equilibria in a setting with private ownership of initial endowments, the reader is referred to Magill and Quinzii (1996, ch. 6).

streams are ranked, i.e., utility at $t = 1$ is left unchanged. These small perturbations leave the set of Drèze equilibria invariant. However, for arbitrarily small perturbations all Drèze equilibria, as well as all other market equilibria, become minimally constrained inefficient.

The notion of minimal constrained efficiency cannot be weakened further, since the planner should at least retain the possibility of changing the production plan and redistributing total consumption at $t = 0$. We conclude that, even in economies with one good per state, arbitrarily small income effects can make it impossible to select a production plan that achieves a market equilibrium that satisfies at least some weak version of constrained efficiency. The question of how to choose a market equilibrium remains open and is briefly discussed at the end of the paper.

The remainder of the paper is organized as follows. In Section 2 we introduce the framework and present the definitions. In Section 3 we give an example showing that minimally constrained efficient market equilibria need not exist. Section 4 contains concluding remarks.

7.2 Framework and definitions

The reason why the market mechanism can be unable to generate allocations that exhibit desirable efficiency properties can be illustrated in a very simple setting. We consider two periods, $t = 0, 1$, and two possible states of nature at $t = 1$, denoted $s = 1$ and $s = 2$, respectively. The unique state at $t = 0$ is included as the state $s = 0$. There is a single good, denoted s, in each state $s = 0, 1, 2$ and there is just one firm. It transforms good 0 into a state dependent output at $t = 1$. We assume that there are no assets other than shares in the firm. The firm has constant returns to scale and makes zero profits. Its technology is given by a family of normalized production plans $(-1, \lambda, 1 - \lambda)$. The production set is

$$Y = \{\alpha(-1, \lambda, 1 - \lambda) \in \mathbb{R}^3 \mid \alpha \geq 0,\ 0 \leq \underline{\lambda} \leq \lambda \leq \overline{\lambda} \leq 1\}\,.$$

There are two types of consumers. Ideally, each type is represented by a continuum of mass 1. For convenience, we refer to each continuum of identical consumers as a single consumer denoted $i = 1, 2$. Consumer i has the initial endowment e^i, consumption set $\mathbb{R}^3_+$, and utility function U^i.

If the firm selects the normalized production plan $(-1, \lambda, 1 - \lambda)$ and consumer i chooses the investment $\alpha^i \geq 0$ in the firm, the resulting consumption bundle is $e^i + \alpha^i(-1, \lambda, 1 - \lambda)$. Consumer i selects α^i so as to maximize utility in the budget set

$$B^i(\lambda) = \{e^i + \alpha^i(-1, \lambda, 1 - \lambda) \in \mathbb{R}^3_+ \mid \alpha^i \geq 0\}\,.$$

If the utility functions are strictly quasiconcave, i's optimal investment $\alpha^i(\lambda)$ is uniquely determined. Agent i consumes $x^i(\lambda) = e^i + \alpha^i(\lambda)(-1, \lambda, 1 - \lambda)$, holds shares $\vartheta^i(\lambda) = \alpha^i(\lambda)/(\alpha^1(\lambda) + \alpha^2(\lambda))$, and the firm produces $y(\lambda) = [\alpha^1(\lambda) + \alpha^2(\lambda)](-1, \lambda, 1 - \lambda)$.

Definition 1. *The allocation* $(y(\lambda), x^1(\lambda), x^2(\lambda))$ *is called a* market equilibrium *iff*

1) $x^i(\lambda) = e^i + \alpha^i(\lambda)(-1, \lambda, 1-\lambda)$*, where* $\alpha^i(\lambda)$ *is* i*'s optimal investment at the production ray* λ*,*

2) $y(\lambda) = [\alpha^1(\lambda) + \alpha^2(\lambda)](-1, \lambda, 1-\lambda) \in Y$.

Market equilibria are the only allocations that the market can achieve. In general, these allocations cannot be Pareto compared and the shareholders face a social choice problem. In order to resolve the problem, Drèze (1974) suggested that shareholders use sidepayments among themselves at $t = 0$ in order to reach unanimity.

A Drèze equilibrium is a market equilibrium in which the production plan of the firm passes the following test: It is impossible for the shareholders to find another production plan and sidepayments at $t = 0$ such that all shareholders are better off if they use their original investment levels and get the sidepayments.[4] More precisely, consider a market equilibrium $(y(\tilde{\lambda}), x^1(\tilde{\lambda}), x^2(\tilde{\lambda}))$ with respect to $\tilde{\lambda}$ and let $\mathcal{I} = \{i \mid \alpha^i(\tilde{\lambda}) > 0\}$. The market equilibrium is a *Drèze equilibrium* if it is impossible to find a normalized production plan $(-1, \lambda, 1-\lambda)$ and a system of sidepayments $(\tau^i)_{i\in\mathcal{I}}$ at $t = 0$ with $\sum_{i\in\mathcal{I}} \tau^i = 0$ such that

$$U^i(e^i + \tau^i(1,0,0) + \alpha^i(\tilde{\lambda})(-1, \lambda, 1-\lambda)) > U^i(x^i(\tilde{\lambda}))$$

for every $i \in \mathcal{I}$. Note that the production plan $(-1, \lambda, 1-\lambda)$ on the left hand side of the above inequality is multiplied by the investment level $\alpha^i(\tilde{\lambda})$ that is optimal at the normalized equilibrium production plan $(-1, \tilde{\lambda}, 1-\tilde{\lambda})$.

We recall the definitions of constrained feasibility and constrained efficiency (cf. Magill and Quinzii, 1996). A commodity vector $x \in \mathbb{R}^3$ is written as $x = (x_0, x_\mathbf{1})$, where $x_0 \in \mathbb{R}$ corresponds to $t = 0$ and $x_\mathbf{1} \in \mathbb{R}^2$ corresponds to $t = 1$. An allocation (y, x^1, x^2) is constrained feasible if it can be implemented by a planner who simultaneously determines the production plan $y = (y_0, y_\mathbf{1}) \in Y$, the shares ϑ^i of all consumers and who, moreover, freely redistributes good 0. More precisely, the allocation $((y_0, y_\mathbf{1}), (x_0^1, x_\mathbf{1}^1), (x_0^2, x_\mathbf{1}^2)) \in Y \times \mathbb{R}^3_+ \times \mathbb{R}^3_+$ is *constrained feasible* if $x_0^1 + x_0^2 = e_0^1 + e_0^2 + y_0$ and there exist shares $\vartheta^i \geq 0$ such that $x_\mathbf{1}^i = \vartheta^i y_\mathbf{1}$ for all i and $\sum_i \vartheta^i = 1$. Note that the set of constrained feasible allocations does not depend on how the aggregate endowment of good 0 is distributed across consumers and that it is, in general, larger than the set of market equilibria. A constrained feasible allocation is called *constrained efficient* if there does not exist a Pareto superior constrained feasible allocation.

In searching for constrained efficient market equilibria we can restrict attention to the set of Drèze equilibria since a constrained efficient market equilibrium is a Drèze equilibrium.

In DDG (2002) we present an example of a finance economy with a unique, but constrained inefficient Drèze equilibrium. Thus, the market is unable to achieve constrained efficiency in the example. Therefore, we are led to ask the question of whether the efficiency requirements can be relaxed such that the market can at least achieve an extremely weak form of constrained efficiency.

[4] In the usual definition of a Drèze equilibrium, shares ϑ^i, and not the investment levels α^i, are taken as fixed when a production plan is evaluated. The two definitions are equivalent.

The planner who can implement constrained efficient allocations is more powerful than the market, since the planner can distribute consumption at $t = 0$ directly and affect consumption at $t = 1$ indirectly by allocating shares to individuals. Clearly, to improve upon a market equilibrium, the planner must be able to compensate the losers of a change of the available asset by reallocating consumption at $t = 0$. Therefore, we cannot deprive the planner of the right to distribute good 0. However, we take away the right to allocate shares. Since the power of a planner who is deprived of this right cannot be further reduced, a constrained feasible allocation is called minimally constrained efficient if it is not possible for a planner who does not possess the right to distribute shares, to Pareto improve upon the allocation.

More precisely, the economy with the weakened planner can be described as follows. First the planner chooses λ. Given λ, each consumer i selects the optimal investment $\alpha^i(\lambda)$ such that the resulting consumption plan $(x_0^i, x_{\mathbf{1}}^i) = e^i + \alpha^i(\lambda)(-1, \lambda, 1-\lambda)$ maximizes i's utility in the budget set associated with i's initial endowment and λ. The planner can redistribute total consumption $\sum_i x_0^i$ at $t = 0$, but cannot affect individual consumption $x_{\mathbf{1}}^i$ at $t = 1$ and λ. That is to say, whenever the planner has chosen λ, the stock market opens and each consumer i chooses $(x_0^i, x_{\mathbf{1}}^i) = e^i + \alpha^i(\lambda)(-1, \lambda, 1-\lambda)$ optimally. Then the stock market is closed and nobody, including the planner, can change $x_{\mathbf{1}}^i$. After the stock market is closed the planner can redistribute good 0.

Definition 2. *A constrained feasible allocation is called* minimally constrained efficient *if there is no Pareto superior allocation* $(\lambda, (c_0^i, x_{\mathbf{1}}^i)_i)$ *satisfying*

(i) $x_{\mathbf{1}}^i = e_{\mathbf{1}}^i + \alpha^i(\lambda)(\lambda, 1-\lambda)$*, where* $\alpha^i(\lambda)$ *is consumer* i*'s optimal investment given* λ*,*
(ii) $\sum_i c_0^i = \sum_i e_0^i - \sum_i \alpha^i(\lambda)$*, and*
(iii) $\sum_i \alpha^i(\lambda)(-1, \lambda, 1-\lambda) \in Y$*.*

Condition (i) says that, after the planner has chosen the production ray, individual consumption at $t = 1$ is determined by the market. Condition (ii) states that the planner can redistribute the aggregate consumption $\sum_i e_0^i - \sum_i \alpha^i(\lambda)$ at $t = 0$. Condition (iii) says that the planner adjusts the level of production to the consumers' aggregate investment.

Our method of defining minimal constrained efficiency can, in principle, also be used if there are several goods in each state. In this case, even equilibria with respect to fixed sets of assets are typically constrained inefficient due to price effects. Therefore, Grossman (1977) weakened the definition of constrained efficiency by introducing a central planner with incomplete coordination. We compare our planner with Grossman's. Grossman's planner cannot act simultaneously in different states, but our planner is not even allowed to act in any state other than $s = 0$. At $s = 0$, our planner is, apart from the ability to choose λ, weaker than Grossman's, since shareholdings and individual consumption at $t = 1$ are determined by individual optimization. Our planner can only redistribute the resources at $s = 0$ that are not used for production, whereas Grossman's planner can also allocate shares.

Numerical computation shows that the unique Drèze equilibrium in the example in DDG (2002) is minimally constrained efficient although it is not constrained efficient. This fact can be explained as follows. The example is driven by strong income effects: The optimal investment of the first consumer depends strongly on his wealth and, therefore, on the sidepayment obtained from the planner. However, in the case of minimal constrained efficiency this effect ceases to play a role since individual investments in shares are, by definition, independent of sidepayments. Since the mechanism driving the example in DDG (2002) cannot be used in the case of minimal constrained efficiency, one would like to know whether at least one Drèze equilibrium in a finance economy is minimally constrained efficient.

7.3 Can the market achieve minimal constrained efficiency?

In order to answer this question we proceed as follows. First we present and discuss a quasilinear example. Due to the existence of a representative consumer, a constrained efficient Drèze equilibrium necessarily exists. Then we introduce small income effects by perturbing the example slightly and analyze how the perturbation affects the efficiency properties of the Drèze equilibria.

In the unperturbed example, consumers have quasilinear utilities given by

$$U^1(x_0, x_1, x_2) = x_0 + x_1^{0.6} \quad \text{and}$$
$$U^2(x_0, x_1, x_2) = x_0 + x_2^{0.6},$$

respectively. We assume $\underline{\lambda} = 0.1, \overline{\lambda} = 0.9$ and $e^1 = e^2 = (2, 0, 0)$. It turns out that the economy under consideration has three Drèze equilibria, A, B and C, corresponding to $\lambda_A = 0.1$, $\lambda_B = 0.5$, and $\lambda_C = 0.9$, respectively.

In the definition of a Drèze equilibrium, shares are kept fixed when shareholders evaluate alternative production plans. In order to gain insight into the consequences of this feature, it is useful to investigate the interior equilibrium B. To do so, we first consider the indirect utility $u^1(2, \lambda)$ that consumer 1 obtains, if the firm chooses the ray λ and if consumer 1 makes the optimal investment $\alpha^1(\lambda) = 0.6(0.6\lambda)^{1.5}$. Since this utility equals $u^1(2, \lambda) = 2 + 0.4(0.6\lambda)^{1.5}$, the function $u^1(2, \cdot)$ is convex. Similarly, the utility level of consumer 2 at λ equals $u^2(2, \lambda) = u^1(2, 1 - \lambda)$ and is convex in λ. As a consequence, shareholders' social surplus associated with the ray λ, $u^1(2, \lambda) + u^2(2, \lambda)$, is convex in λ. Due to the symmetry between $u^1(2, \lambda)$ and $u^2(2, \lambda)$, the social surplus has a critical point at $\lambda_B = 0.5$, which must be a global minimum (see Fig 7.1).

Observe that the situation changes drastically if the shareholders are deprived of the possibility of adjusting their shares, or, equivalently, their investment levels, when λ_B is tested against some alternative λ. Consider consumer 1 who wants to choose the investment level $\alpha^1(\lambda)$ in proportion to $\lambda^{1.5}$. If α^1 is now taken as fixed at its value at $\lambda_B = 0.5$, then the utility reached at ray λ equals $\tilde{u}^1(2, \lambda) = c_0 + c_1\lambda^{0.6}$ with $c_1 > 0$, whereas the indirect utility with share adjustment is a function of the type $u^1(2, \lambda) = c_0' + c_1'\lambda^{1.5}$ with $c_1' > 0$. Thus, by disregarding how consumer 1's individual investment level $\alpha^1(\lambda)$ varies with λ, the originally convex function $u^1(2, \cdot)$ is turned into a concave function $\tilde{u}^1(2, \cdot)$. As a consequence, $\tilde{u}^1(2, \cdot) + \tilde{u}^2(2, \cdot)$ is a

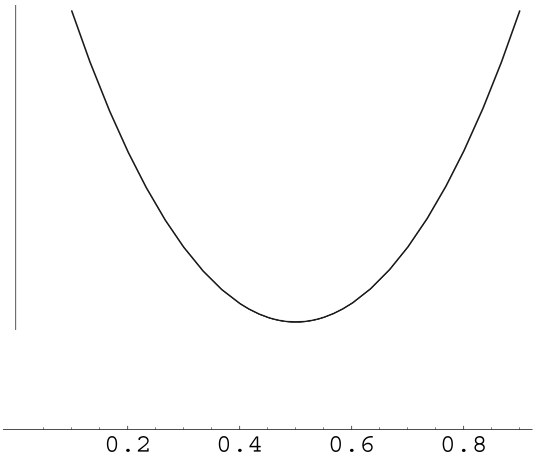

Fig. 7.1. Surplus minimum at the Drèze equilibrium $\lambda_B = 0.5$

concave function and the critical point $\lambda = 0.5$ becomes a maximum. For this reason, λ_B yields a Drèze equilibrium. Clearly, the utility sum $\tilde{u}^1(2,\cdot)+\tilde{u}^2(2,\cdot)$ constructed by fixing the shares does not represent owners' welfare at alternative production rays correctly.

At the Drèze equilibria A and C, consumers' social surplus is maximized. Hence, A and C are constrained efficient.

Now we perturb the quasilinear example by altering the utility derived from consumption at $t = 0$ without changing the utility obtained from consumption at $t = 1$. In particular, the utility functions stay additively separable after perturbation. Let

$$U_a^1(x_0, x_1, x_2) = x_0 + ax_0^2 + x_1^{0.6}, \qquad U_a^2(x_0, x_1, x_2) = x_0 + ax_0^2 + x_2^{0.6} \quad (7.1)$$

where $0 < a \leq 0.1$. It is easy to show that i's utility function is quasiconcave in the relevant range.

As in the unperturbed example, the production ray varies in the interval $[0.1, 0.9]$ and there are three Drèze equilibria corresponding to $\lambda_A = 0.1$, $\lambda_B = 0.5$, and $\lambda_C = 0.9$, respectively. However, the boundary equilibria are no longer constrained efficient for any $a > 0$. Moreover, the boundary equilibria are, as all other equilibria, not even minimally constrained efficient.

Proposition 1. *For arbitrarily small $a > 0$, no market equilibrium associated with some ray λ is minimally constrained efficient.*

Proof.. Consider any ray λ and the corresponding market equilibrium allocation. Clearly, the equilibrium corresponding to $\lambda_B = 0.5$ is not minimally constrained efficient. Therefore, let $\lambda \neq 0.5$. We show that the production ray $1 - \lambda$, together with a suitable reallocation of consumption at $t = 0$ is preferred to λ by both types of consumers. Due to symmetry we can assume $\lambda < 0.5$.

Agent i consumes $x^i(\lambda) \in B^i(\lambda)$ when the ray λ is chosen. If λ is replaced by $1-\lambda$, agent i consumes $x^i(1-\lambda)$ and achieves the utility $U_a^i(x^i(1-\lambda)) \neq U_a^i(x^i(\lambda))$.

Let τ^i be the amount of good 0 required in addition to $x^i(1-\lambda)$ in order to let i achieve the original utility level $U_a^i(x^i(\lambda))$. More precisely,

$$U_a^1(x^1(1-\lambda)+\tau^1(1,0,0)) = U_a^1(x^1(\lambda)) \tag{7.2}$$

and

$$U_a^2(x^2(1-\lambda)+\tau^2(1,0,0)) = U_a^2(x^2(\lambda))\,. \tag{7.3}$$

Since $\lambda < 0.5 < 1-\lambda$, we have $\tau^1 < 0$ and $\tau^2 > 0$. Moreover, by symmetry, $x_0^1(\lambda) = x_0^2(1-\lambda)$ and

$$U_a^1(x^1(\lambda)) = U_a^2(x^2(1-\lambda))\,, \qquad U_a^2(x^2(\lambda)) = U_a^1(x^1(1-\lambda))\,. \tag{7.4}$$

We add (7.2) and (7.3), use symmetry and the utility specifications (7.1), and obtain

$$(\tau^1+\tau^2) + a((\tau^1)^2+(\tau^2)^2) + 2a(\tau^1 x_0^1(1-\lambda)+\tau^2 x_0^1(\lambda)) = 0\,. \tag{7.5}$$

Since calculation of consumer 1's optimal shares yields that the demand for good zero is strictly decreasing, we have $x_0^1(\lambda) > x_0^1(1-\lambda) > 0$.

Assume that $\tau^1+\tau^2 \geq 0$ and, hence, $\tau^2 \geq |\tau^1|$. Then $\tau^2 x_0^1(\lambda)) > |\tau^1 x_0^1(1-\lambda)|$. Therefore, the left hand side of (7.5) must be strictly positive for every $a > 0$, which is a contradiction. We conclude that $\tau^1+\tau^2 < 0$. Hence, the equilibrium corresponding to λ is not minimally constrained efficient. □

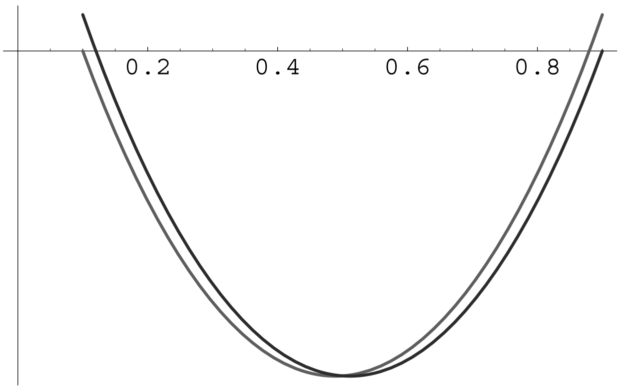

Fig. 7.2. Intersecting total "saving" functions

Figure 7.2 illustrates the argument. Take the equilibrium at 0.1 and consider the sidepayment $\tau^1(\lambda)$ necessary to keep consumer 1 at the utility level $U_a^1(x^1(0.1))$ if the ray 0.1 is replaced by the ray λ. That is to say, $\tau^1(\lambda)$ is given by

$$U_a^1(x^1(\lambda)+\tau^1(\lambda)(1,0,0)) = U_a^1(x^1(0.1))\,.$$

Let $\tau^2(\lambda)$ be defined in a similar way. Thus, $\tau^1(\lambda)+\tau^2(\lambda)$ specifies the total amount of compensation required to maintain the utility levels achieved at 0.1.

The relationship to Figure 7.1 becomes clearer if the compensation is replaced by $-(\tau^1(\lambda)+\tau^2(\lambda))$, which is the amount of good 0 that can be saved at λ while keeping consumer i on the utility level $U_a^i(x^i(0.1))$. This total "saving" function becomes positive at $\lambda = 0.9$, which indicates that the equilibrium with respect to $\lambda = 0.1$ is not minimally constrained efficient. A similar saving function can be defined if the other boundary $\lambda = 0.9$ is taken as the reference point. If a goes to 0, both curves in Figure 7.2 approach the social surplus curve depicted in Figure 7.1 (up to a constant).

We discuss the relationship between the first order condition for minimal constrained efficiency, the first order condition for constrained efficiency, and Drèze equilibria. Let $\tau^i(\lambda)$ be necessary to keep consumer i at the utility level reached at some given interior ray $\bar{\lambda}$. If $\bar{\lambda}$ is minimally constrained efficient then $\sum_i \tau^i(\lambda)$ takes its minimal value 0 at $\bar{\lambda}$ and $d(\sum_i \tau^i(\bar{\lambda}))/d(\lambda) = 0$.

Let $\sigma^i(\lambda)$ denote the sidepayment necessary to keep consumer i at the utility level reached at $\bar{\lambda}$ if sidepayments can be invested in shares. If $\bar{\lambda}$ is constrained efficient then $d(\sum_i \sigma^i(\bar{\lambda}))/d(\lambda) = 0$. By definition, we have $\sigma^i(\lambda) \leq \tau^i(\lambda)$ for any λ with equality at $\bar{\lambda}$. Assume now that $\bar{\lambda}$ is minimally constrained efficient but not necessarily constrained efficient. Since $\sum_i \sigma^i(\lambda) \leq \sum_i \tau^i(\lambda)$ for all λ with equality at $\bar{\lambda}$ the functions $\sum_i \sigma^i(\lambda)$ and $\sum_i \tau^i(\lambda)$ have the same slope at $\bar{\lambda}$. Thus, the first order condition $d(\sum_i \sigma^i(\bar{\lambda}))/d(\lambda) = 0$ for constrained efficiency is satisfied at the minimally constrained efficient $\bar{\lambda}$. In our setting, it is not difficult to show that the ray $\bar{\lambda}$ induces a Drèze equilibrium if and only if $d(\sum_i \sigma^i(\bar{\lambda}))/d(\lambda) = 0$. Therefore, we conclude that $\bar{\lambda}$ is minimally constrained efficient only if it is a Drèze equilibrium.[5]

The nonexistence of constrained efficient and minimally constrained efficient market equilibria is caused by the following facts. First, the example is built upon a nonconvexity. In the unperturbed, quasilinear example, the nonconvexity can be described as follows. The amount of good 0 initially available in the economy just suffices to maintain the utility profile $(u^1(2, 0.1),\ u^2(2, 0.1))$ reached at the boundary point $\lambda = 0.1$, if the other boundary point $\lambda = 0.9$ is chosen. However, if the firm implements any ray λ strictly between 0.1 and 0.9, this amount is insufficient. Second, as soon as the perturbation parameter a becomes positive, the graphs of the two saving functions intersect each other. To maintain the profile $(u_a^1(2, 0.1),\ u_a^2(2, 0.1))$ at $\lambda = 0.9$, one can dispense with a positive amount of good 0. A similar statement holds if the boundary points are interchanged (cf. Fig.7.2). These two features cannot be ruled out in general. Therefore, one cannot expect the market to be able to achieve minimally constrained efficient outcomes.[6]

The allocations attainable by the market depend on the initial allocation of endowments. To obtain a situation in which a constrained efficient market equilibrium exists in the perturbed example, a lump sum redistribution of initial endowments is required. Markets do not perform such redistributions and thus, are less powerful

[5] The proof of the above Proposition is independent of these arguments since we consider all symmetric pairs of rays $\lambda, 1-\lambda$ with $\lambda \neq 0.5$ and show that neither λ nor $1-\lambda$ is minimally constrained efficient.

[6] It has been emphasized in the literature on compensation criteria à la Hicks and Kaldor that intersecting utility possibility frontiers often entail inconsistent policy recommendations (see, e.g., Gravel, 2001).

than even the very weak planner discussed in the context of minimal constrained efficiency. The importance of the initially determined distribution of wealth in nonconvex environments was first pointed out by Guesnerie (1975) in the framework of complete markets and nonconvex production sets. Guesnerie showed that all marginal cost pricing equilibria can be inefficient, even though Pareto efficiency requires prices to equal marginal costs.

7.4 Concluding remarks

We have seen that shareholders' social surplus can reach its minimum at a Drèze equilibrium if all shareholders have quasilinear utilities. This is due to the fact that the definition of a Drèze equilibrium only takes welfare changes of first order into account. Thus, no distinction is made between an interior maximum and any other critical point.

In the quasilinear case, a constrained efficient Drèze equilibrium exists. Therefore, it is tempting to refine the Drèze equilibria in order to rule out constrained inefficient allocations. However, our example shows that this endeavor can fail to provide any solution as soon as one deviates from the quasilinear setting: Arbitrarily small income effects render all market equilibria constrained inefficient.

Moreover, even if the efficiency requirements are substantially reduced, they can remain unfulfilled at every market equilibrium in a finance economy. In our example the stock market cannot even achieve a minimally constrained efficient outcome if the quasilinear setting is abandoned. Hence, the existence of a constrained efficient equilibrium in the quasilinear economy should be viewed as an artifact lacking any robustness.

Clearly, there are economies in which the problem does not arise. For example, Drèze equilibria are constrained efficient if there is only one firm and if every consumer's indirect utility function is quasiconcave. This function describes the maximum amount of utility the consumer can derive from a production decision at different levels of wealth at $t = 0$. The indirect utility functions underlying Figure 7.1 are not quasiconcave. This is due to the fact that the specification of the direct utility functions U^i makes optimal shareholdings sufficiently sensitive to changes in the production ray.[7] Since the indirect utility depends on how the optimal number of shares that an individual holds varies with the asset span and individual wealth at $t = 0$, it is, unless attention is restricted to particularly simple examples, quite difficult to state economically meaningful conditions ensuring the quasiconcavity of indirect utility functions. We do not think that imposing restrictions on consumers' characteristics presents a promising approach to overcome the problem of nonexistence of constrained efficient market equilibria.

It has been suggested to us to use lotteries instead of deterministic allocations. A similar approach has been successfully applied in other settings. Cole and Prescott

[7] If the power 0.6 in the definition of U^i is replaced by a number below 0.5, quasiconcavity of the indirect utility function u^i results.

(1997), for instance, use random allocations to analyze equilibria in economies with clubs. Club membership is indivisible and lotteries are used to restore convexity. Lotteries have also been used to overcome the nonconvexity of the set of feasible allocations in economies with adverse selection. In that case, the nonconvexity is due to individual incentive constraints and eliminated by introducing random allocations; see Prescott and Townsend (1984). In this paper, the difficulty is not due to a nonconvexity on the individual level but to a pure public good problem.

In our framework, random allocations could be introduced by making the production decision stochastic and letting consumers choose their investments contingent on the realization. More specifically, consider the set $U = \{(u^1, u^2) \leq (u^1(2,\lambda), u^2(2,\lambda)) \mid \lambda \in [0.1, 0.9]\}$ of vectors that are below a utility profile attained at some market equilibrium in the quasilinear example. The set U is nonconvex. Let the production ray become random and consumers have von Neumann-Morgenstern utility functions. If consumers are allowed to choose their investments after they have learned the realization of λ the set U is convexified. More precisely, the convex hull of U is generated by the two profiles $(u^1(2,\lambda), u^2(2,\lambda))$ associated with the boundary equilibria $\lambda = 0.1$ and $\lambda = 0.9$. In comparison to the deterministic market equilibrium at $\lambda = 0.5$, both consumers are better off in expectation if the firm chooses a symmetric lottery over $\lambda = 0.1$ and $\lambda = 0.9$.

In the example the procedure corresponds to the introduction of a veil of ignorance. Before the lottery takes place it is not known whose favorite production ray will be realized. This ex ante viewpoint is appropriate for certain fairness considerations, but appears unnatural in the analysis of the efficiency of equilibria in economies with incomplete markets. The introduction of lotteries does not provide a genuine extension of the Arrow-Debreu-McKenzie theory to the case of incomplete markets. Furthermore, introducing lotteries amounts to making markets more complete. Our goal, however, is to analyze efficiency issues in a model with a given, small set of assets.[8] Since the introduction of lotteries over production plans is difficult to justify on economic grounds and since it changes the nature of the underlying problem in an essential way, we do not think that the use of lotteries lends itself to the present framework.

Majority voting presents another way to overcome the social choice problems faced by shareholders. For properties of corporate control by majority voting, see DeMarzo (1993) and Geraats and Haller (1998). Apart from problems such as equilibrium existence, agenda control, non sincere voting etc., the following point deserves attention. Since the voting outcome depends on power, it need not reflect welfare properly. The point is easily understood in the context of the quasilinear example in Section 3. To break ties, a third quasilinear consumer with arbitrarily small weight is introduced, whose utility increases if the ray λ approaches 0.5. This additional consumer becomes the median voter. Due to symmetry, majority voting leads to $\lambda = 0.5$ if every shareholder has one vote. Moreover, it is not difficult to modify

[8] In addition, even if the set of market equilibria is convexified, it differs substantially from the set of constrained efficient allocations. We do not see how one would obtain an analogue to the first welfare theorem.

the example such that voting according to the one share-one vote rule with after trade voting and rational expectations yields the same outcome. However, in this case the median voter needs more than minimal weight and the technology Y needs to be truncated in accordance with the distribution of weights in order to give the median voter overwhelming power. The median voter's optimal choice, though, is the welfare minimum.[9] Thus, majority voting should be seen as a modelling device that is better suited for positive than for normative purposes.

Instead of examining whether a proposed production plan can be unanimously improved upon after sidepayments are made, one can compare the gains, expressed in units of good 0, that are obtained from any production plan in comparison to a given reference point. In the perturbed quasilinear example the point of zero production, that is to say, the allocation (e^1, e^2) of initial endowments, can be used for reference. Consumer i's surplus $S^i(\lambda)$ is given by the amount of good 0 consumer i needs in excess of e^i to obtain the same utility level as if the firm chose the ray λ. The total surplus $\sum_i S^i(\lambda)$ associated with some market equilibrium can then be maximized. In the perturbed quasilinear example the maximum is taken at both boundary points $\lambda = 0.1$ and $\lambda = 0.9$. Thus, the same outcome as in the quasilinear case is obtained.

A major advantage of this approach lies in the fact that it relies on the maximization of continuous functions rather than maximization of incomplete, intransitive, and nonconvex relations. The surplus maximum is characterized as follows: It presents the minimum amount of good 0 needed in the absence of the firm in order to be able to compensate all consumers such that they can attain every utility profile that is induced by some production decision. Clearly, this type of surplus maximization, which is motivated by the lack of constrained efficient market equilibria, does not aim at achieving constrained efficiency and its theoretical foundation remains controversial.

The surplus function described above can be viewed as a particular social welfare function. To overcome the problem of the nonexistence of constrained or even minimally constrained efficient market equilibria, one might also resort to any other social welfare function. However, it is a priori unclear which welfare function is particularly well suited for this purpose.[10]

A less radical procedure suggesting itself in the perturbed quasilinear example is the choice of the boundary equilibria $\lambda = 0.1$ or $\lambda = 0.9$ on the basis that they are "less inefficient" than, say, $\lambda = 0.5$. To define the degree of inefficiency, interpersonal utility comparisons are not required.

The last three approaches provide welfare oriented methods that may be used to overcome the problem presented in this paper. In each case a particular function is optimized. These approaches require a large amount of information and are far

[9] If $0.4 = \underline{\lambda} \leq \lambda \leq 0.6 = \overline{\lambda}$ and the three groups whose favorite rays are $\lambda = 0.4$, $\lambda = 0.5$, and $\lambda = 0.6$, respectively, are of equal size then the unique outcome of the one share-one vote rule is the social minimum $\lambda = 0.5$.

[10] In another context involving lotteries, Dhillon and Mertens (1999) argue in favor of relative utilitarianism.

more complex than the usual profit maximization in General Equilibrium Theory with complete markets. They would change the character of the theory considerably.

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8

A Competitive Model of Economic Geography

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Summary. Most of the literature argues that competitive analysis has nothing interesting to say about location. This paper argues, to the contrary, that a competitive model *can* have something interesting to say about location, provided that locations are *not* identical and transportation costs are *not* zero. To do this, it constructs a competitive intertemporal general equilibrium model and applies it to a suggestive example of migration.

Key words: Economic geography.

JEL Classification Numbers: D5, R0.

8.1 Introduction

How should we model the location of economic activity? Most of the literature has taken the point of view that competitive analysis has nothing to say about location. Krugman [7], for instance, writes "Essentially, to say anything useful or interesting about the location of economic activity in space, it is necessary to get away from the constant-returns, perfect competition approach that still dominates most economic analysis." This view arises in part from a theorem of Starrett [14]: if all locations are identical and transportation costs are zero, then perfect competition (constant returns to scale in production, perfect markets, price-taking agents) must lead economic activity to be uniformly distributed in space.

We think the literature has taken the wrong lesson from Starrett's theorem. Locations are not all the same, and transportation costs are not zero, so the question, in our minds, is not whether a competitive model can have anything interesting to say about a world in which locations are identical and transportation costs are zero, but rather whether a competitive model can have anything interesting to say about a world in which locations are *not* identical and transportation costs are *not* zero. Put differently, the question is not whether competition can lead to a heterogeneous present from a homogenous past, but whether competition can lead to the kind of heterogeneity we actually see in the present from the kind of heterogeneity we see (or imagine) in the past. No one has tried seriously to address this question.

The purpose of this paper is to construct a competitive model that incorporates differences in location and non-zero transportation costs, and to use this model to explore a suggestive example. Our (tentative) conclusion is that competitive models *can* have something interesting to say.[1]

From the point of view of Debreu [4], location — like time and state of nature — should be incorporated simply by viewing location as part of the description of a commodity, and transportation should be incorporated simply as a particular kind of production. However, this approach is really not satisfactory. The problem is not with locating and transporting *commodities*, but rather with locating and transporting *people*. The perspective of our model is that, as has been true throughout most of history, individuals are only able to work and consume in the same location where they reside, and changing that location requires resources and time, and may be unpleasant or even dangerous. As we show in an example, this perspective is especially useful for thinking about migration.

Three other features of our approach merit comment here. The first is that we build a model with a finite number of locations and a continuum of people. Since the surface of the earth appears continuous and its population is surely finite, these choices might seem quite backward, but we believe they are natural. We posit a finite number of locations because we view locations as neighborhoods, cities, counties, or even countries — but not as tiny plots of land in the inner city. We posit a continuum of agents because it eases the modeling of competition and the treatment of indivisibilities inherent in locational choices. We do not view our continuum model as the limiting case of models with large finite numbers of agents but rather as simply a convenient approximation of a fixed model with a large, but definite and finite, number of agents.[2] (In fact, we adopt a framework in which the population is described entirely in statistical terms, without explicit reference to the number of agents.) We believe our model is quite consistent with the way data is reported and with computational work that applies Scarf's [12] algorithm to locations; see Arnott and MacKinnon [1] or Richter [10,11] for example.

The second is that we treat all goods as potentially mobile, but view mobility as a matter of degree. Land is perfectly immobile, raw materials and the products of agriculture and manufacture are typically mobile — but the cost of transportation may be different for different goods, and will be determined endogenously at equilibrium.

The third is that we take account of time and uncertainty, and allow for both to enter into agents' decision-making. The role of time is quite obvious in our example, which involves migration; uncertainty plays no role in the present paper, but will be treated in subsequent work.

[1] We don't suggest that constant returns perfect competition is the whole story. See Fujita, Krugman and Venables [5] for analysis based on increasing returns and monopolistic competition.

[2] As Berliant [2] points out, in the locational context there are logical problems with viewing a continuum model as a limit of large finite models; see also Papageorgiu and Pines [9] and Berliant and ten Raa [3].

Following this Introduction, Section 2 presents our locational model, Section 3 states and proves our existence theorem, and Section 4 presents the example.

8.2 The Locational Model

In this Section, we lay out the basics of our model and present a simple example to illustrate the ideas.

8.2.1 Sites and Trips

We take as given a finite set S of *sites*. We view the description of a site as including time and state of nature as well as physical location. If there is only one site, the geography disappears so we assume henceforth that there are at least two sites.

Each consumer is located by an initial specification of site (see below) and by subsequent choices of site. Because changes of site may require resources, it is convenient to view the changes of site, rather than sites, as the actual objects of choice. We therefore define a *trip* to be a pair $t = (s, s')$ where $s, s' \in S, s \neq s'$: s is the starting point, s' is the ending point. We take as given a set of trips $T \subset S \times S$. A *path* from site s to site s' is a finite collection of trips

$$t_1 = (s_0, s_1), t_2 = (s_1, s_2) \ldots, t_n = (s_{n-1}, s_n)$$

such that $s_0 = s$ and $s_n = s'$. Note that there may be many paths between two given sites — or none.

8.2.2 Consumption Goods

Consumption goods are described, just as Debreu [4] suggests, by physical characteristics, location, date, and state of nature. The last three of these are summarized in the site; we assume that physical characteristics belong to a finite set L. Thus, consumption goods are indexed by $S \times L$. We abuse notation and write S, L for the cardinalities of the sets of sites and of physical characteristics, as well as for the sets themselves. (And we frequently say that L consumption goods are available at each site.) Thus the space of consumption bundles is $\mathbb{R}^{S \times L} = \mathbb{R}^{SL}$. For simplicity, we assume that consumption goods are perfectly divisible, although there would be no difficulty in allowing for (some) indivisible goods (with appropriate assumptions on preferences and endowments).

8.2.3 Objects of Choice

Consumption bundles and trips are objects of choice and are priced, so the choice space and price space are

$$\mathcal{X} = \mathbb{R}^{SL} \times \mathbb{R}^{T}$$

Write π_{SL}, π_T for the projections of the choice space $\mathcal{X}$ onto the first factor and second factor (respectively). It is convenient to use functional notation and view choices x and prices p as functions

$$x, p : (S \times L) \cup T \rightarrow \mathbb{R}$$

Write x_{SL}, p_{SL}, x_T, p_T for the restrictions of choices and prices. Write $\delta_{(s,\ell)}, \delta_t$ for the consumption bundles representing 1 unit of the consumption good (s, ℓ) or 1 unit of the trip t, respectively.

8.2.4 Production

We follow McKenzie [8] in describing production in terms of an aggregate production possibility set $Y \subset \mathcal{X}$; an element $y \in Y$ is an *aggregate activity vector*. Thus we allow for production of consumption goods at a single site, for transport of consumption goods between sites, and for transport of individuals between sites. We make appropriate modifications of standard assumptions:

P1 Y is a closed convex cone
P2 $-\mathcal{X}_+ \subset Y$
P3 $\pi_{SL}(Y) \cap \mathbb{R}^{SL}_+ = \{0\}$
P4 if $y = (y_{SL}, y_T) \in Y$ then $y' = (y_{SL}, y_T^+) \in Y$

The second assumption means that we allow for free disposal in production, the third means that there is no free production of consumption goods (but note that we allow for free production of trips), and the fourth means that trips are not used as inputs to production.

8.2.5 Consumers

A *consumer* is characterized by a *choice set* $X \subset \mathcal{X}$, an *endowment* $e \in \mathbb{R}^{SL} \times \{0\}$, an *initial site* $s \in S$, and a *weak preference relation* $\succeq$ on X, with associated strict preference relation $\succ$. Note that both consumption bundles and trips enter into preferences, and that consumers are endowed with consumption goods but not with trips.

For our purposes, we insist that the choice of a trip is indivisible and that consumers can choose at most one trip between given sites (keep in mind that sites are dated):

C1 for all $x \in X$: $x_T(T) \subset \{0, 1\}$

Note that a consumer's choice of trips might also be subject to further restrictions (depending on interpretation).

We insist that consumption can only take place at sites where the consumer is located. To express this idea, say that the site s' is *accessible* from the site s given a choice function x_T if there is a path $t_1, \ldots, t_n$ from s to s' such that $x_T(t_i) = 1$ for each i. If s is the initial site then a consumer who chooses x occupies the site s' (at some point during the period modeled) exactly when s' is accessible from s given x_T. Thus our requirement is formalized by:

C2 for all $x \in X$: if s' is not accessible from s given x_T then $x_{SL}(s', \cdot) \equiv 0$

Subject to these two restrictions, we assume only that consumption sets are closed subsets of the positive orthant (so that only negative consumption is not permitted):

C3 $X \subset \mathbb{R}_+^{SL} \times \mathbb{R}_+^T$ is a closed set

Preferences satisfy the usual requirements:

C4 the weak preference relation $\succeq$ is reflexive, transitive, complete, and has closed graph in $X \times X$
C5 the strict preference relation $\succ$ is irreflexive, transitive, and has open graph in $X \times X$
C6 preferences are locally non-satiated in consumption goods; that is, for each $x = (x_{SL}, x_T) \in X$ and each $\varepsilon > 0$ there is a vector $x'_{SL} \in \mathbb{R}^{SL}$ such that $x' = (x'_{SL}, x_T) \in X$, $\|x'_{SL} - x_{SL}\| < \varepsilon$, and $x' \succ x$

We do not require that the endowment e belongs to the choice set X, but we do need to require that the consumer can survive without trade. To this end, say that $x \in X$ is a *survival choice* if $x - e \in Y$. We require

C7 there exists at least one survival choice

Note that, if production plans make non-positive profits (which will necessarily be the case at equilibrium), then a survival choice will be in the consumer's budget set with respect to every price system.

Finally, we require that the consumer will never make a choice which involves no consumption goods:

C8 if $x \in X$ and $x_{SL} \equiv 0$ then there is a survival choice x' such that $x' \succ x$

8.2.6 The Economy

For our purposes, it is convenient to view the *consumer sector* of the economy entirely in statistical terms, as suggested by Hart, Hildenbrand and Kohlberg [6].[3] That is, a consumer sector is a probability measure μ on the space $\mathcal{C}$ of consumer characteristics. (We give $\mathcal{C}$ the topology of closed convergence, viewing the components of the 4-tuple $(X, e, s, \succeq)$ of consumer characteristics as closed subsets of $\mathcal{X}$, $\mathbb{R}^{SL}$, S, $\mathcal{X} \times \mathcal{X}$, respectively.) As usual, the interpretation we have in mind is that, for any set Q of consumer characteristics, $\mu(Q)$ represents the fraction of the total consumer population whose characteristics lie in Q.

An *economy* is a pair $\mathcal{E} = \langle \mu, Y \rangle$ consisting of a consumer sector and an aggregate production set.

We assume throughout that all goods are potentially available in the aggregate

[3] We emphasize that this modeling choice is a convenience, rather than a requirement; we could easily formulate the model and establish existence in a framework with a non-atomic measure space of agents. Indeed, we could easily formulate the model in a framework with a finite number of agents, although the indivisible nature of choices and the non-convexity of consumption sets and preferences would mean that equilibrium need not exist.

E1 $\left(Y + \int e\, d\mu\right) \cap \left(\mathbb{R}^{SL}_{++} \times \mathbb{R}^{T}_{++}\right) \neq \emptyset$

8.2.7 Equilibrium

An *equilibrium* for the economy $\mathcal{E} = \langle \mu, Y \rangle$ consists of a price vector $p \in (\mathbb{R}^{SL}_{+} \times \mathbb{R}^{T}_{+}) \setminus \{0\}$, a probability measure ν on $\mathcal{C} \times \mathcal{X}$, and a production vector $y \in Y$ such that

(1) the marginal of ν on $\mathcal{C}$ is μ
(2) $\int (x - e)\, d\nu = y$
(markets clear)
(3) $\nu(\{(X, e, s, \succeq, x) : x \notin X$ or $p \cdot x > p \cdot e\}) = 0$
(almost all consumers choose in their budget set)
(4) $\nu(\{(X, e, s, \succeq, x) : \exists x' \in X, p \cdot x' \leq p \cdot e \,\&\, x' \succ x\}) = 0$
(almost all consumers optimize)
(5) $p \cdot y = \max\{p \cdot y' : y' \in Y\} = 0$
(production profits are maximized)

A *quasi-equilibrium* differs from an equilibrium in that only the following weaker version of (4) is satisfied:

$(4')\nu(\{(X, e, s, \succeq, x) : \exists x' \in X, p \cdot x' < p \cdot e \,\&\, x \succ x'\}) = 0$
(almost all consumers quasi-optimize)

We caution the reader that *some* commodity and trip prices may be 0 at equilibrium. (The price of a commodity may be 0 if no one chooses to locate at the site at which that commodity is available.) However, as the following simple proposition shows, not *all* commodity prices can be 0 at equilibrium (or quasi-equilibrium).

Proposition *If (p, ν, y) is a quasi-equilibrium for the economy $\mathcal{E}$ then the price of some consumption good is strictly positive.*

Proof Suppose not. By definition, not all prices are zero, so the price of some trip t^* must be strictly positive. Assumption **E1** guarantees that t^* is produced, and Assumption **P4** guarantees that t^* can be produced from consumption goods alone. By supposition, these consumption goods have price zero, so t^* can be produced at a positive profit. Since Y is a cone, this means that arbitrarily large production profits are possible, and hence that no production plan can maximize profit; this contradicts the definition of quasi-equilibrium. □

8.3 Existence of Equilibrium

Our existence result is the following:

Theorem *Every economy $\mathcal{E} = \langle \mu, Y \rangle$ satisfying the assumptions* **P1 - P4***,* **C1 - C8***,* **E1** *above admits a quasi-equilibrium.*

Before beginning the proof, we record some notation and a useful lemma. If E, F are topological spaces, we write $M(E), M(F)$ for the spaces of (regular Borel) measures on E, F. If $f : E \to F$ is a continuous function and $\mu \in M(E)$ is a regular Borel measure on E, we write $f_*(\mu)$ for the direct image measure on F, whose value on a Borel set $G \subset F$ is

$$f_*(\mu)(G) = \mu(f^{-1}(G))$$

Notice that if $E = F \times F'$ is a product and f is the projection on the first factor, then $f_*(\mu)$ is the marginal of μ on F.

Lemma *Let A be a separable metric space, let B be a compact metric space and let $\pi : A \times B \to A$ be the projection on the first factor. Let $C \subset A \times B$ be a closed set such that $\pi(C) = A$. If μ is a positive measure on A then*

$$\Gamma = \{\nu \in M(C) : \pi_*(\nu) = \mu\}$$

is a non-empty, weak-star compact convex subset of $M(C)$.

Proof Convexity of Γ is trivial, and compactness is straightforward. To see that Γ is not empty, suppose first that μ is compactly supported, with support $\operatorname{supp}\mu$. Choose a sequence $\{\mu_n\}$ of atomic measures, each of which has support a finite subset of $\operatorname{supp}\mu$, such that $\mu_n \to \mu$ weak star. Since B is compact so is $A^* = \pi^{-1}(\operatorname{supp}\mu) \cap C$, and $\pi(A^*) \supset \operatorname{supp}\mu$. For each n, write $\mu_n = \sum_k a_n^k \delta_{x_n^k}$, choose points $c_n^k \in A^*$ with $\pi(c_n^k) = x_n^k$, and set $\nu_n = \sum_k a_n^k \delta_{c_n^k}$. It is evident that $\pi_*(\nu_n) = \mu_n$ and that any weak star limit point ν of the sequence $\{\nu_n\}$ is a measure supported on $A^* \subset C$ for which $\pi_*(\nu) = \mu$.

Now consider an arbitrary measure μ. Since A is a separable metric space, every measure on A is tight. Hence we can choose a sequence $\{K^i\}$ of *disjoint* compact subsets of A such that $\mu(\bigcup K^i) = \mu(A)$. Setting $\mu^i = \mu|K^i$ and applying the result in the compactly supported case yields measures ν^i supported on $C \cap \pi^{-1}(K^i)$ such that $\pi_*(\nu^i) = \mu^i$. Setting $\nu = \sum_i \nu^i$ yields a measure ν on C such that $\pi_*(\nu) = \mu$, as desired. □

We now turn to the proof of the Theorem.

Proof of Theorem Let $\mathcal{E} = \langle \mu, Y \rangle$ be an economy satisfying the assumptions. We construct perturbed economies $\mathcal{E}_n = (\mu_n, Y_n)$, find equilibria for these perturbed economies, and then obtain a quasi-equilibrium for $\mathcal{E}$ by a limiting argument.

Step 1 We first construct the perturbed economies $\mathcal{E}_n$. To construct the consumption sector μ_n, first define $\tau^n : \mathcal{C} \to \mathcal{C}$ by

$$\tau^n(X, e, s, \succeq) = (X, \frac{1}{n}\mathbf{1} + \inf\{e + \frac{1}{n}\mathbf{1}, n\mathbf{1}\}, s, \succeq)$$

Set $\mu_n = \tau^n_*(\mu)$. To construct the production sector Y_n, first note that, by assumption, Y is a closed convex cone and $\pi_{SL}(Y) \cap \mathbb{R}_+^{SL} = \{0\}$ (no free production of consumption goods). Hence we may we can find $d > 0$ such that

$\|\pi_{SL}(y)^-\| \geq \delta\|\pi_{SL}(y)^+\|$ for each $y \in Y$. (That is, the marginal rate of transformation for consumption goods is bounded.) Define Y_n by

$$Y_n = \{y \in \mathbb{R}^{SL} \times \mathbb{R}^T : \exists y' \in Y, y'_T = y_T, \|y - y'\| \leq \frac{d}{2n}\|y^-\|\}$$

It is easily checked that $\mathcal{E}_n$ satisfies the assumptions **P1 - P4**, **C1 - C8**, **E1**.

Step 2 We construct a compact price simplex for $\mathcal{E}_n$. Define Y_n° to be the polar cone of Y_n; that is

$$Y_n^\circ = \{p \in \mathbb{R}^{SL} \times \mathbb{R}^T : p \cdot y \leq 0 \text{ for all } y \in Y_n\}$$

Because $Y_n \supset -(\mathbb{R}_+^{SL} \times \mathbb{R}_+^T)$, it follows that $Y_n^\circ \subset \mathbb{R}_+^{SL} \times \mathbb{R}_+^T$. Define a normalized price simplex by

$$\Delta_n = \{p \in Y_n^\circ : \sum_{(s,\ell) \in S \times L} p(s,\ell) + \sum_{t \in T} p(t) = 1\}$$

Note that Δ_n is a compact convex subset of $\mathbb{R}_+^{SL} \times \mathbb{R}_+^T$.

We claim that, for $p \in \Delta_n$, prices of consumption goods are bounded away from 0: that is, there is some $\varepsilon > 0$ such that $p(s,\ell) \geq \varepsilon$ for each $p \in \Delta_n$. To see this, suppose first that $p \in \Delta_n$ and $p_{SL} \equiv 0$. Our normalization guarantees that $p(t) > 0$ for some trip t. Assumption **E1** guarantees that t is produced, and Assumption **P4** guarantees that t can be produced from consumption goods: there is a $y \in Y_n$ such that $\pi_T(y^-) = 0$ and $y^+(t) > 0$. Since the prices of consumption goods are identically 0 and the prices of trips are non-negative, it follows that $p \cdot y > 0$, which contradicts the fact that $p \in Y_n^\circ$. We conclude that there is no $p \in \Delta_n$ for which $p_{SL} \equiv 0$. Because Δ_n is compact, it follows immediately that there is an $\varepsilon_1 > 0$ such that $\sum_{(s,\ell)} p(s,\ell) \geq \varepsilon_1 > 0$ for each $p \in \Delta_n$. Set $\varepsilon = \frac{\varepsilon_1}{SL}\frac{d}{2n}$. Fix $p \in \Delta_n$ and $(s^*,\ell^*) \in S \times L$. Because $\sum p(s,\ell) \geq \varepsilon_1$, there is at least one (s',ℓ') for which $p(s',\ell') \geq \frac{\varepsilon_1}{SL}$. By Assumption **P2**, $-\delta_{(s^,\ell^*)} \in Y$; by construction, $(\frac{d}{2n}\delta_{(s',\ell')} - \delta_{(s*,\ell*)}) \in Y_n$. Because $p \in Y_n^\circ$, it follows that

$$\frac{d}{2n}p(s',\ell') - p(s^*,\ell*) = p \cdot (\frac{d}{2n}\delta_{(s',\ell')} - \delta(s^*,\ell^*)) \leq 0$$

whence

$$p(s^*,\ell^*) \geq \frac{\varepsilon_1}{SL}\frac{d}{2n} = \varepsilon$$

as required.

Step 3 Now we construct an equilibrium for $\mathcal{E}_n$ as a fixed point of an excess demand correspondence. By construction, endowments in the support of μ_n are bounded. Since the consumption components of prices in Δ_n are bounded away from 0, we can find a compact set $B \subset \mathcal{X}$ that contains all the budget-feasible choices for a consumer whose characteristics lie in $\operatorname{supp}\mu_n$; that is: if $p \cdot x \leq p \cdot e$ for some $(X, e, s, \succeq) \in \operatorname{supp}\mu_n$ then $x \in B$. For $p \in \Delta_n$, set

$$C(p) = \{(X, e, s, \succeq, x) \in \mathcal{C} \times B : p \cdot x \leq p \cdot e,\ \nexists x' \in X, p \cdot x' \leq p \cdot e, x' \succ x\}$$

Note that $(X, e, s, \succeq, x) \in C(p)$ exactly when x is an optimal choice in the budget set of a consumer with characteristics $(X, e, s, \succeq)$. Apply the above Lemma with $A = \mathcal{C}$ (the space of consumer characteristics), $B = B$, $C = C(p)$ to conclude that the set of measures

$$\Gamma(p) = \{\nu \in M(C(p)) : \pi_*(\nu) = \mu\}$$

is compact, convex, and non-empty. It is easily checked (using compactness of B) that the correspondence $p \mapsto \Gamma(p)$ is upper hemi-continuous.[4]

Set $\widehat{B} = B - B = \{b - b' : b, b' \in B\}$, and define correspondences

$$Z : \Delta_n \to \widehat{B}\ ;\ \ V : \widehat{B} \to \Delta_n\ ;\ \ \Phi : \Delta_n \times \widehat{B} \to \Delta_n \times \widehat{B}$$

by

$$\begin{aligned} Z(p) &= \left\{ \int (x - e)\, d\nu : \nu \in \Gamma(p) \right\} \\ V(z) &= \arg\max\{p \cdot z : p \in \Delta_n\} \\ \Phi(p, z) &= Z(p) \times V(z) \end{aligned}$$

Note that $Z(p)$ is the set of aggregate excess demands at the price p. The reader will easily verify (using continuity and linearity of integration, and the properties of the correspondence Γ) that Z, V and Φ are upper hemi-continuous and have non-empty compact convex values.[5]

Because $\Delta_n \times \widehat{B}$ is compact and convex and Φ is an upper hemi-continuous correspondence with non-empty, compact convex values, Φ has a fixed point (p_n, z_n). By definition, $z_n = \int (x - e) d\nu_n$ for some $\nu_n \in \Gamma(p)$.

We claim that (p_n, ν_n, z_n) constitute an equilibrium for $\mathcal{E}_n$. To see this, we need only show that $z_n \in Y_n$, so suppose not. Then there is a price p such that $p \cdot z_n > \sup\{p_n \cdot y : y \in Y_n\}$. Because $0 \in Y_n$, it follows that $p \cdot z_n > 0$. Because Y_n is a cone, it follows that $p \cdot y \leq 0$ for each $y \in Y_n$ and hence that $p \in Y_0^\circ$. Normalizing if necessary, we may assume that $p \in Y_n^\circ$. On the other hand, an optimizing consumer whose preferences are locally non-satiated spends all his income. Integrating across consumers implies Walras's Law in the aggregate: $p_n \cdot z_n = 0$. Combining these facts and recalling that (p_n, z_n) is a fixed point of Φ, we obtain

$$p \cdot z_n > 0 = p_n \cdot z_n = \sup\{p' \cdot z_n : p' \in \Delta_n\}$$

[4] The argument, which we leave to the reader, is entirely standard with the exception of two points. 1) Because endowments do not belong to consumption sets, they do not belong to budget sets either. However, Assumption **C7** guarantees that budget sets are always non-empty. 2) Because the choice of trips is indivisible, the budget correspondence of a consumer is *not* continuous in prices. However, because of Assumption **C8**, the individually rational part of the budget correspondence is continuous, whence the optimal choice correspondence is upper hemi-continuous.

[5] The arguments follow familiar lines, with the exceptions noted in the previous footnote.

This is a contradiction, so we conclude that $z_n \in Y_n$, and hence that (p_n, ν_n, z_n) constitute an equilibrium for $\mathcal{E}_n$.

Step 4 To construct a quasi-equilibrium for $\mathcal{E}$ as a limit of the quasi-equilibria just constructed for the perturbed economies $\mathcal{E}_n$, our first task is to show that the sequence $\{\nu_n^*\}$ of equilibrium measures is tight.

To this end, set $\bar{e} = \int e\, d\mu$. Because $\pi_{SL}(Y_1) \cap \mathrm{I\!R}^{SL} = \{0\}$, we may choose a $\bar{y} \in Y_1$ such that if $y \in Y_1$ and $\pi_{SL}(y)^- \leq \bar{e}$ then $\pi_{SL}(y)^+ \leq \pi_{SL}(\bar{y})^+$.

For each positive integer m and each $(s,l) \in S \times L$, set

$$H_m(s,l) = \{(X, e, s, \succ, x) \in \mathcal{C} \times \mathcal{X} : x(s,l) > m\}$$

Note that

$$\int_{H_m(s,l)} x(s,l)\, d\nu_n^* > m\nu_n^*(H_m(s,l)) \tag{8.1}$$

On the other hand, since ν_n^* is an equilibrium measure,

$$\int (x - e)\, d\nu_n^* = y_n \in Y_n \tag{8.2}$$

Our construction guarantees that

$$\int e\, d\nu_n^* = \int e\, d\mu_n \to \int e\, d\mu \tag{8.3}$$

Since consumption is non-negative and $Y_n \subset Y_1$ for each n, our construction also guarantees that

$$\int x\, d\nu_n^* = \int e\, d\nu_n^* + y_n \leq \bar{e} + \bar{y}^+ \tag{8.4}$$

Combining the equations and inequalities (8.1)-(8.4)yields

$$\nu_n^*(H_m(s,l)) \leq \frac{1}{m}[\bar{e}(s,l) + \bar{y}^+(s,l)] \tag{8.5}$$

Now let $\epsilon > 0$. Since the single measure μ is tight, we may choose a compact subset $K \subset \mathcal{C}$ such that $\mu(\mathcal{C} \setminus K) < \epsilon/2$. Set $H_m = \bigcup_{(s,l)} H_m(s,l)$. In view of the inequality (8.5), if m is sufficiently large, then $\nu_n^*(H_m) < \epsilon/2$. Setting

$$K^* = \{(X, e, s, \succeq, x) \in \mathcal{C} \times \mathcal{X} : e \in K, x \notin H_m\}$$

we obtain a compact subset of $\mathcal{C} \times \mathcal{X}$ such that $\nu_n^*(\mathcal{C} \times \mathcal{X} \setminus K^*) < \epsilon$ for each n. Hence the sequence $\{\nu_n^*\}$ of equilibrium measures is tight.

Step 5 We now obtain a quasi-equilibrium for $\mathcal{E}$. Note that all the equilibrium prices p_n lie in the simplex

$$\Delta = \{p \in -Y^\circ : \sum y(s,l) = 1\}$$

which is a compact subset of the non-negative orthant $\mathcal{X}_+$. Hence some subsequence of the price sequence (p_n) converges. A tight sequence of measures contains a convergent subsequence. Passing to an appropriate subsequence, therefore, we may assume there is a price $p \in \Delta$ and a measure ν on $\mathcal{X} \times$X such that $p_n \to p$ and $\nu_n \to \nu$. Weak star convergence implies that

$$y_n = \int (x-e)d\nu_n \to \int (x-e)d\nu = y$$

Because $y_n \in Y_n$ and each Y_n is closed, it follows that $y \in Y_n$ for each n and hence that $y \in Y = \bigcap_n Y_n$. It is now easily verified that (ν, p, y) constitutes a quasi-equilibrium for the economy $\mathcal{E}$. □

Note that we have only established the existence of a quasi-equilibrium, not of an equilibrium. Because trips are indivisible and agents can only consume at sites they occupy, it is not hard to find simple examples of quasi-equilibria that are not equilibria; indeed, it is not hard to find economies that admit no equilibrium at all. It seems difficult to write down simple and economically natural assumptions sufficient to guarantee that every quasi-equilibrium is an equilibrium, or even that an equilibrium exists — although the example below does enjoy these properties.

8.4 Example

In this Section we present an example that addresses migration in a way rather different from the literature (compare Sjaastaad [13] for instance), and demonstrates our thesis that a small inhomogeneity in locations can lead to a large and striking inhomogeneity at equilibrium: substantial migration flows, both forward *and* backward.

There are 2 locations (East and West) and 3 dates (0, 1, 2), but all consumers are located in the East at date 0, so we identify 5 sites $E0$, $E1$, $W1$, $E2$, $W2$ and 6 trips:

$$t_1 = (E0, E1), t_2 = (E0, W1)$$
$$t_3 = (E1, E2), t_4 = (E1, W2)$$
$$t_5 = (W1, W2), t_6 = (W1, E2)$$

That is, consumers, who are located in the East at date 0, can remain in the East at date 1 or migrate to the West at date 1; consumers who remain in the East at date 1 can remain in the East at date 2 or migrate to the West at date 2; consumers who have moved to the West at date 1 can remain in the West at date 2 or return to the East at date 2. (Migration occurs between dates.) The possible choices of paths and consequent choices of location through time are:

$\pi_1 = (t_1, t_3)$: East at dates 0, 1, 2
$\pi_2 = (t_1, t_4)$: East at dates 0, 1; West at date 2
$\pi_3 = (t_2, t_5)$: East at date 0, West at dates 1, 2
$\pi_4 = (t_2, t_6)$: East at date 0, West at date 1, East at date 2

There are 2 commodities at each site, land and corn; we write $E0c$, $E0\ell$ for corn, land at site $E0$, and so forth. Agents are endowed with (claims to) k acres of land at each site, but no corn. Corn can be produced using land as the sole input: 1 acre of land yields 1 ton of corn at the same date and location. Corn is perishable, and cannot be transported from one site to another. Each trip between distinct sites uses as input

τ tons of corn at the initial site. Agents derive utility only from the consumption of corn, according to the utility function

$$U(x) = \ln x(E0c) + \beta[\ln x(E1c) + \alpha \ln x(W1c)] + \beta^2[\ln x(E2c) + \alpha \ln x(W2c)]$$

(Recall that $x(E0c)$ represents corn consumed in the East at date 0, and so forth). The parameter β ($0 < \beta \leq 1$) represents time discounting, the parameter $\alpha > 0$ represents the weighting of West versus East: if $\alpha < 1$, agents prefer the East (other things equal); if $\alpha > 1$ agents prefer the West. To solve for equilibrium, let ρ_j ($j = 1, 2, 3, 4$) be the fraction of agents who choose the path π_j .

Because agents derive no utility from consuming land directly, all land will be used for corn production. Moreover, zero profit in corn production implies that the price of 1 ton of corn equals the price of 1 acre of land, so we henceforth suppress land prices entirely. Write

$$p = (p(E0c), p(E1c), p(E2c), p(W1c), p(W2c))$$

for the vector of corn prices, and normalize so that $p(E0c) = 1$. Given these prices for corn and the implied prices for land, all agents hold wealth:

$$w(p) = [1 + p(E1c) + p(E2c) + p(W1c) + p(W2c)]k$$

Write $w_j(p)$ for the net wealth of an agent who chooses path π_j. If τ is small enough then at equilibrium all sites will be occupied, whence all corn will be consumed. Hence market clearing at each site yields:

- Site $E0$

$$\rho_1 \frac{w_1(p)}{1 + \beta + \beta^2} + \rho_2 \frac{w_2(p)}{1 + \beta + \beta^2 \alpha} + \rho_3 \frac{w_3(p)}{1 + \beta\alpha + \beta^2 \alpha} + \rho_4 \frac{w_4(p)}{1 + \beta\alpha + \beta^2} + \rho_3 \tau + \rho_4 \tau = k \tag{8.6}$$

- Site $E1$:

$$\rho_1 \frac{\beta w_1(p)}{(1 + \beta + \beta^2) p(E1c)} + \rho_2 \frac{\beta w_2(p)}{(1 + \beta + \beta^2 \alpha) p(E1c)} + \rho_2 \tau = k \tag{8.7}$$

- Site $E2$:

$$\rho_1 \frac{\beta^2 w_1(p)}{(1 + \beta + \beta^2) p(E2c)} + \rho_4 \frac{\beta^2 w_4(p)}{(1 + \beta\alpha + \beta^2) p(E2c)} = k \tag{8.8}$$

- Site $W1$:

$$\rho_3 \frac{\beta\alpha w_3(p)}{(1 + \beta\alpha + \beta^2 \alpha) p(W1c)} + \rho_4 \frac{\beta\alpha w_4(p)}{(1 + \beta\alpha + \beta^2) p(W1c)} + \rho_4 \tau = k \tag{8.9}$$

- Site $W2$:

$$\rho_2 \frac{\beta^2 \alpha w_2(p)}{(1+\beta+\beta^2\alpha)p(W2c)} + \rho_3 \frac{\beta^2 \alpha w_3(p)}{(1+\beta\alpha+\beta^2\alpha)p(W2c)} = k \tag{8.10}$$

Because consumers have identical preferences and endowments, they must attain equal levels of utility in equilibrium. Direct calculation yields indirect utility functions $v_j(p)$ conditional on trip chosen, expressed as functions of corn prices:

$$\begin{aligned}
v_1(p) &= \ln\left(\frac{w_1(p)}{1+\beta+\beta^2}\right) + \beta \ln\left(\frac{\beta w_1(p)}{(1+\beta+\beta^2)p(E1c)}\right) \\
&\quad + \beta^2 \ln\left(\frac{\beta^2 w_1(p)}{(1+\beta+\beta^2)p(E2c)}\right) &(8.11)\\
v_2(p) &= \ln\left(\frac{w_2(p)}{1+\beta+\beta^2\alpha}\right) + \beta \ln\left(\frac{\beta w_2(p)}{(1+\beta+\beta^2\alpha)p(E1c)}\right) \\
&\quad + \beta^2\alpha \ln\left(\frac{\beta^2 \alpha w_2(p)}{(1+\beta+\beta^2\alpha)p(W2c)}\right) &(8.12)\\
v_3(p) &= \ln\left(\frac{w_3(p)}{1+\beta\alpha+\beta^2\alpha}\right) + \beta\alpha \ln\left(\frac{\beta\alpha w_3(p)}{(1+\beta\alpha+\beta^2\alpha)p(W1c)}\right) \\
&\quad + \beta^2\alpha \ln\left(\frac{\beta^2 \alpha w_3(p)}{(1+\beta\alpha+\beta^2\alpha)p(W2c)}\right) &(8.13)\\
v_4(p) &= \ln\left(\frac{w_4(p)}{1+\beta\alpha+\beta^2}\right) + \beta\alpha \ln\left(\frac{\beta\alpha w_4(p)}{(1+\beta\alpha+\beta^2)p(W1c)}\right) \\
&\quad + \beta^2 \ln\left(\frac{\beta^2 w_4(p)}{(1+\beta\alpha+\beta^2)p(E2c)}\right) &(8.14)
\end{aligned}$$

At equilibrium, identical consumers obtain the same utility, so:

$$v_1(p) = v_2(p) = v_3(p) = v_4(p) \tag{8.15}$$

Finally, the fractions of consumers choosing each path must add to 1:

$$\rho_1 + \rho_2 + \rho_3 + \rho_4 = 1 \tag{8.16}$$

Boiling down equations (8.6)-(8.16) yields 9 equations in 8 unknowns: corn prices in 4 sites (recall the normalization $p(E0c) = 1$) and the 4 fractions ρ_j. However, Walras' Law means that one of these equations are redundant, leaving 8 independent equations in 8 unknowns. An equilibrium is determined by a solution to these equations having the additional property that each of the fractions ρ_j lies between 0 and 1. Except for special values of the parameters it does not seem possible to solve these equations in closed form, but it is not difficult to solve numerically. To ease the computational burden, we take $\beta = 1$, so consumers do not discount the future. We also fix $k = 10$, just for purposes of illustration.

First consider the homogeneous case: $\tau = 0$ and $\alpha = 1$. In this case, half the population moves to the West at date 1, equalizing land rents and corn production in both locations from date 1 on. Migration flow between dates 1 and 2 is indeterminate (consumers do not care about location and travel is costless). This is competition in a world without spatial differentiation or transportation cost, and the outcome is perfectly homogeneous, exactly as Starrett's [14] theorem predicts.

Second consider a small cost of moving: $\tau = 0.01$ (this is 0.1% of the production of each agent's land endowment at each site), but $\alpha = 1$ (so that agents remain indifferent between East and West). As shown in the middle rows of Tables 8.1–8.4, the result is essentially the same:[6] at equilibrium, half the population moves to the West at date 1, but no one moves thereafter; land and corn prices are equalized at all sites with twice as much land in production, so consumption of corn per capita doubles as well. Third consider $\tau = 0.01, \alpha \neq 1$. As we see in Table 8.1, if $\alpha = 0.92$ (consumers prefer the East), slightly more than half of all consumers spend their whole lives in the East; the rest migrate West in date 1 and remain there in date 2. Agents who migrate West pay less for land and corn and, as shown in Tables 8.2–8.4, consume more corn, compensating them for the undesirability of the West and the cost of migrating. If $\alpha = 1.08$ (consumers prefer the West), slightly more than half of all consumers migrate West in date 1 and stay there; all the rest remain in the East throughout. Agents who migrate West pay more for land and corn, and hence consume less corn, but are compensated by the desirability of the West. (The imperfect asymmetry of parameters and outcomes reflects the initial asymmetry that all consumers begin in the East.)

Table 8.1. Paths and prices ($\beta = 1, k = 10, \tau = .01$)

α	ρ_1	ρ_2	ρ_3	ρ_4	$p(E1c)$	$p(E2c)$	$p(W1c)$	$p(W2c)$
0.92	0.56	0.00	0.44	0.00	0.54	0.54	0.42	0.42
0.94	0.54	0.00	0.46	0.00	0.53	0.53	0.44	0.44
0.96	0.39	0.13	0.34	0.13	0.52	0.52	0.46	0.46
0.98	0.45	0.06	0.43	0.06	0.51	0.51	0.48	0.48
1.00	0.50	0.00	0.50	0.00	0.50	0.50	0.50	0.50
1.02	0.42	0.07	0.44	0.07	0.49	0.49	0.52	0.52
1.04	0.36	0.11	0.41	0.11	0.48	0.48	0.54	0.54
1.06	0.12	0.34	0.20	0.34	0.47	0.47	0.56	0.56
1.08	0.45	0.00	0.55	0.00	0.46	0.46	0.58	0.58

All of this seems reasonable, if not particularly surprising. However, for values of α closer to 1, a much more interesting phenomenon occurs: *some consumers migrate from East to West in date 1 and then migrate back at date 2*, to be replaced by new

[6] To the precision presented in the Tables, the result is exactly the same, but the small cost of moving does make a difference: for example,

$$p(E1c) \approx 0.5000001 \quad p(E2c) \approx 0.499750 \quad p(W1c) \approx 0.499500 \quad p(W2c) \approx 0.499750$$

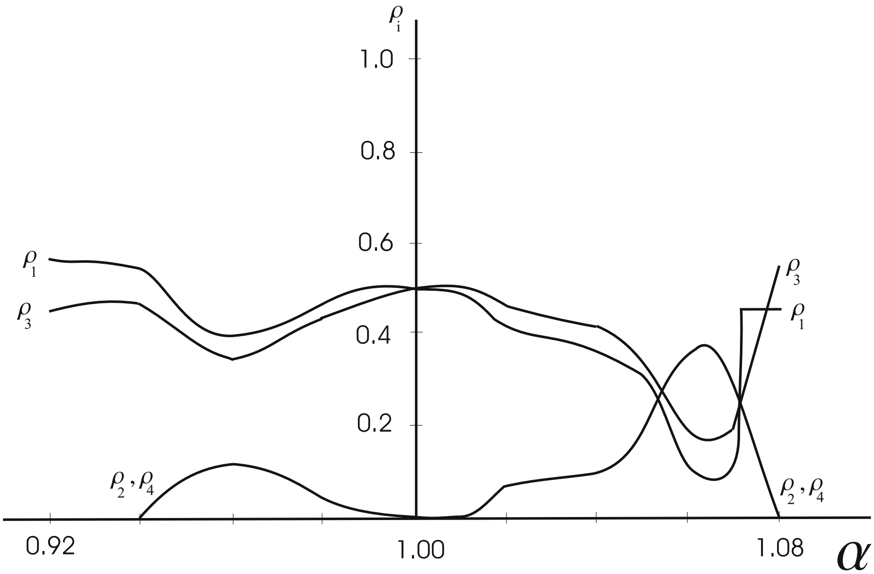

Fig. 8.1. Path probability vs. α

migrants from the East. As shown in Table 8.1 and Figure 8.1, for $\alpha < 1$ (the East is preferred), the backward flow of migration reaches a peak ($\rho_2 = \rho_4 = 0.13$) at around $\alpha = 0.96$. A similar phenomenon occurs for $\alpha > 1$, with the offsetting flows reaching a peak ($\rho_2 = \rho_4 = 0.34$) at around $\alpha = 1.06$. (Again, note the imperfect symmetry between these parameter values.) Both of these outcomes reflect a general equilibrium phenomenon: In the first instance, Easterners migrate West to take advantage of low prices/abundant land in the West, but later return to the more attractive East. In the second instance, Easterners migrate to the more attractive West, but return because Western land is expensive.

Table 8.2. Corn consumption, date 0 ($\beta = 1, k = 10, \tau = .01$)

α	$x_1(E0)\rho_1$	$x_2(E0)$	$x_3(E0)$	$x_4(E0)$
0.92	9.8	10.0	10.3	10.0
0.94	9.8	10.0	10.2	10.0
0.96	9.9	10.0	10.1	10.0
0.98	9.9	10.0	10.1	10.0
1.00	10.0	10.0	10.0	10.0
1.02	10.1	10.0	9.9	10.0
1.04	10.1	10.0	9.9	10.0
1.06	10.2	10.0	9.8	10.0
1.08	10.3	10.0	9.8	10.0

Table 8.3. Corn consumption, date 1 ($\beta = 1, k = 10, \tau = .01$)

α	$x_1(E1c)$	$x_2(E1c)$	$x_3(W1c)$	$X_4(W1c)$
0.92	18.0	18.5	22.5	21.9
0.94	18.5	18.9	21.8	21.3
0.96	19.0	19.2	21.2	20.9
0.98	19.5	19.6	20.6	20.4
1.00	20.0	20.0	20.0	20.0
1.02	20.5	20.4	19.5	19.6
1.04	21.1	20.8	19.0	19.3
1.06	21.7	21.2	18.6	18.9
1.08	22.3	21.7	18.1	18.6

Table 8.4. Corn consumption, date 2 ($\beta = 1, k = 10, \tau = .01$)

α	$x_1(E2c)$	$x_2(W2c)$	$x_3(W2c)$	$x_4(E2c)$
0.92	18.0	21.9	22.5	18.5
0.94	18.5	21.3	21.8	18.9
0.96	19.0	20.9	21.1	19.2
0.98	19.5	20.4	20.5	19.6
1.00	20.0	20.0	20.0	20.0
1.02	20.5	19.6	19.5	20.4
1.04	21.1	19.3	19.0	20.8
1.06	21.7	18.9	18.6	21.3
1.08	22.3	18.6	18.1	21.7

Reverse migration seems striking — and unexpected, although well-documented in the empirical literature — especially because it might seem to suggest that resources are wasted and that social welfare could be improved by eliminating reverse migration and making appropriate transfers. Of course, no resources are in fact wasted; as usual, equilibrium is Pareto optimal.

This example echoes a point made in the Introduction. As Starrett's theorem tells us, perfect competition and perfect homogeneity must lead to a homogenous outcome. As this example demonstrates, however, perfect competition and just a little heterogeneity (in this case, a relatively small preference for one location over the other) can lead to a very heterogeneous outcome indeed.

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9

The Organization of Production, Consumption and Learning

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Summary. This paper provides an extension of general equilibrium theory that incorporates the actions of individuals both as demanders and suppliers of goods and as members of firms, schools, social groups, and contractual relationships. The central notion of the paper is a *group*: a collection of individuals associated with one another for some purpose. The model takes as primitive an exogenous set of group types, interpretable as (potential) firms, schools, social groups, contracts etc. The types of schools and firms that materialize in equilibrium, as well as the way that agents acquire skills, are determined endogenously in a competitive market, as are the contracts they enter into, and the production and consumption of private commodities. Equilibrium exists and the core coincides with the set of equilibrium states. Examples and Applications illustrate the flexibility and power of the framework.[4]

Key words: Clubs, Team production, Organization of firms.

JEL Classification Numbers: D2, D51, D71.

9.1 Introduction

In the usual general equilibrium model, individuals interact with the market but not with each other. In this paper we develop a model in which individuals interact both

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with the market and with each other — in firms, schools, social groups, and contractual relationships.

The central notion of this paper is a *group*: a collection of individuals who interact with one another for some purpose. (We could also call a group a *relationship* or an *organization*.) A group is determined by the characteristics of its members, by the inputs it uses and the outputs it produces, by the services it provides to its members, its infrastructure, and its governance or organizational structure. Some groups are productive (firms), some are educational (schools), some are social, some are contractual, and some have all of these aspects. Individuals typically belong to many groups.

In our model, there are two broad classes of commodity: the standard commodities of general equilibrium theory (private goods for short) and memberships in groups. Both private goods and group memberships enter into preferences, are objects of choice, and are priced. Private goods (but not group memberships) may be used as inputs or produced as outputs by a group. Within a group, memberships are distinguished by their *membership characteristic*. The prices of memberships within a group may be interpreted as the sharing of costs and revenues and as transfers among members. At equilibrium, the markets for private goods clear, taking into account the inputs and outputs of the groups that form, and membership choices are consistent across the population.

With standard assumptions (including a continuum of agents), equilibrium exists and passes a test of perfect competition (coincidence of core states and equilibrium states). The patterns of consumption and groups that emerge — the firms, schools, organizations and social and contractual relationships that form — are determined endogenously in a competitive market.

The model presented here modifies the clubs model presented in Ellickson, Grodal, Scotchmer and Zame [6] in two ways:

- We allow groups to produce outputs of standard commodities.
- Characteristics are associated with memberships, not individuals, and we allow for joint restrictions on choices of private goods and group memberships.

The first modification allows us to model the production of physical outputs. Groups can also produce services, but a service is not a standard commodity, and hence not an output of the group. The second modification allows much more flexibility in how we interpret memberships and groups. Characteristics can describe both roles within groups and attributes of individuals required to fill those roles. These attributes can be either acquired or innate.

The changes to the formal model required to accommodate these extensions are quite modest. The first change requires only that we replace the input vector of our earlier model with an input-output vector. The second change requires only that we drop the assumption in our earlier model that agents must have the same characteristic in every group to which they belong, instead allowing agents to choose different membership characteristics in different groups.

Although the modifications to the formal structure are modest, they greatly extend the scope of our framework. Groups and memberships can be interpreted in many ways. When the group is a "firm" it is natural to interpret members as "workers" or "supervisors," according to their roles, and to interpret (negative) prices for memberships as "wages." When the group is a "school" it is natural to interpret members as "teachers" or "students" and to interpret (negative) prices for teacher memberships as "salaries" and (positive) prices for student memberships as "tuition." When the group is a service, it is natural to interpret members as either "providers" or "clients" and to interpret prices as "fees", which are negative for providers and positive for clients. In all these situations, membership prices will reflect the market values of the inputs used and outputs produced, but membership prices will also reflect externalities within the group. As we stress below, firms that offer disagreeable working conditions or uncongenial co-workers will be forced to pay higher wages in order to attract employees. The examples and applications we present in Section 9.5 illustrate these and other instances of our model.

Membership characteristics embody both roles in groups and the skills necessary to qualify for (fulfill) those roles. Some skills are innate, and others must be learned – in private study, in schools, or in apprenticeships. We model learning as the ways agents qualify for memberships. For example, for skills that are not innate:

- Agents can qualify for a membership by consuming certain standard goods such as a home computer or a textbook.
- Agents can qualify for a membership by choosing other memberships such as attending school or serving an apprenticeship.

These possibilities are built into consumption sets. Consumption sets will typically differ across agents to reflect different capabilities to qualify for memberships, and for some agents some memberships may be simply beyond their reach. In the capitalist example of Section 6.3, for instance, the capitalist member must bring capital to the group.

There is no standard model of learning in general equilibrium theory, so we cannot compare our model of learning to a standard model. However there are standard models of production, and we should say something about what sets our treatment apart.

- In McKenzie's [14] formulation, there are no firms — only an aggregate constant-returns-to-scale aggregate production technology — and no profits. In the Arrow-Debreu [1] formulation, there are firms and profits, but the firms have no members — only shareholders.
- In both the McKenzie and Arrow-Debreu formulations, labor is a commodity just like every other commodity, and is priced just like every other commodity. In our model, labor is not a commodity. Workers are members of a firm, and their membership (which is a commodity) is priced in the same way as any other membership with compensation varying according to the role the member plays and the skills he brings to the job.
- In the standard model, there is no sense in which workers belong to a firm or care about working conditions. In our model, members of a group might care about every aspect of the group environment.

As in our paper [6], the economies considered here have a continuum of agents. This framework handles smoothly the difficulties that arise because club memberships are indivisible and because membership choices must be consistent across the population. For instance, if a type of firm requires two programmers and a lawyer, then there must be twice as many agents who choose to be programmers in that type of firm as agents who choose to be lawyers in that type of firm. The consistency problem must be solved in a context where agents can belong to many groups, as may be necessary for acquiring skills in some groups and using those skills in other groups.

Our main formal results are that equilibrium exists, that equilibrium states belong to the core, and that core states can be decentralized by prices (core equivalence). Because the present model is closely related to our earlier model, the proofs of these results require only small changes from the proofs of the corresponding results in [6], and so are omitted. (For economies with a finite number of agents, the techniques of Ellickson, Grodal, Scotchmer and Zame [7] could be adapted to prove that approximate equilibria exist, that approximate equilibrium states belong to the core, and that approximate core states can be approximately decentralized by prices.) We view the framework — rather than the theorems or the proofs — as the most important contribution of the present paper.

The theory we develop has some features in common with the competitive theory of labor management set forth by Keiding [11] and Drèze [5], but there are many important differences. As do we, Keiding [11] works in a continuum model and accommodates differentiated labor skills. However, in Keiding's model, labor skills are given exogenously and fixed (rather than acquired), there are no firms (agents have access to a single production technology), and agents care only about their own consumption of private goods and the fraction of their labor endowment delivered to the firm. As a result, there is no matching problem. Private goods are traded in competitive markets and labor is supplied cooperatively, but in equilibrium, it is "as if" there are competitive prices for the non-marketed labor services. In Drèze [5], labor skills are differentiated, and each group of agents has access to an exogenously given production set, but agents do not care about the composition of the groups in which they work, and again there is no matching problem.

Our model also has something in common with cooperative theories of coalition production, but again there are many important differences. Böhm [3], for instance specifies a production possibility set for every coalition of agents but focuses on the core, where the coalition of the whole forms and only the production possibility set of the coalition of the whole is used. In this sense, at least, there are no firms and no matching problem. Our model is closer in spirit to Ichiishi [10], in the sense that he focuses on the core of a coalition structure, so individuals really do "belong" to firms, and treats labor as supplied through membership in a firm (rather than as a standard commodity as Böhm [3] does). However, Ichiishi's emphasis is on a cooperative solution, which reflects his purpose, rather than on a competitive price-taking solution, as in our model. Equally importantly, Ichiishi's model reflects a view of the world in which each agent chooses to belong to a single productive coalition, whereas our model reflects a view of the world in which each agent may choose to belong to many productive groups (indeed, it may be necessary to belong to one group in order to acquire the skills necessary to belong to another group). In Ichiishi's model, therefore, it is appropriate to consider firm structures that are partitions of the set of agents, while in our model it is appropriate to consider firm structures that are *not* partitions of the set of agents.

Several other papers bear a closer relationship to the present paper. Makowski [12] interprets clubs as organizations formed by entrepreneurs. Cole and Prescott [4] provides an integration of club theory and general equilibrium theory, which accomplishes some of the same things as [6], but their approach is to view the objects of choice as divisible lotteries over club memberships and consumption bundles, rather than as indivisible club memberships and divisible consumption bundles. Closest in spirit is Prescott and Townsend [16], which views clubs as productive units and focuses on moral hazard.

In Section 2 we preview our model with a whimsical example. The model is presented in Section 3, and Section 4 presents the main Theorems. Section 5 presents examples that show some of the power of our model. The first example, which formalizes the whimsical example of Section 2, shows how learning can be modeled through apprenticeship, and how groups can be interpreted as firms producing services. The second example, which concerns the transition in the industrial revolution from home production to factory production during the industrial revolution, shows how agents' preferences over working conditions influence the nature of equilibrium. The third example, which builds on the second, shows how our model can be used to articulate the organization of the firm. The fourth example shows how contractual issues can be represented in our framework and how competition influences the contracts chosen in equilibrium.

9.2 A Venetian Holiday

A whimsical example may help the reader understand the formal model to come. Consider trips on a Venetian gondola. Each trip requires the services of two gondoliers: one in the front, and one in the back. Each trip can accommodate two passengers: one in the front and one in the back. (For simplicity we assume that trips actually require two passengers, although it is certainly possible to imagine trips with a single passenger, or even none.) The trip may promise silence or it may promise singing by the rear gondolier. Trips may take place in the morning or in the afternoon.

To code gondola rides as groups in our framework, we must specify inputs and outputs, and characteristics of the memberships of each group. In the present context, the input is the use of a gondola, and there is no output (because we code the gondola ride as part of the description of the group). We distinguish 4 group types: the first consists of a front and a rear gondolier and a front and a rear passenger, with a specification that the trip will take place in the morning and promise silence; the second consists of a front and a rear gondolier and a front and a rear passenger, with a specification that the trip will take place in the morning and promise singing; the third and fourth substitute "afternoon" for "morning." (Note that we follow the usual general equilibrium practice of incorporating time by dating the commodities — or services in this case.) In the first and the third group type there are the same family of membership characteristics: front passenger, rear passenger, front gondolier, and rear gondolier. In the second and fourth group type the membership characteristics are front passenger, rear passenger, front gondolier, and rear singing gondolier

Hence we in total have 5 membership characteristics. Using obvious notation, we write gm, gsm, ga, gsa for the 4 group types (writing gsa to represent a gondola ride, with singing, in the afternoon, and so forth) and we distinguish 16 (kinds of) memberships in the 4 group types:

$$(FG, gm), (RG, gm), (FP, gm), (RP, gm)$$
$$(FG, gsm), (RGS, gsm), (FP, gsm), (RP, gsm)$$
$$(FG, ga), (RG, ga), (FP, ga), (RP, ga)$$
$$(FG, gsa), (RGS, gsa), (FP, gsa), (RP, gsa)$$

Note that these 16 kinds of memberships are distinct: front and rear gondoliers have different responsibilities, front and rear passengers have different views of the scenery, gondoliers and/or passengers may prefer silence or singing, afternoons are different from mornings. In our framework, memberships are objects of choice and are priced, and these 16 memberships might all have different prices.

To this point, we have said nothing about the feasibility of choices for various individuals. It is probably true that no special ability is required of passengers — apart from their physical presence in Venice — but some special ability is surely required of gondoliers. (It might even be that different abilities are required of front and rear gondoliers, a possibility we ignore here.) We might therefore imagine that society consists of two sub-populations: Venetians, who are born knowing how to operate a gondola (or have acquired that skill before our story opens), and Tourists, who do not have and cannot acquire that skill. Our formalization of this distinction is that consumption sets of Tourists allow the choice of passenger memberships but not of gondolier memberships, while consumption sets of Venetians allow the choice of gondolier memberships but not of passenger memberships.[5]

Of course, some additional special ability is also required to sing. Perhaps a class of Venetians is born with this ability (or have acquired that skill before our story opens), another class is not born with it but can acquire it by serving as the front gondolier on a singing trip, and yet a third class is not born with it and cannot acquire it. Our formalization is again in terms of consumption sets. For members of the first class, consumption sets allow either gondolier choice. For members of the third class, consumption sets do not allow choice of either (RGS, gsm) or (RGS, gsa): a non-singing Venetian cannot make a choice that requires singing. For members of the second class, the consumption set precludes the choice (RGS, gsm) and only allows the choice (RGS, gsa) in conjunction with the choice (FG, gsm): a Venetian who is not born with the ability to sing cannot make a choice that requires singing in the

[5] For simplicity, we assume that Venetians cannot choose to be passengers rather than gondoliers.

morning, and can only make a choice that requires singing in the afternoon if s/he acquires, in the morning, the ability to sing.

In our equilibrium notion, the 16 kinds of memberships in gondola rides are priced, and agents optimize given these prices The equilibrium conditions require that prices within each type of group must sum to the cost of inputs, and that membership choices are consistent across the population. (In particular, equal numbers of Tourists (respectively, Venetians) choose front and rear memberships in the morning, and so forth.) It is natural to guess that, at equilibrium, passengers pay positive prices, gondoliers pay negative prices (that is, gondoliers are paid by the passengers), and these prices generate a net surplus that exactly covers the cost of the gondola. We verify these guesses in Section 9.5.1 below.

9.3 General Equilibrium with Groups

We first extend our club model in [6] so that it applies to many different organizations: firms, contracts, social clubs, schools, etc. In order to make the present paper as self-contained as possible, we repeat some definitions and some motivation, highlighting the differences in square brackets. Instead of the terms *club types* and *clubs* used in the previous paper, we use here the terms *group types* and *groups*.

9.3.1 Private Goods

There are $N \geq 1$ divisible, publicly traded private goods. Although we allow all kinds of private goods, we typically have physical goods in mind, rather than labor or other services, because we typically model these in the description of group types.

9.3.2 Groups and Memberships

Groups are described by an exogenous set of *group types*. To define group types, we begin with finite sets Ω of *membership characteristics* and Γ of *organizational characteristics*. A *group type* is a triple (π, γ, y) consisting of a *profile* $\pi : \Omega \to \mathbf{Z}_+ = \{0, 1, \ldots\}$, an organizational characteristic $\gamma \in \Gamma$, and an *input-output vector* $y \in \mathbf{R}^N$. We take as given a finite set of possible group types $\mathcal{G} = \{(\pi, \gamma, y)\}$.

As usual, we interpret negative components of y as inputs and positive components of y as outputs. We allow for the possibility that $y < 0$ (so that group formation requires inputs but produces no outputs, which would typically be the case for a group whose purpose is to provide a service) and for the possibility that $y > 0$ (so that production requires only the efforts of the members). [In our earlier paper we insisted that $y \leq 0$ so that club formation might require inputs but yielded no outputs.] For $\omega \in \Omega$, $\pi(\omega)$ represents the number of members with the membership characteristic ω that the group is required to have, and so $|\pi| = \sum_{\omega \in \Omega} \pi(\omega)$ is the total number of members that the group is required to have.

A membership characteristic might be anything that matters to the individuals who comprise a group or to the activity in which the group is engaged. In particular, a membership characteristic can encompass personal qualities (intelligence, appearance, personality, etc.), roles within the group (teacher, student, supervisor, skilled laborer, unskilled laborer, etc.) and skills (singing, dancing, language, etc.). Formally, all membership characteristics are acquired; membership characteristics that are innate (height for instance) are encompassed in our specification of consumption sets below. [In our earlier paper, we insisted that membership characteristics be innate and fixed.] Importantly, membership characteristics are observable and contractible. We emphasize that Ω is simply an abstract finite set. In particular, Ω is not a vector space and need not have any linear structure.

An organizational characteristic might be anything that matters to the potential member of a group apart from the characteristics of the members in the group and the input-output vector. In particular, an organizational characteristic can encompass the activity within the group (that is, the process used to produce output), the organizational hierarchy within the group, and the duties of each potential member of the group.[6]

A *membership* is an opening in a particular group type corresponding to a particular membership characteristic. Formally, a membership is a pair $m = (\omega, (\pi, \gamma, y))$ such that $(\pi, \gamma, y) \in \mathcal{G}$ and $\pi(\omega) \geq 1$. We write $\mathcal{M}$ for the set of memberships; because the set $\mathcal{G}$ of group types is finite so is the set $\mathcal{M}$ of memberships.

Just as an agent chooses a bundle of private goods (possibly none), so an agent chooses a bundle of memberships (possibly none); to distinguish memberships from private goods we refer to a bundle of memberships as a *list*. Formally, a membership list is a function $\ell : \mathcal{M} \to \{0, 1, \ldots\}$; $\ell(m)$ specifies the number of memberships of type $m = (\omega, (\pi, \gamma, y))$. Write **Lists** for the set of lists.

9.3.3 Agents

The set of agents is a nonatomic finite measure space $(A, \mathcal{F}, \lambda)$. That is, A is a set, $\mathcal{F}$ is a σ-algebra of subsets of A, and λ is a non-atomic measure on $\mathcal{F}$ with $\lambda(A) < \infty$.

A complete description of an agent $a \in A$ consists of a consumption set, an endowment vector of private goods and a utility function. Agent a's *consumption set* X_a specifies the feasible pairs of bundles of private goods and lists of memberships that the agent may choose. We assume that X_a has the following properties:

- $X_a \subset \mathbf{R}^N_+ \times$ **Lists**
- if $(x_a, \mu_a) \in X_a$ and $x'_a \geq x_a$ then $(x'_a, \mu_a) \in X_a$
- there exists $M > 0$ such that for all $a \in A$ and all $(x_a, \mu_a) \in X_a$

$$\sum_{m \in \mathcal{M}} \mu_a(m) \leq M$$

[6] We could probably dispense with organizational characteristics, coding everything into membership characteristics. However, distinguishing the characteristics of the organization from the characteristics of its members seems quite natural in applications, so we have chosen to build the distinction into the theory.

The first two requirements are familiar: private good consumption must be non-negative, and increased consumption of private goods is always possible. [In our earlier paper, we insisted that non-negativity be the only constraint on consumption of private goods.] The last assumption provides a bound on the number of memberships that each agent can choose. In particular, for each agent $a \in A$ the set

$$\mathbf{Lists}_a = \{\mu_a \in \mathbf{Lists} : \text{ there exists } x_a \in \mathbf{R}_+^N \text{ s.t. } (x_a, \mu_a) \in X_a\}$$

is finite.

A choice (x_a, μ_a) is in X_a if it is possible for agent a to consume the bundle of private goods x_a and fulfill the requirements of the memberships specified by μ_a. The consumption set X_a encodes restrictions on the choices that a can make with respect to both private goods and memberships. For instance, we encode the information that a is short (and that this condition is immutable) by insisting that if $(x_a, \mu_a) \in X_a$ then $\mu_a(\omega, g) = 0$ whenever the characteristic ω includes being tall. We encode the information that b cannot read (but that this condition is remediable) by insisting that if $(x_b, \mu_b) \in X_b$ and $\mu_b(\omega, g) > 0$ for some characteristic ω that includes being able to read then $\mu_b(\omega', g') > 0$ for some membership (ω', g') that teaches b to read. And we encode the information that c must own a violin to be concertmaster of an orchestra by insisting that if $(x_c, \mu_c) \in X_c$ and $\mu_c(\omega, g) > 0$ for a concertmaster membership then x_c includes at least 1 violin. (It is for this reason that we do not assume that $X_a = \mathbf{R}_+^N \times \mathbf{Lists}_a$.)

Much of the flexibility of our model arises from the fact that an agent can choose (memberships with) different characteristics in different groups. In particular, we allow an agent to fill different roles in different groups — to provide barber services in one group and receive them in another. The possibility of filling different roles in different groups is essential to the way we model the acquisition of skills, which will typically be acquired in one venue and applied in another. Of course, the acquisition of skills usually precedes their application; we incorporate the temporal element by viewing the date or time period as part of the description of a group, just as traditional general equilibrium theory frequently views the date or time period as part of the description of a commodity.

Agent a's *endowment* is $(e_a, 0) \in X_a$. Note that agents are endowed with private goods but not with group memberships, and that survival without group memberships is possible.

Agent a's *utility function* $u_a : X_a \to \mathbf{R}$ is defined over private goods consumptions and lists of group membership. We assume throughout that, for each $\mu_a \in \mathbf{Lists}$

$$u_a(\cdot, \mu_a) : \{x_a | (x_a, \mu_a) \in X_a\} \to \mathbf{R}$$

is continuous and strictly monotone; i.e., utility is strictly increasing in consumption of private goods. We make no assumptions about the way in which utility depends on the choice of group memberships.

9.3.4 Economies

An *economy* $\mathcal{E}$ is a mapping $a \mapsto (X_a, e_a, u_a)$ for which:

- the consumption set correspondence $a \mapsto X_a$ is a measurable correspondence
- the endowment mapping $a \mapsto e_a$ is an integrable function
- the utility mapping $(a, x, \ell) \mapsto u_a(x, \ell)$ is a jointly measurable function of its arguments

For convenience, we will sometimes make the simplifying assumption that the *aggregate endowment* $\bar{e} = \int_A e_a \, d\lambda(a)$ is strictly positive, so all private goods are represented in the aggregate. (Admittedly, this is not a very satisfactory assumption in a production economy.)

9.3.5 States

A *state* of an economy is a measurable mapping

$$(x, \mu) : A \to \mathbf{R}^N \times \mathbf{R}^{\mathcal{M}}$$

A state specifies choices of private goods and of group memberships for each agent. Feasibility of a state of the economy is defined as feasibility of the consumption and production plans and consistent matching of agents.

Our example of singing gondoliers provides a convenient framework for understanding consistent matching of agents. A gondola ride requires a front and rear gondolier and a front and rear passenger. Consistent matching means that there are no trips with empty seats. If agents can only choose one gondola trip, then consistent matching could be expressed by the requirement that the space of agents who choose some gondola trip can be partitioned into a disjoint family of four-member sets, each containing one front gondolier, one rear gondolier, one front passenger and one rear passenger. In our example, however, some agents will choose more than one trip (because they learn to sing in the morning), and a simple description in terms of partitions will not work. Instead, we can express consistent matching by the requirement that the "number" of agents who choose to be a front gondolier on some trip equal the "number" of agents who choose to be a rear gondolier on some trip, and so forth.[7] In a continuum framework, it is not meaningful to speak of "numbers" when the sets in question are infinite, but we can express the same idea by requiring that the fraction of the entire population who choose to be a front gondolier on some trip equal the fraction of the entire population who choose to be a rear gondolier on some trip, and so forth.

It is convenient to formalize this idea in terms of the aggregate of choices. To this end, define an *aggregate membership vector* to be an element $\bar{\mu} \in \mathbf{R}^{\mathcal{M}}$. An aggregate membership vector $\bar{\mu}$ is *consistent* if for every group type $(\pi, \gamma, y) \in \mathcal{G}$, there is a real number $\alpha(\pi, \gamma, y)$ such that

$$\bar{\mu}(\omega, (\pi, \gamma, y)) = \alpha(\pi, \gamma, y)\pi(\omega)$$

[7] In a finite economy, this is not quite enough, because we must be careful that no agent chooses to be both a front gondolier and a rear gondolier on the same trip, but in the continuum framework this problem does not arise; see the discussion in Ellickson, Grodal, Scotchmer and Zame [6].

for each $\omega \in \Omega$. Given a measurable set $B \subset A$ and a measurable choice function $\mu : B \to$ **Lists**, we say that μ is *consistent for* B if the aggregate membership vector $\int_B \mu_a \, d\lambda(a)$ is consistent.

The state (x, μ) is *feasible for* the measurable subset $B \subset A$ if

(i) **Individual feasibility**

$$(x_a, \mu_a) \in X_a \ \text{ for each } a \in B$$

(ii) **Material balance**

$$\int_B x_a \, d\lambda(a) \leq \int_B e_a \, d\lambda(a) + \int_B \left[\sum_{(\omega,(\pi,\gamma,y))} \mu_a(\omega, (\pi, \gamma, y)) \frac{y}{|\pi|} \right] d\lambda(a)$$

(iii) **Consistency** $\int_B \mu_a \, d\lambda(a)$ is consistent for B.

That is, (x, μ) is feasible for B if individuals choose in their consumption sets, private consumption does not exceed the sum of endowments and net production, and agents are matched consistently. The state (x, μ) is *feasible* if it is feasible for the set A itself. If (x, μ) is a state of the economy, $m = (\omega, g)$ is a membership and $\mu_a(m) > 0$, we say *a chooses* the membership m.

If (x, μ) is consistent for B and $\int_B \mu_a \, d\lambda(a) = \alpha(\pi, \gamma, y)\pi(\omega)$ for each $\omega \in \Omega$, then material balance is equivalent to the assertion that

$$\int_B x_a \, d\lambda(a) \leq \int_B e_a \, d\lambda(a) + \sum_{(\pi,\gamma,y)\in\mathcal{G}} \alpha(\pi, \gamma, y) \, y$$

Because members of a group care only about the membership characteristics of other members, and not about their identities, it is not necessary to identify the agents belonging to each individual group.

9.3.6 Group Equilibrium

Both private goods and group memberships are priced, so prices (p, q) lie in $\mathbf{R}^N \times \mathbf{R}^{\mathcal{M}}$; p is the vector of prices for private goods and q is the vector of prices for group memberships. Because utility functions are assumed monotone in private goods, private goods prices will be positive in equilibrium, but prices of group memberships may be positive, negative or zero. Because a membership specifies both a group type and a membership characteristic, membership prices depend both on the group type and the membership characteristic.

A *group equilibrium* consists of prices $(p, q) \in \mathbf{R}^N_+ \times \mathbf{R}^{\mathcal{M}}$ with $p \neq 0$ and a feasible state (x, μ) such that

(1) **Budget balance for group types** For each $(\pi, \gamma, y) \in \mathcal{G}$:

$$\sum_{\omega\in\Omega} \pi(\omega) \ \ q(\omega, (\pi, \gamma, y)) + p \cdot y = 0$$

(2) **Budget feasibility for agents** For almost all $a \in A$:

$$(p, q) \cdot (x_a, \mu_a) \le p \cdot e_a$$

(3) **Optimization by agents** For almost all $a \in A$:

$$(x'_a, \mu'_a) \in X_a \text{ and } u_a(x'_a, \mu'_a) > u_a(x_a, \mu_a) \Rightarrow (p, q) \cdot (x'_a, \mu'_a) > p \cdot e_a$$

Thus, at an equilibrium the sum of membership prices in a given group type just balances the value of the input-output vector, and individuals optimize subject to their budget constraints. (Recall that feasibility of the state (x, μ) already entails material balance.) A *group quasi-equilibrium* satisfies (1), (2) and the weaker condition

(3′)**Quasi-Optimization** For almost all $a \in A$:

$$(x'_a, \mu'_a) \in X_a \text{ and } u_a(x'_a, \mu'_a) > u_a(x_a, \mu_a) \Rightarrow (p, q) \cdot (x'_a, \mu'_a) \ge p \cdot e_a$$

As usual, the difference between equilibrium and a quasi-equilibrium is that at the latter agents do not necessarily optimize in their budget sets but only choose consumption bundles that are not dominated by any consumption bundle that costs strictly less than their wealth. Evidently every equilibrium is a quasi-equilibrium; we give conditions in Section 4 that guarantee that every quasi-equilibrium is an equilibrium.

9.3.7 Pricing Relevant Characteristics

The description of a membership in a group is very detailed: it includes the characteristics of the given membership and of the other memberships in the group, the purpose and organization of the group, and the input-output vector. Because we allow individuals to care about all these aspects, we must allow prices to depend on all these aspects as well; if we did not, we could easily find examples in which some core states could not be decentralized by prices and indeed in which equilibrium did not exist. However, in certain circumstances, it may happen that some aspects of membership are irrelevant to agents in the economy, and in those circumstance we can conclude that prices do not distinguish between such aspects. Indeed, if, in equilibrium, there is a set of positive measure of agents, each of whom chooses (in equilibrium) the membership m and finds the membership m' to be a perfect substitute for m, then m' must be at least as expensive as m: $q(m') \ge q(m)$. And if there is also a set of positive measure of agents, each of whom chooses the membership m' and finds the membership m to be a perfect substitute for m', then m, m' must have the same price: $q(m') = q(m)$.

The pricing of irrelevant aspects highlights one of the distinctions between the present framework, in which memberships chosen by a particular individual may display different characteristics in different groups, and the framework in [6], in which individuals bring the same characteristics to each group to which they belong. As a consequence, in our earlier framework irrelevant aspects of an individual may seem relevant to membership prices, when in fact they are not. Our present framework facilitates a much more direct connection between membership prices and the attributes of memberships that matter.

To illustrate, consider writing a paper with two coauthors, one Danish and the other English, and imagine the only thing that matters is that both authors speak the same language. In the present framework, we need only consider two types of group, d (Danish-speaking) and e (English-speaking), and two types of membership, D (Danish-speaking) and E (English speaking). Assuming coauthorship requires no inputs, we would, with the obvious notation, identify the group types as $((D, D), d, 0)$, $((E, E), e, 0)$ and the memberships as $(D, ((D, D), d, 0))$, $(E, ((E, E), e, 0))$. In the framework of [7], however, we must formally distinguish individuals who speak both Danish and English from individuals who speak only one language or the other, and must then consider six kinds of partnership and eight kinds of membership; with the obvious notation the partnerships would be

$$((D, D), d, 0) \qquad ((E, E), e, 0) \qquad ((DE, D), d, 0)$$

$$((DE, E), e, 0)) \qquad ((DE, DE), d, 0)) \qquad ((DE, DE), e, 0)$$

and the memberships would be

$$(D, ((D, D), d, 0)) \qquad (E, ((E, E), e, 0)) \qquad (DE, ((DE, D), d, 0))$$

$$(D, ((DE, D), d, 0)) \qquad (DE, ((DE, E), e, 0))) \qquad (E, ((DE, E), e, 0))$$

$$(DE, ((DE, DE), d, 0)) \qquad (DE, ((DE, DE), e, 0))$$

Of course, if there are eight kinds of memberships there must formally be eight membership prices. However, if no one cares whether their partner can speak two languages, many of these memberships will be perfect substitutes and the corresponding equilibrium membership prices will coincide: $q(DE, ((DE, D), d, 0)) = q(D, ((D, D), d, 0))$ (assuming both partnerships are chosen in equilibrium), etc.

9.4 Theorems

We say the feasible state (x, μ) is *Pareto optimal* if there is no feasible state (x', μ') such that $u_a(x'_a, \mu'_a) > u_a(x_a, \mu_a)$ for almost every $a \in A$; we say (x, μ) is *strongly Pareto optimal* if there is no feasible state (x', μ') such that $u_a(x'_a, \mu'_a) \geq u_a(x_a, \mu_a)$ for almost every $a \in A$ and $u_b(x'_b, \mu'_b) > u_b(x_b, \mu_b)$ for all b in some subset $B \subset A$ of positive measure. Similarly, we say the feasible state (x, μ) is in the *core* if there is no subset $B \subset A$ of positive measure and state (x', μ') that is feasible for B such that $u_b(x'_b, \mu'_b) > u_b(x_b, \mu_b)$ for almost every $b \in B$; we say (x, μ) is in the *strong core* if there is no subset $B \subset A$ of positive measure and state (x', μ') that is feasible for B such that $u_b(x'_b, \mu'_b) \geq u_b(x_b, \mu_b)$ for almost every $b \in B$ and $u_c(x'_c, \mu'_c) > u_c(x_c, \mu_c)$ for every c in some subset $C \subset B$ of positive measure.

Of course the Pareto set contains the strong Pareto set and the core contains the strong core. For pure exchange economies in which consumption sets are the positive orthant and preferences are strictly monotone, the Pareto set and strong Pareto set coincide and the core and strong core coincide. In our context, however, the strong Pareto set may be a proper subset of the Pareto set and the strong core may be a proper subset of the core.[8] However a natural assumption provides a simple way around this problem.

Say that *endowments are desirable* if for every agent a and every consumption choice $(x_a, \mu_a) \in X_a$ for which $u_a(x_a, \mu_a) > u_a(e_a, 0)$, there exists $x'_a \leq x_a, x'_a \neq x_a$ such that $(x'_a, \mu_a) \in X_a$.[9]

Proposition 1. *If endowments are desirable then the strong Pareto set coincides with the Pareto set and the strong core coincides with the core.*

The First Welfare Theorem follows by the usual straightforward argument (but the Second Welfare Theorem may fail; see [6]).

Theorem 1. *Every group equilibrium state belongs to the core and in particular is Pareto optimal. If endowments are desirable then every group equilibrium state belongs to the strong core and in particular is strongly Pareto optimal.*

As in the exchange case, a quasi-equilibrium (x, μ) need not be an equilibrium if a positive measure set of agents B are in the minimum expenditure situation. (That is, for agents $b \in B$ there is no bundle of private commodities x'_b such that $x'_b \leq x_b, x'_b \neq x_b$ and $(x'_b, \mu_b) \in X_b$.) In the exchange case, irreducibility rules out this possibility; a similarly-motivated condition will rule it out in our setting also.

Let $\mathcal{E}$ be a group economy and let (x, μ) be a feasible state. Let $I \subset \{1, \cdots, N\}$ be a non-empty set of private goods. Say that the feasible state (x, μ) is a *minimum consumption configuration for good* i if for almost all agents $a \in A$ there does not exist a bundle x'_a of private goods such that $x'_a \leq x_a, x'_{ai} < x_{ai}$ and $(x'_a, \mu_a) \in X_a$. (If $(0, \mu_a) \in X_a$ then a feasible state is a minimum consumption configuration for good i only if the entire social endowment of i is used in group formation.) Say that (x, μ) is *group linked* if for every partition $\{1, \cdots, N\} = I \cup J$ of the set of consumption goods for which (x, μ) is a minimum expenditure configuration for

[8] In the exchange case, the arguments for equality of the Pareto set and strong Pareto set, and for equality of the core and strong core, are familiar. The essential point is that if we are given allocations x, x' such that x is weakly preferred to x by some set of agents and strictly preferred by some subset of these agents, then we can tax the latter agents and redistribute the proceeds, obtaining an allocation x'' that is strictly preferred to x by all agents in the set. In our context, however, given states $(x, \mu), (x', \mu')$ such that (x', μ') is weakly preferred to (x, μ) by some set of agents and strictly preferred by some subset of those agents, we may find that in the state (x', μ') the latter group of agents consume no private goods — or consume a bundle that is minimal in their consumption set, given the group membership choices — and hence cannot be taxed.

[9] The reader familiar with [6] will note that we have adapted this definition to allow general consumption sets.

each good $i \in I$, then for almost every $a \in A$ there is a real number $r \in \mathbf{R}$ and an index $j \in J$ such that

$$u_a(e_a + r\delta_j, 0) > u_a(x_a, \mu_a)$$

(As usual, we write δ_j for the consumption bundle consisting of one unit of the private good j and nothing else.) We say that $\mathcal{E}$ is *group irreducible* if every feasible allocation is group linked.[10]

Proposition 2. *If $\mathcal{E}$ is group irreducible then every quasi-equilibrium is an equilibrium.*

In our continuum framework, equilibrium exists and passes a familiar test of perfect competition: coincidence of the core with the set of equilibrium states.

Theorem 2. *If $\mathcal{E}$ is group irreducible and endowments are desirable and uniformly bounded above then it admits a group equilibrium.*

Theorem 3. *If $\mathcal{E}$ is group irreducible and endowments are desirable and uniformly bounded above then the core coincides with the set of group equilibrium states.*

The (omitted) proofs of these results follow closely the proofs of the corresponding results in [6]; the only changes necessary are the very minor ones necessary to incorporate the small differences in formal structures.

- In our earlier work we allow for inputs to group formation; here we allow for inputs and outputs. This difference requires only that we extend our accounting to keep track of inputs and outputs.
- In our earlier work we insist that consumption sets be of the form $X_a = \mathbf{R}^{\mathbf{L}} \times \mathbf{Lists_a}$; here we allow for general consumption sets. This difference requires only that we be more careful about the distinction between quasi-equilibrium and equilibrium.
- In our earlier work we insist that agents choose only memberships corresponding to a particular (given and immutable) external characteristic; here the characteristics are attached to the memberships instead of to the agents, and the agents are allowed to assume different characteristics in different groups. However, aside from allowing more general consumption sets, this difference requires no changes in the argument.

9.5 Examples and Applications

In this Section, we give a series of examples to illustrate some of the flexibility of our model: groups can be interpreted as apprenticeships, as firms producing personal services that are not traded on the market, as firms producing physical goods that are traded on the market, or as relationships governed by contracts — and some groups

[10] The reader familiar with [6] will note that we have adapted the definitions of club linked and club irreducible to take into account that we allow general consumption sets.

possess several of these aspects. The flexibility of our model relies heavily on the possibility that an agent can belong to several groups.

Example 9.5.1 elaborates our whimsical Venetian holiday (Section 2). The example illustrates that some skills are innate but that others can be acquired, and that skills can be acquired in one group (an apprenticeship, in this case) and applied in another group. Of course, the acquisition of skills typically precedes the application of those skills — a feature that is modeled by dating the groups.

In Example 9.5.1, the groups produce services which are not traded. In Examples 9.5.2 and 9.5.3, the groups produce physical goods which are traded. These examples illustrate how our model of production differs from the standard general equilibrium model (see our discussion in the Introduction). In the setting considered here, which is motivated by Mokyr's [15] description of the historical shift of production from homes to factories during the industrial revolution 1760-1830, the difference is crucial because it enables us to incorporate the tension between the unpleasantness of working conditions and the productivity gains that characterized factory production. As Example 9.5.2 shows, the resolution of this tension depends on parameters of the economy: if factory production is sufficiently more efficient then it may drive out home production — but for an open set of parameters, factory production and home production can co-exist.

Example 9.5.3 provides another difference between our model and the standard model. In the standard model, capital is purchased on the market just as is any other input, so there is no role for "capitalists." Example 9.5.3 provides such a role by differentiating between a worker-managed firm, in which capital is provided equally by all the members, and a capitalist-managed firm, in which capital is provided only by the lone capitalist. In our example, there is again a tension between the unpleasantness of working for a capitalist and the productivity gains possible in that organizational form. As before, the resolution of this tension depends on parameters of the economy: if capitalist-managed firms are sufficiently more efficient they may drive out worker-managed firms — but for an open set of parameters, capitalist-managed firms and worker-managed firms can co-exist.

Finally, Example 9.5.4 shows how a group can be interpreted as a contract, and why that is useful. Contracts are typically cast as bargaining problems, with the influence of the market appearing only in reservation payoffs. As this example illustrates, however, our framework permits us to analyze contracting directly within a competitive environment. In particular, we show how the terms of contracts will be determined by competitive market forces, and contrast the competitive outcome with the outcomes possible when the terms of contracts are determined by bargaining among the parties.

9.5.1 Venetian Holiday

As a simple illustration of the way our model works — especially the intertemporal acquisition of skills — and of the computation of equilibrium, we flesh out our whimsical example of Venetian gondola rides; see Section 9.2 and Subsection 9.3.7.

We identify 5 membership characteristics: front and rear passengers, front and rear gondoliers, and rear gondoliers who sing: FP, RP, FG, RG, RGS.[11] Similarly, we identify 4 group types: non-singing gondola trips in the morning and afternoon, singing gondola trips in the morning and afternoon. For simplicity, we assume gondola rides require no inputs (ignoring the necessary gondola), so with the obvious abuse of notation we have:

$$\begin{aligned} gm &= ((FP, RP, FG, RG), m, 0) \\ ga &= ((FP, RP, FG, RG), a, 0) \\ gsm &= ((FP, RP, FG, RGS), m, 0) \\ gsa &= ((FP, RP, FG, RGS), a, 0) \end{aligned}$$

Note that morning and afternoon trips are distinguished only by the organizational characteristic m, a.

There are a continuum of agents, of three kinds: Tourists T, Venetians who can sing VS, and Venetians who cannot sing V; with population masses $\lambda(T), \lambda(VS)$, $\lambda(V)$ respectively. All agents are endowed with four units of the single consumption good.

Tourists can choose any non-negative quantity of the private good, and at most one gondola trip — as either a front or rear passenger — in the morning and at most one gondola trip in the afternoon. Write

$$\mathbf{Lists_T} = \Big\{\ell : \ell(FP, gm) + \ell(RP, gm) + \ell(FP, gsm) + \ell(RP, gsm) \leq 1, \\ \ell(FP, ga) + \ell(RP, ga) + \ell(FP, gsa) + \ell(RP, gsa) \leq 1\Big\}$$

so that $X_t = \mathbf{R}_+ \times \mathbf{Lists_T}$ for each $t \in T$. Tourists care about consumption during the day, and about gondola rides; a gondola ride (as front or rear passenger, in the morning or afternoon) without singing doubles utility, a gondola ride with singing quadruples utility, but additional rides are of no value. Hence for each $t \in T$ and $(x, \ell) \in X_t$:

$$u_t(x, \ell) = \begin{cases} x & \text{if} \quad \sum_{\omega,g} \ell(\omega, g) = 0 \\ 2x & \text{if} \quad \sum_{\omega} [\ell(\omega, gm) + \ell(\omega, ga)] \geq 1 \\ & \text{but} \ \sum_{\omega} [\ell(\omega, gsm) + \ell(\omega, gsa)] = 0 \\ 4x & \text{if} \quad \sum_{\omega} [\ell(\omega, gsm) + \ell(\omega, gsa)] \geq 1 \end{cases}$$

Venetians who can sing can choose any non-negative quantity of the private good, and can choose at most one gondola trip — as a front or rear gondolier, but not as a passenger[12] — in the morning and one in the afternoon. Write

[11] Although it seems natural to distinguish rear gondoliers who sing, it is not really necessary, since membership in the group type which promises singing would distinguish them equally well. This and other modeling choices are largely matters of convenience and taste.

[12] Alternatively, we could allow Venetians to choose to be passengers but to derive no utility from such a choice; this would complicate notation, but lead to the same equilibrium outcomes.

$$\textbf{Lists}_{\textbf{VS}} = \Big\{\ell : \ell(FG,gm) + \ell(RG,gm) + \ell(FG,gsm) + \ell(RGS,gsm) \leq 1,$$
$$\ell(FG,ga) + \ell(RG,ga) + \ell(FG,gsa) + \ell(RGS,gsa) \leq 1\Big\}$$

so that $X_v = \mathbf{R}_+ \times \textbf{Lists}_{\textbf{VS}}$ for each $v \in VS$.

Venetians who cannot sing can choose any non-negative quantity of the private good, and can choose at most one gondola trip — as a front or rear gondolier — in the morning and one in the afternoon, but cannot choose to be a singing gondolier in the morning, and cannot choose to be a singing gondolier in the afternoon unless they have chosen to be a front gondolier in a singing gondola trip in the morning. Write

$$\textbf{Lists}_{\textbf{V}} = \Big\{\ell : \ell(FG,gm) + \ell(RG,gm) + \ell(FG,gsm) + \ell(RGS,gsm) \leq 1,$$
$$\ell(FG,ga) + \ell(RG,ga) + \ell(FG,gsa) + \ell(RGS,gsa) \leq 1,$$
$$\ell(RGS,gsm) = 0,\ \ell(RGS,gsa) \leq \ell(FG,gsm)\Big\}$$

so that $X_v = \mathbf{R}_+ \times \textbf{Lists}_{\textbf{V}}$ for each $v \in V$.

Venetians care about consumption but suffer disutility from providing services:

$$\begin{aligned} u_v(x,\ell) = x &- A[\ell(FG,gm) + \ell(FG,gsm) + \ell(FG,ga) + \ell(FG,gsa)] \\ &- B[\ell(RG,gm) + \ell(RG,ga)] \\ &- C[\ell(RGS,gsm) + \ell(RGS,gsa)] \end{aligned}$$

We take $0 \leq A \leq B \leq C$: the front of the gondolier is a less difficult post than the rear, and singing is additionally onerous.

The equilibrium prices and choices will depend on the proportion of each subpopulation and on the disutility parameters A, B, C. We assume here that $\lambda(T) > 2\lambda(V \cup VS)$ (which guarantees that some tourists obtain no rides), that $\lambda(V) > \lambda(VS)$ (which guarantees that not all Venetians who cannot sing can learn how), and that $0 \leq A \leq 1$, $A \leq B \leq A+1$ and $B \leq C \leq B+2$ (which guarantees that Venetians have the proper incentives to provide singing and non-singing trips, in the morning and in the afternoon, and to learn to sing).

To solve for equilibrium, we rely on two observations: a) in the relevant range, this is a transferable utility economy, so the equilibrium state maximizes social welfare, and b) at equilibrium, agents who are *ex ante* identical must obtain the same utility.

Taking the consumption good as numeraire, with price 1, let q be the equilibrium membership price function. Budget balance for group types entails:

$$\begin{aligned} q(FG,gm) + q(RG,gm) + q(FP,gm) + q(RP,gm) &= 0 \\ q(FG,gsm) + q(RGS,gsm) + q(FP,gsm) + q(RP,gsm) &= 0 \\ q(FG,ga) + q(RG,ga) + q(FP,ga) + q(RP,ga) &= 0 \\ q(FG,gsa) + q(RGS,gsa) + q(FP,gsa) + q(RP,gsa) &= 0 \end{aligned}$$

Some Venetians who cannot sing will not be able to learn; these Venetians provide non-singing services in the afternoon and in the morning, in the front or in the rear; Venetians who cannot sing but do learn provide services in the front of a singing ride in the morning and in the rear of a singing ride in the afternoon. All these must obtain the same utility, so:

$$\begin{aligned}
&[-q(FG, gm) - A] + [-q(FG, ga) - A] \\
&= [-q(FG, gm) - A] + [-q(FG, gsa) - A] \\
&= [-q(FG, gm) - A] + [-q(RG, ga) - B] \\
&= [-q(RG, gm) - B] + [-q(FG, ga) - A] \\
&= [-q(RG, gm) - B] + [-q(FG, gsa) - A] \\
&= [-q(RG, gm) - B] + [-q(RG, ga) - B] \\
&= [-q(FG, gsm) - A] + [-q(RGS, gsa) - C]
\end{aligned}$$

Some tourists obtain no rides, consume their endowments and obtain utility 4, so all tourists obtain utility 4. Keeping in mind that tourists are indifferent between morning and afternoon rides and front and rear seating, it follows that

$$q(FP, gm) = q(RP, gm) = q(FP, ga) = q(RP, ga) = 2$$
$$q(FP, gsm) = q(RP, gsm) = q(FP, gsa) = q(RP, gsa) = 3$$

From the equations above, a little straightforward algebra yields the remaining membership prices (remember that negative prices are wages):

$$\begin{aligned}
q(FG, gm) &= -2 + \frac{1}{2}(B - A) \\
q(RG, gm) &= -2 - \frac{1}{2}(B - A) \\
q(FG, ga) &= -2 + \frac{1}{2}(B - A) \\
q(RG, ga) &= -2 - \frac{1}{2}(B - A) \\
q(FG, gsm) &= -(C - B) + \frac{1}{2}(B - A) \\
q(RGS, gsm) &= -6 + (C - B) - \frac{1}{2}(B - A) \\
q(FG, gsa) &= -2 + \frac{1}{2}(B - A) \\
q(RGS, gsa) &= -4 - \frac{1}{2}(B - A)
\end{aligned}$$

(Our assumptions about the disutility parameters guarantee that providing gondolier services is preferred to not working.)

Note that if $B > \frac{1}{3}(2C + A)$ then $q(FG, gsm) > 0$: learning to sing is so valuable that some Venetians pay for singing lessons when they join the crew in the morning. Because the equilibrium is Pareto optimal, this human capital is efficiently acquired.

9.5.2 The Factory System

In this application we address the rise of the factory system, using the stimulating discussion in Mokyr [15] as motivation. According to Mokyr, when the site of production shifted from homes to factories during the industrial revolution of 1760 –1830, the consequences for the worker were profound. Other things equal, workers preferred working at home, but factories offered productivity gains through on-site training and team production.

We assume a continuum of agents and 2 commodities. The set of membership characteristics is $\Omega = \{W, M\}$ where W represents a worker and M a manager. There are two types of productive enterprise:

- In *domestic production* a firm consists of 2 workers, each working at home, and a manager. The group type is $g_1 = (\pi_1, \gamma_1, y_1)$ where $\pi_1 = (2, 1)$ and $y_1 = (-3, 3)$.
- In *factory production* a firm consists of two workers, working in a centralized factory, and an on-site manager. The group type is $g_2 = (\pi_2, \gamma_2, y_2)$ where $\pi_2 = (2, 1)$ and $y_2 = (-6, \alpha)$.

The 2-worker factory requires twice the input of the 2-worker firm, 3 units of materials (as with domestic production) and 3 units for building and equipment. Each domestic firm produces 3 units of output; each factory produces output $\alpha > 0$. We will refer to the parameter α as factory productivity.

All agents are *ex ante* identical. Each has endowment $e = (k, 0)$. We refer to k as the (per capita) wealth of the economy. No agent can join more than one firm. Preferences are described by the utility function:[13]

$$u(x, \ell) = \begin{cases} 8\sqrt{x_1 x_2} & \text{if the agent chooses no memberships} \\ 6\sqrt{x_1 x_2} & \text{if he chooses a membership of type } (M, g_1) \\ 4\sqrt{x_1 x_2} & \text{if he chooses membership } (W, g_1) \text{ or } (M, g_2) \\ 2\sqrt{x_1 x_2} & \text{if he chooses membership } (W, g_2) \end{cases}$$

Equilibrium is described by prices p_1, p_2 for inputs and outputs and q for memberships, and choices for all agents; there is no loss in normalizing so that $p_1 = 1$. Since agents are *ex ante* identical, choices can be described by the fractions ρ_0 of agents choosing no membership and ρ_j of agents choosing a membership in a firm of type g_j ($j = 1, 2$). We defer the calculations, and first describe equilibrium.

The nature of equilibrium depends on the endowment parameter k and the productivity parameter α. Figure 9.1 which subdivides the parameter space

[13] Interpretation: workers do not like the regimen of factory life; managers have to appear on the factory floor to monitor, whereas with domestic production monitoring is not as demanding. This is simply an interpretation. One of the virtues of our approach is that what goes on inside a firm does not have to be modeled explicitly. Of course, it would be very interesting to connect our approach to the extensive literature on firm organization in which the monitoring and team production technology is made quite explicit. But that is for another time.

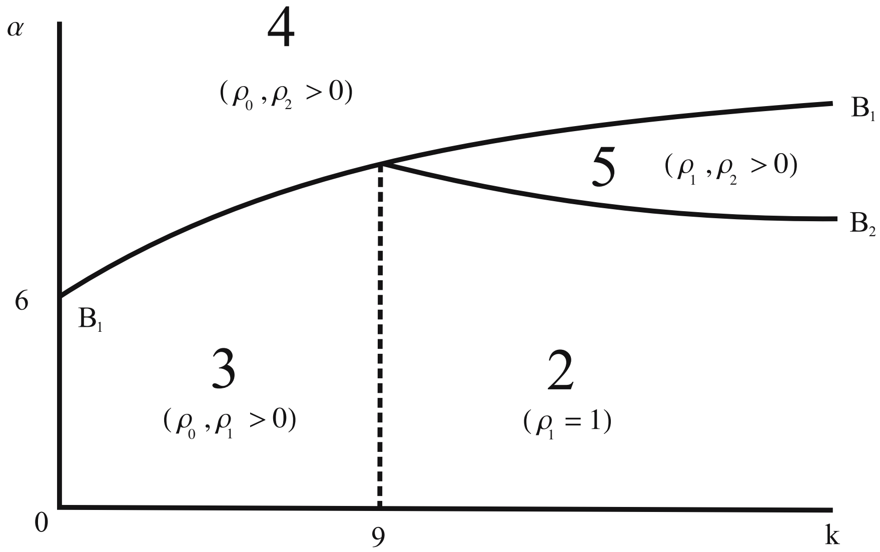

Fig. 9.1. The factory system

$$\{(k, \alpha) \in \mathbf{R}^2 \mid (k, \alpha) > 0\}$$

into a number of regions, captures the most important features of the equilibrium correspondence.

The curve labeled B_1 is the graph of the function

$$\alpha = B_1(k) := \frac{54 + 63k}{9 + 7k} \qquad k > 0 \tag{9.1}$$

Similarly, the curve labeled B_2, which intersects the graph of B_1 at the point $(k, \alpha) = (9, 69/8)$, is the graph of the function

$$\alpha = B_2(k) := \frac{33k - 21}{4k - 4} \qquad k \geq 9 \tag{9.2}$$

(Note that B_2 is defined only for $k \geq 9$.)

Below the curve $\min\{B_1, B_2\}$ (Regions 2 and 3), where factory productivity is low, there is no factory production: agents either engage in domestic production or choose not to work. Above the curve B_1 (Region 4), where factory productivity is high, there is no domestic production: agents either engage in factory production or choose not to work.

On the curve B_1 (Region 1) and, for wealth $k > 9$, in the area bounded by the curves B_1 and B_2 (Region 5) domestic production and factory production coexist. According to Mokyr, such a split between working in factories and working at home was characteristic of most industries in which factories appeared during the Industrial

Revolution. The most detailed evidence he presents comes from a 1906 census in France. Of workers working either in factories or at home, the per cent working in factories varied across industries as follows:[14]

- >90 %: Chemicals, glass and pottery, iron and steel, printing, rubber and paper.
- 80 % – 89 %: Food processing, fine metals and jewelry, metalwork, textiles.
- 60 % – 69 %: Stone-cutting, wood and carpentry.
- 50 % – 59 %: Leather, straw and baskets.
- 30 % – 39 %: Apparel making.

Apparently even as late as the beginning of the Twentieth Century, the tug of working in the home was still powerful enough to offset the advantages of working in a factory.

To calculate equilibrium, it is convenient to work with indirect utilities. Recalling that $p_1 = 1$, indirect utility takes the form

$$v(p,q,k,\omega) = \begin{cases} V(0) := 4k/\sqrt{p_2} & \text{if no memberships} \\ V(M,g_1) := 3(k - q(M,g_1))/\sqrt{p_2} & \text{if } \omega = (M,g_1) \\ V(W,g_1) := 2(k - q(W,g_1))/\sqrt{p_2} & \text{if } \omega = (W,g_1) \\ V(M,g_2) := 2(k - q(M,g_2))/\sqrt{p_2} & \text{if } \omega = (M,g_2) \\ V(W,g_2) := (k - q(W.g_2))/\sqrt{p_2} & \text{if } \omega = (W,g_2) \end{cases}$$

Recall that ρ_0 is the fraction of agents choosing no membership and ρ_j is the fraction choosing a membership in a firm of type g_j $(j = 1,2)$. Budget balance requires

$$2q(W,g_1) + q(M,g_1) + 3p_2 - 3 = 0 \tag{9.3}$$

and

$$2q(W,g_2) + q(M,g_2) + \alpha p_2 - 6 = 0 \tag{9.4}$$

for firms of type g_1 and g_2 respectively. Market clearing for private commodity 1 requires

$$\begin{aligned} k = \rho_0 \left[\frac{k}{2}\right] &+ \rho_1 \left[\frac{2}{3}\left(\frac{k - q(W,g_1)}{2}\right) + \frac{1}{3}\left(\frac{k - q(M,g_1)}{2}\right)\right] \\ &+ \rho_2 \left[\frac{2}{3}\left(\frac{k - q(W,g_2)}{2}\right) + \frac{1}{3}\left(\frac{k - q(M,g_2)}{2}\right)\right] \\ &+ \rho_1 + 2\rho_2 \end{aligned}$$

To the left of the equality sign is the per capita endowment of commodity 1. On the right of the equality sign, the first two lines represent per capita demand by consumers and the third line per capita demand for inputs by firms. Substituting $\rho_0 + \rho_1 + \rho_2 = 1$ and the budget-balance equations, this market-clearing equation simplifies to

[14] See Mokyr [15], p. 151, Table 2.

$$(p_2 + 1)\rho_1 + \left(\frac{\alpha p_2 + 6}{3}\right)\rho_2 = k \tag{9.5}$$

If $\rho_1 > 0$, then agents must be indifferent between working for or managing firms of type g_1. Setting $V(W, g_1) = V(M, g_1)$ implies

$$-2q(W, g_1) + 3q(M, g_1) = k \tag{9.6}$$

Similarly, if $\rho_2 > 0$, then equating $V(W, g_2) = V(M, g_2)$ yields

$$-q(W, g_2) + 2q(M, g_2) = k \tag{9.7}$$

Equations 9.3–9.7 provide the main ingredients for characterizing the equilibrium correspondence mapping the wealth-productivity parameters (k, α) to prices

$$(p_2, q(W, g_1), q(M, g_1), q(W, g_2), q(M, g_2))$$

and to the fraction ρ_0 of agents choosing not to work, the fraction ρ_1 engaging in domestic production as a worker or manager, and the fraction ρ_2 engaging in factory production as a worker or manager.

To derive the equilibrium correspondence, we begin with the case in which $\rho_0, \rho_1, \rho_2 > 0$: a positive fraction of agents belong to each type of firm and a positive fraction choose leisure. As we now show, this case corresponds to the curve B_1, the graph of equation 9.1, which we call Region 1. Agents are indifferent between leisure, working for or managing a firm of type g_1, and working for or managing a firm of type g_2. Solving

$$V(0) = V(W, g_1) = V(M, g_1) = V(W, g_2) = V(M, g_2)$$

yields equilibrium membership prices

$$q(W, g_1) = -k \quad q(M, g_1) = -\frac{k}{3} \quad q(W, g_2) = -3k \quad q(M, g_2) = -k$$

Equations 9.3 and 9.4 (budget balance for firms of type g_1 and g_2) imply

$$p_2 = \frac{9 + 7k}{9} = \frac{6 + 7k}{\alpha}$$

and hence

$$\alpha = B_1(k) := \frac{54 + 63k}{9 + 7k}$$

which is equation 9.1. From the market-clearing equation 9.5, the ρ_j must satisfy

$$(18 + 7k)\rho_1 + (36 + 21k)\rho_2 = 9k$$

as well as $\rho_0 + \rho_1 + \rho_2 = 1$. Solving these two equations for ρ_1 and ρ_2 as functions of ρ_0 yields

$$\rho_1 = \frac{12k + 36 - (21k + 36)\rho_0}{18 + 14k}$$
$$\rho_2 = \frac{2k - 18 + (7k + 18)\rho_0}{18 + 14k}$$

Imposing the restrictions $0 \leq \rho_1, \rho_2 \leq 1$ implies that

$$\rho_0 \in \left[\frac{18 - 2k}{18 + 7k}, \frac{12 + 4k}{12 + 7k}\right] \qquad \text{if} \quad k \leq 9$$

and

$$\rho_0 \in \left[0, \frac{12 + 4k}{12 + 7k}\right] \qquad \text{if} \quad k > 9$$

This characterizes the set of assignments (ρ_0, ρ_1, ρ_2) along the curve B_1.

Below $\min\{B_1, B_2\}$ no agents are engaged in factory production: either all are engaged in domestic production (Region 2) or they are split between engaging in domestic production and not working (Region 3). Suppose first that all agents are engaged in domestic production: $\rho_1 = 1; \rho_0 = \rho_2 = 0$. The market-clearing equation 9.5 implies $p_2 = k - 1$. Equations 9.3 and 9.6 imply

$$q(W, g_1) = \frac{9 - 5k}{4} \qquad q(M, g_1) = \frac{3 - k}{2}$$

The inequalities $V(W, g_1) \geq V(W, g_2)$ and $V(M, g_1) \geq V(M, g_2)$ imply

$$q(W, g_2) \geq \frac{9 - 7k}{2} \qquad q(M, g_2) \geq \frac{9 - 5k}{4}$$

Since $V(W, g_1) \geq V(0)$ implies $k \geq 9$, combining the inequalities above with equation 9.4 implies that

$$\alpha \leq B_2(k) := \frac{33k - 21}{4k - 4}, \qquad k \geq 9$$

which confirms that this case corresponds to Region 2.

Suppose instead that $\rho_2 = 0$ and $\rho_0, \rho_1 > 0$: some agents are engaged in domestic production and the rest choose leisure. The equalities $V(W, g_1) = V(M, g_1) = V(0)$ imply that

$$q(W, g_1) = -k \qquad q(M, g_1) = -\frac{k}{3}$$

while the inequalities $V(0) \geq V(W, g_2)$ and $V(0) \geq V(M, g_2)$ imply

$$q(W, g_2) \geq -3k \qquad q(M, g_2) \geq -k$$

Equation 9.3 implies

$$p_2 = \frac{9 + 7k}{9}$$

Combining equation 9.4 with the above inequalities, we conclude that

$$\alpha \leq B_1(k)$$

Since we must have $k \leq 9$ to rule out the preceding case, this case coincides with Region 3. Equation 9.5 yields the equilibrium fraction of agents belonging to firms of type 1,

$$\rho_1 = \frac{9k}{18 + 7k}$$

ρ_1 increases as agent wealth k increases, reaching 1 when $k = 9$.

Above the curve B_1, no agents are engaged in domestic production. We consider first the case in which a positive fraction work or manage factories and a positive fraction do not work: $\rho_1 = 0$ and $\rho_0, \rho_2 > 0$. The equalities $V(W, g_2) = V(M, g_2) = V(0)$ imply

$$q(W, g_2) = -3k \qquad q(M, g_2) = -k$$

The inequalities $V(W, g_2) \geq V(W, g_1)$ and $V(M, g_2) \geq V(M, g_1)$ imply

$$q(W, g_1) \geq -k \qquad q(M, g_1) \geq -\frac{k}{3}$$

Equation 9.4 implies

$$p_2 = \frac{6 + 7k}{\alpha}$$

When factory productivity α increases, per capita wealth k held fixed, the equilibrium price of the produced good falls. Equation 9.5 implies

$$\rho_2 = \frac{3k}{12 + 7k}$$

so that, as per capita wealth k increases, factory employment increases. From equation 9.3 and the above equalities and inequalities for prices, we conclude this case applies if

$$\alpha > B_2(k)$$

corresponding to Region 4 in Figure 9.1.

Because ρ_2 approaches 3/7 in the limit as $k \to \infty$, full employment in factory production cannot occur. It is also easy to verify this directly. If $\rho_2 = 1$ (and so $\rho_0 = \rho_1 = 0$), then equation 9.5 requires

$$p_2 = \frac{3k - 6}{\alpha}$$

Equations 9.4 and 9.7 imply

$$q(W, g_2) = \frac{24 - 7k}{5} \qquad q(M, g_2) = \frac{12 - k}{5}$$

But $V(W, g_2) = V(M, g_2) \geq V(0)$ implies $k \leq -3$, which is impossible.

The final possibility is that agents engage simultaneously in factory and domestic production.[15] We already know this occurs along the curve B_1, but along that line a positive fraction of agents also choose not to work. Suppose $\rho_0 = 0$ but $\rho_1, \rho_2 > 0$. Equating $V(W, g_1) = V(M, g_2)$ implies

$$q(W, g_1) = q(M, g_2)$$

Combining this equality with equations 9.3–9.7 yields

$$q(W, g_1) = \frac{(9-k)\alpha - 18k - 54}{8\alpha - 45} \qquad q(M, g_1) = \frac{(6+2k)\alpha - 27k - 36}{8\alpha - 45}$$
$$q(W, g_2) = \frac{(18-10k)\alpha + 9k - 108}{8\alpha - 45} \qquad q(M, g_2) = \frac{(9-k)\alpha - 18k - 54}{8\alpha - 45}$$

$$p_2 = \frac{3 + 21k}{8\alpha - 45}$$

and

$$\rho_1 = \frac{90 - 45k - 17\alpha + k\alpha}{48 + 21k - 9\alpha - 7k\alpha} \qquad \rho_2 = \frac{-42 + 66k + 8\alpha - 8k\alpha}{48 + 21k - 9\alpha - 7k\alpha}$$

The restriction $0 \leq \rho_1, \rho_2 \leq 1$ implies that $k \geq 9$ and $\alpha \in [B_1(k), B_2(k)]$, so this case corresponds to Region 5 of Figure 1.

To this point, we have assumed — as in Mokyr — that agents care about where they work. What difference does it make if we eliminate this assumption? Suppose preferences are described by the utility function

$$u(x, \ell) = \begin{cases} 8\sqrt{x_1 x_2} & \text{if the agent chooses no memberships} \\ 6\sqrt{x_1 x_2} & \text{if he chooses a membership of type } (M, g_1) \text{ or } (M, g_2) \\ 4\sqrt{x_1 x_2} & \text{if he chooses membership } (W, g_1) \text{ or } (W, g_2) \end{cases}$$

Utility depends on whether you are a manager or a worker, but not on whether the productive activity takes place at home or on the factory floor.

Notice that, in contrast to the preceding example, firms acquire no inputs from the market. Instead capital is provided by the workers (in firms of type g_1) or by an entrepreneur (in firms of type g_2), an obligation we build into consumption sets. An agent choosing a membership (W, g_1) is required to choose a nonnegative consumption vector (x_1, x_2) such that $x_1 \geq 1$ — but the first unit consumed of commodity 1 contributes nothing to utility. Similarly, an agent choosing a membership (M, g_2) must consume at least 3 units of commodity 1, but the first three units contribute nothing to utility.

Many qualitative features of the equilibrium correspondence are preserved. Proceeding just as before leads to the phase diagram illustrated in Figure 9.2. Once again

[15] There is one other logical possibility, $\rho_0 = 1$ and $\rho_1 = \rho_2 = 0$, but it is easy to show this can never occur.

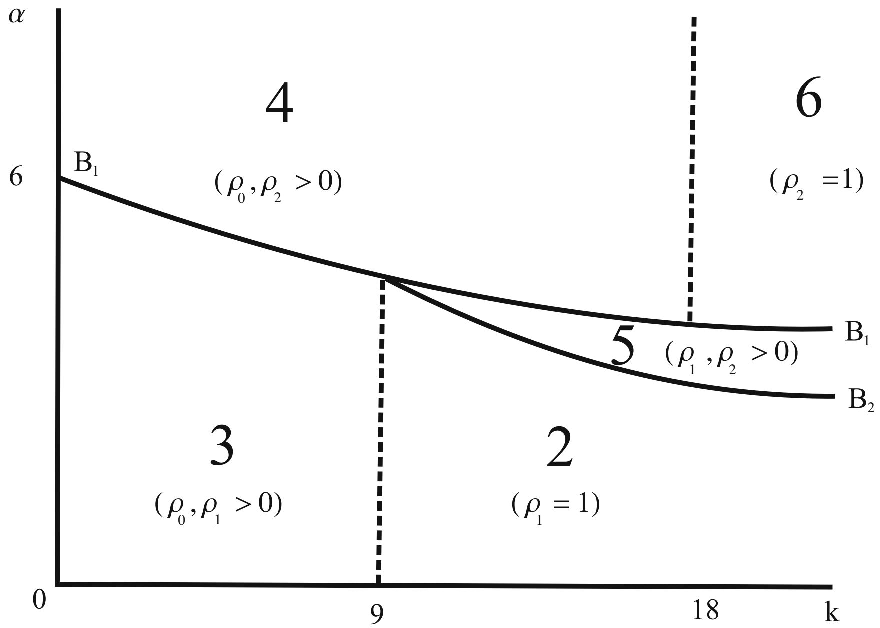

Fig. 9.2. No preference distinctions

the fundamental feature of the phase diagram is the boundary defined by the graphs of two functions,

$$\alpha = B_1(k) := \begin{cases} \frac{21k+54}{7k+9} & \text{if } 0 \leq k \leq 18 \\ \frac{3k-6}{k-3} & \text{if } k > 18 \end{cases}$$

and

$$\alpha = B_2(k) := \frac{3k}{k-1} \qquad k \geq 9$$

As before, there is no factory production below $\min\{B_1, B_2\}$ and no domestic production above B_1. The regions of this figure have the same interpretation as the corresponding regions of Figure 9.1, but there is now a Region 6 corresponding to full employment in factory work ($\rho_2 = 1$, $\rho_0 = \rho_1 = 0$). (The boundary between Regions 5 and 6 is the vertical dotted line at $k = 18$.) Details of the equilibrium correspondence are left to the interested reader.

Nevertheless, caring about where you work does make a qualitative impact on the nature of equilibrium. Consider equilibria in which domestic and factory production coexist (corresponding to wealth-productivity parameters (k, α) in region 5 or on B_1). As the reader can easily verify, under the initial specification of preferences — where agents care about their workplace — managers and workers are compensated for working in factories:

$$q(W, g_1) > q(W, g_2) \qquad \text{and} \quad q(M, g_1) > q(M, g_2)$$

In contrast, when agents do not care about their workplace, there are no compensating differentials:

$$q(W, g_1) = q(W, g_2) \qquad \text{and} \quad q(M, g_1) = q(M, g_2)$$

Caring about where you work has other consequences as well. In Figure 9.2 the curve B_1 slopes downward: the threshold level for shifting from domestic to factory production decreases as economies acquire additional wealth for building the factories. This reflects the effect of indivisibilities at the plant level. Each factory requires 6 units of commodity 1 as input; as k increases, this indivisibility matters less and less.

In Figure 9.1, on the other hand, B_1 has an upward slope. With greater wealth, economies require more productivity from factories to compensate for inferior working conditions, and this trumps the influence of indivisibility.

More subtly, in Figure 9.1 the separation between curves B_1 and B_2 remains in the limit as k approaches infinity,

$$\lim_{k \to \infty} B_1(k) = 9 > \frac{33}{4} = \lim_{k \to \infty} B_2(k)$$

but in Figure 9.2 these curves converge:

$$\lim_{k \to \infty} B_1(k) = \lim_{k \to \infty} B_2(k) = 3$$

In the latter case, the gap disappears as the difference in material input cost between factory and domestic production becomes negligible relative to per capita wealth. But in the former case, where agents care about where they work, not only the cost of material inputs but also the cost of workers and of managers differs between domestic firms and factories. In contrast to material inputs, the difference in the cost of human resources does not become negligible as wealth increases — wealthy agents demand better working conditions.

9.5.3 Capitalists

In the preceding application no one owns a firm. Capital, the material inputs a manager and his workers require if they are to form a going concern, is acquired from the "market," not from any specific agent. In this application we articulate a role for the "capitalist," an agent who supplies capital to a particular firm.

All agents are *ex ante* identical with endowment $e = (k, 0)$. As in application 5.2, we distinguish between workers and managers: $\Omega = \{W, M\}$. There are two types of firm. A firm of type g_1 is worker managed. There are 3 workers, each contributing 1 unit of capital (commodity 1) as well as his labor; the firm is managed cooperatively without a formal "manager." Formally:

$$g_1 = (\pi_1, \gamma_1, y_1) \qquad \pi_1 = (3, 0) \qquad y_1 = (0, 3)$$

Firms of type g_2 are owned and managed by an entrepreneur who supplies all 3 units of the capital and also serves as a manager of two workers who contribute nothing but their labor. Formally:

$$g_2 = (\pi_2, \gamma_2, y_2) \qquad \pi_2 = (2, 1) \qquad y_1 = (0, \alpha)$$

Preferences are described by the utility function:

$$u(x, \ell) = \begin{cases} 8\sqrt{x_1 x_2} & \text{if the agent chooses no membership} \\ 6\sqrt{(x_1 - 1)x_2} & \text{if the agent chooses } m = (W, g_1) \\ 6\sqrt{(x_1 - 3)x_2} & \text{if the agent chooses } m = (M, g_2) \\ 4\sqrt{x_1 x_2} & \text{if the agent chooses } m = (W, g_2) \end{cases}$$

Workers prefer the working conditions of a worker-managed firm, other things being equal, but self-monitoring and joint supply of capital may be less efficient than having a single owner who controls his workers. The parameter α measures the relative efficiency of the entrepreneurial firm.

Notice that, in contrast to the preceding example, firms acquire no inputs from the market. Instead capital is provided by the workers (in firms of type g_1) or by an entrepreneur (in firms of type g_2), an obligation we build into consumption sets. An agent choosing a membership (W, g_1) is required to choose a nonnegative consumption vector (x_1, x_2) such that $x_1 \geq 1$, and the first unit consumed of commodity 1 contributes nothing to utility. Similarly, an agent choosing a membership (M, g_2) must consume at least 3 units of commodity 1 and the first three units contribute nothing to utility.

As before, let ρ_0 denote the fraction of agents choosing no membership and ρ_j the fraction choosing a membership in a firm of type j ($j = 1, 2$), and normalize so that $p_1 = 1$. We omit the computation of equilibrium, which parallels that of Example 9.5.2. Figure 9.3, which subdivides the parameter space

$$\{(k, \alpha) \in \mathbf{R}^2 \mid (k, \alpha) > 0\}$$

into a number of regions, captures the most important features of the equilibrium correspondence. The curve labeled B_1 is the graph of the equation

$$\alpha = B_1(k) := \begin{cases} \frac{9+7k}{3+k} & \text{if } k \leq 27/2 \\ \frac{12k-24}{2k-5} & \text{if } k > 27/2 \end{cases} \tag{9.8}$$

The curve labeled B_2, which intersects the graph of B_1 at the point $(k, \alpha) = (9/2, 27/5)$, is the graph of the equation

$$\alpha = B_2(k) := \frac{5k - 9}{k - 2} \qquad k \geq 9/2 \tag{9.9}$$

(Note that B_2 is defined only for $k \geq 9/2$.)

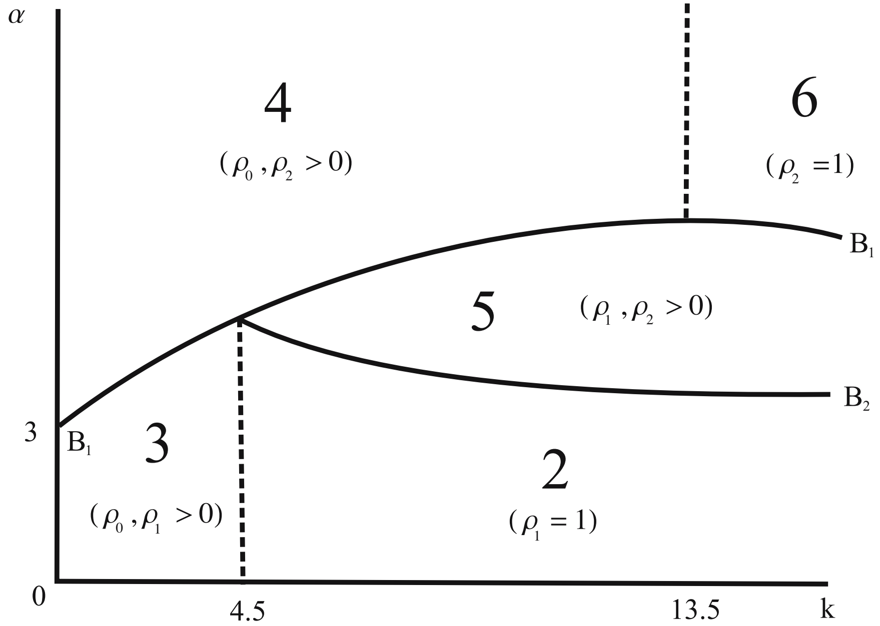

Fig. 9.3. Phase diagram: capitalists

Below the curve $\min\{B_1, B_2\}$ (Regions 2 and 3), where the productivity of entrepreneurial firms is low, there are no entrepreneurial firms: agents either belong to a worker-managed firms or choose not to work. Above the curve B_1 (Regions 4 and 6), where the productivity of entrepreneurial firms is high, there are no worker-managed firms: agents either are members of entrepreneurial firms or they choose not to work. On the curve B_1 and, for wealth $k > 9/2$, in the area bounded by the curves B_1 and B_2, worker-managed and entrepreneurial firms coexist. Region 5, along with the curve B_1, is where worker-managed firms and entrepreneurial firms coexist.

For equilibria in this region, it would, at least in principle, be possible to test for the presence of a trade-off between "economic democracy" and the efficiencies of a more hierarchically organized firm. We are unaware of any study of that sort comparable to Mokyr's comparison of domestic and factory production.

This application has ruled out by fiat the possibility of acquiring capital from the market. This seems reasonable since otherwise capitalists are providing capital without insisting on control. How then would we propose capturing the publicly-owned corporation? By recognizing that shareholders are also members of the group, supplying capital but delegating control to a board of directors. The advantage to such an investor is the opportunity to diversify risk by holding relatively small stakes in many different firms; the disadvantage is the introduction of an agency problem. We leave such an extension for another time.

9.5.4 Contracts

Here we give a simple example illustrating the interpretation of groups as contracts and the effect of competition on the choice of contracts.

We consider an economy in two dates $0, 1$. There is a single good (grain), which can be consumed at either date but can also be planted at date 0. Two methods of planting are possible:

a) Two agents can work side-by-side, planting 2 units of grain at date 0 and harvesting 2α units of grain. Because the agents work side-by-side, they *must* share the harvest equally, obtaining α units of grain each. We refer to this arrangement as *partnership*.
b) Two agents can work in sequence, the first planting 2 units of grain at date 0, the second harvesting β units. Because the agents work in sequence, the second agent cannot be prevented from eating the entire harvest. We refer to this arrangement as *ownership*, to the second agent as the *entrepreneur* and to the first agent as the *worker*.

We view each of these choices as indivisible and full-time, so each agent can choose to participate in only one (or neither of course).[16] We take output levels α, β as parameters.

Agents are *ex ante* identical. Agents are endowed with 4 units of grain at each date; their utility for consumption patterns over time is:

$$U(c_0, c_1) = \sqrt{c_0 c_1}$$

Aside from the contractual arrangements for planting grain, *no intertemporal contracts are enforceable*; in particular, there is no market at date 0 for consumption at date 1.

We model this story as an atemporal economy with one good, embedding consumption in the second date into utility functions. Formally, we distinguish three membership characteristics P, E, W (partner, entrepreneur, worker) and two group types

- partnership $\mathcal{P} = (\pi_{\mathcal{P}}, \gamma_{\mathcal{P}}, -2)$ where

$$\pi_{\mathcal{P}}(\beta) = \begin{cases} 2 \text{ if } \beta = P \\ 0 \text{ if } \beta = E \\ 0 \text{ if } \beta = W \end{cases}$$

- ownership $\mathcal{O} = (\pi_{\mathcal{O}}, \gamma_{\mathcal{O}}, -2)$ where

$$\pi_{\mathcal{O}}(\beta) = \begin{cases} 0 \text{ if } \beta = P \\ 1 \text{ if } \beta = E \\ 1 \text{ if } \beta = W \end{cases}$$

[16] We might allow for part-time participation, or for smaller-scale planting, but this would add substantial complication without adding much interest.

Consumption sets permit consumption of any non-negative quantity of the (date 0) private good and choice of at most one membership. In this formulation, utility for date 0 consumption and membership choice reflects the *actual* consumption at date 0 and *implicit consumption* at date 1:

$$\begin{aligned} u(c, 0) &= \sqrt{c \cdot 4} \\ u(c, (P, \mathcal{P})) &= \sqrt{c \cdot (4 + \alpha)} \\ u(c, (E, \mathcal{O})) &= \sqrt{c \cdot 4} \\ u(c, (W, \mathcal{O})) &= \sqrt{c \cdot (4 + \beta)} \end{aligned}$$

To solve for the equilibrium (which will depend on the productivity parameters α, β) we take the private good as numeraire. Budget balance entails that membership prices sums to the cost of the input, so equilibrium prices satisfy:

$$\begin{aligned} p &= 1 \\ q(P, \mathcal{P}) &= 1 \\ q(E, \mathcal{O}) + q(W, \mathcal{O}) &= 2 \end{aligned}$$

If membership prices are q, then the (indirect) utility of agents choosing various memberships is

$$\begin{aligned} \hat{u}(0) &= 4 \\ \hat{u}(P, \mathcal{P}) &= \sqrt{[4 - q(P, \mathcal{P})][4 + \alpha]} \\ \hat{u}(W, \mathcal{O}) &= \sqrt{[4 - q(W, \mathcal{O})][4]} \\ \hat{u}(E, \mathcal{O}) &= \sqrt{[4 - q(E, \mathcal{O})][4 + \beta]} \end{aligned}$$

(An agent choosing a partnership will pay the price $q(P, \mathcal{P})$, hence consume $4 - q(P, \mathcal{P})$ units of grain at date 0 and $4 + \alpha$ units of grain at date 1, and so forth.)

Keep in mind that some of these memberships may not be chosen at equilibrium. Indeed, the space of parameter values can be decomposed into three (open) regions and their boundaries; within each region, only a single kind of membership is chosen; see Figure 9.4.

I In this region, neither partnership nor ownership are chosen. In order that this be the case, the indirect utility $\hat{u}(0)$ must be at least as large as every other indirect utility. Simple algebra gives:

$$\begin{aligned} \hat{u}(0) \geq \hat{u}(P, \mathcal{P}) &\Leftrightarrow \alpha \leq \frac{4}{3} \\ \hat{u}(0) \geq \hat{u}(W, \mathcal{O}) &\Leftrightarrow q(W, \mathcal{O}) \geq 0 \\ \hat{u}(0) \geq \hat{u}(E, \mathcal{O}) &\Leftrightarrow q(E, \mathcal{O}) \geq \frac{4\beta}{4 + \beta} \end{aligned}$$

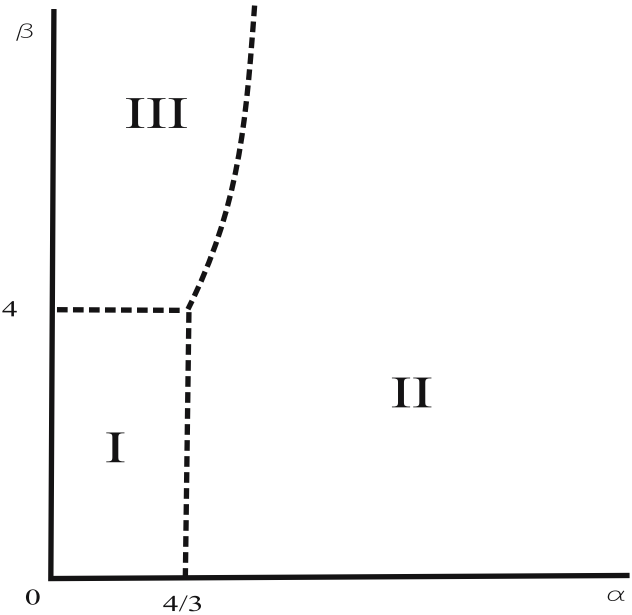

Fig. 9.4. Competitive contracting.

Keeping in mind that $q(W,\mathcal{O}) + q(E,\mathcal{O}) = 2$ and solving, we conclude that this region is the rectangle bounded by

$$0 \le \alpha \le \tfrac{4}{3}$$
$$0 \le \beta \le 4$$

II In this region only partnership is chosen. In order that this must be the case, the indirect utility $\hat{u}(P,\mathcal{P})$ must be at least as large as every other indirect utility. Simple algebra gives:

$$\hat{u}(0) \le \hat{u}(P,\mathcal{P}) \Leftrightarrow \alpha \ge \frac{4}{3}$$
$$\hat{u}(P,\mathcal{P}) \ge \hat{u}(W,\mathcal{O}) \Leftrightarrow q(W,\mathcal{O}) \ge 1 - \frac{3}{4}\alpha$$
$$\hat{u}(P,\mathcal{P}) \ge \hat{u}(E,\mathcal{O}) \Leftrightarrow q(E,\mathcal{O}) \ge \frac{\alpha + 4 + 4(\beta - \alpha)}{4+\beta}$$

Solving, we conclude that this region is infinite above and to the right:

$$0 \le \beta \le \frac{8\alpha}{4-\alpha} \quad \text{for} \quad \frac{4}{3} \le \alpha \le 4$$
$$\beta \text{ arbitrary} \quad \text{for} \quad 4 < \alpha$$

(If $\alpha > 4$ then there is no wage rate the owner can afford to pay from endowment that will make the worker prefer working to being in a partnership.)

III In this region, only ownership is chosen. This region is the complement of the union of regions I, II and we can describe it as:

$$0 \leq \alpha \leq 4$$
$$\beta \geq \min\left\{4, \tfrac{8\alpha}{4-\alpha}\right\}$$

In this region, half the agents choose to be entrepreneurs and half choose to be workers. Because *ex ante* identical agents obtain the same utility in equilibrium, the membership prices are:

$$q(W, \mathcal{O}) = -\frac{2\beta - 8}{8 + \beta}$$
$$q(E, \mathcal{O}) = +\frac{2\beta - 8}{8 + \beta} + 2$$

Remember that negative membership prices are wages: the entrepreneur pays the worker the wage $\frac{2\beta-8}{8+\beta}$ *and* bears the cost of planting.

It is instructive to contrast the competitive environment discussed above to an environment in which there are only two agents. If there are only two agents, there is no reason to view equilibrium as the appropriate solution notion. Rather, it seems that we should permit as a solution any efficient, individually rational configuration.

Given a specified transfer t from the entrepreneur to the worker, straightforward calculations (as above) let us compare ownership, partnership and working alone.

(i) the worker prefers ownership to working alone if $t > 0$
(ii) the entrepreneur prefers ownership to working alone if

$$t < \frac{2\beta - 8}{4 + \beta}$$

(iii) the worker prefers ownership to partnership if

$$t > \frac{3}{4}\alpha - 1$$

(iv) the entrepreneur prefers ownership to partnership if

$$t < \frac{2\beta - 3\alpha - 4}{4 + \beta}$$

From this, we can identify the regions in which various arrangements are individually rational and efficient:

IV Ownership is individually rational and efficient if there is a transfer t (from the entrepreneur to the worker) so that both the entrepreneur and the worker prefer the relationship to working alone and at least one of them prefers the relationship to partnership. Thus ownership is individually rational and efficient if either: there exists a transfer t for which (i), (ii) and (iii) are satisfied, OR there exists a transfer t for which (i), (ii) and (iv) are satisfied. After a little algebra, we if there is a transfer t for which (i), (ii) and (iv) are satisfied then there is a transfer for which (i), (ii) and (iii) are satisfied, and that ownership is efficient and individually rational if

$$\beta > 4 \text{ and } \beta > \frac{3}{2}\alpha + 2$$

V Partnership is individually rational if $3(4 + \alpha) > 16$; equivalently, if $\alpha > \frac{4}{3}$. Partnership is efficient if there does not exist a transfer for which *both* (iii) or (iv) are satisfied. After a little algebra, we see that partnership is individually rational and efficient if

$$\alpha > \frac{4}{3} \qquad \text{and } \beta < \frac{8\alpha}{4 - \alpha}$$

Note that this region coincides with Region **II** above.

VI Working alone is individually rational and efficient only in the complement of the union of regions **IV, V**.

Figure 9.5 provides a sketch of these regions. Note that regions **IV, V** overlap. In the intersection of these regions, *both* ownership and partnership are individually rational and efficient contractual relationships — and either might be chosen in a world with only two agents. However, as we have seen above, (except for parameters in a set of measure 0) only one contractual arrangement survives in a competitive market.

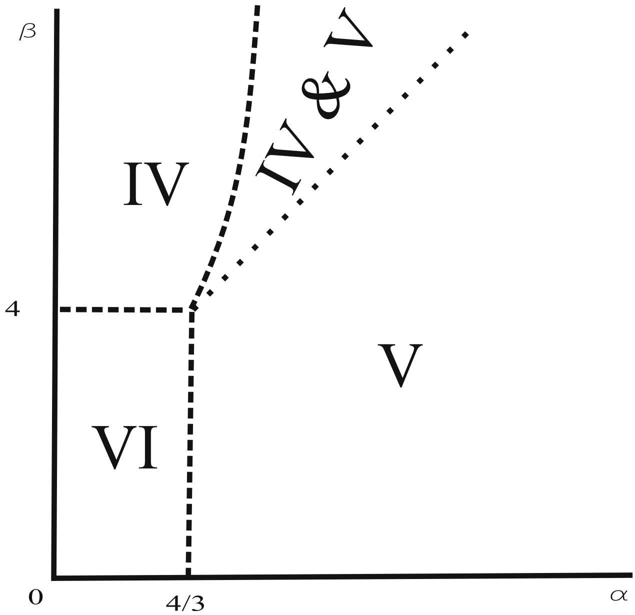

Fig. 9.5. Bilateral contracting.

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10

Household Inefficiency and Equilibrium Efficiency*

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Summary. Collective consumption decisions taken by the members of a household may prove inefficient. The impact on market performance depends on whether household inefficiencies are caused by inefficient net trades with the market or by inefficient distribution of resources within households. Inefficient net trades might be consistent with global efficiency. Inefficient internal distribution always results in inefficient equilibrium allocations. This leads us to consider competitive forces as disciplinary device for households. Competition of households for both resources and members can eliminate or reduce inefficient internal distribution.

Key words: General equilibrium, Household decisions, Household formation.

JEL Classification Numbers: D10, D51, D62.

10.1 Introduction

With few exceptions, both theoretical and empirical economics have treated households as if they were single consumers. Chiappori (1988, 1992), who is primilary interested in testable implications regarding household demand, presents a model of collective rationality of households as an alternative to the model where households are treated like single consumers. This raises the question of whether distinguishing between a household and its members makes any difference. Our main focus here and elsewhere is normative. It lies on the impact that the nature of collective household

* Sections 10.2 to 10.5 of this chapter are taken from Gersbach and Haller (2005). Section 6 contains further results obtained in Gersbach and Haller (2003). It addresses issues raised during seminar presentations at the University of Texas, Austin, and the Center for Economic Studies (CES), Munich. The hospitality and financial support of CES, the Institute for Advanced Studies (IHS), Vienna, and the Institute of Economics, University of Copenhagen, is gratefully acknowledged. We thank Clive Bell for helpful comments and two referees for thoughtful suggestions.

decisions has on market performance. The issue at hand is to what extent competitive exchange among multi-member households leads to a Pareto-optimal allocation, i.e. an efficient market outcome. The classical answer is in the affirmative: market outcomes are efficient. Obviously, this welfare conclusion persists if multi-member households are treated like single consumers. But what if they are not, if each household member has her own preferences and efficiency, both at the household and the economy level, is defined in terms of these individual preferences? According to Haller (2000), the answer still is in the affirmative as long as each household makes an optimal (efficient) choice subject to its budget constraint and, by doing so, exhausts its budget.

Here we start from the opposite assumption that collective household decision-making could be prone to severe frictions and, as a consequence, to inefficiencies. Then a new and perhaps more challenging question arises: How is market performance affected by inefficient household decisions? One intriguing possibility is that inefficiencies at the micro level neutralize each other so that the resulting market allocation is efficient.[3] The more likely scenario is that inefficiencies at the micro level cause global inefficiency. In the sequel, two specific types of inefficient household decisions will be isolated. The first type of household inefficiency results from an inefficient net trade with the market and does not rule out global efficiency. The second type of household inefficiency results from an inefficient distribution of the household's aggregate consumption to individual household members and always causes global inefficiency. While in general one would expect the two types of inefficiency to coexist, it turns out that considerable insight can already be gained from investigating each type in isolation. Both types of inefficiencies are considered in Section 10.4, after introducing a model with fixed household structure in Section 10.2 and restating the first welfare theorem in Section 10.3.

In Section 10.5 we address the main question: To what extent inefficient internal distribution — which always leads to an inefficient equilibrium allocation — will be eliminated if households compete for resources and members. The latter requires choice of household affiliation and, therefore, a variable household structure. We extend the model so that an allocation consists of an allocation of commodities plus a household structure, that is, a partition of the population into households. In the absence of externalities, the threat of leaving a multi-person household and forming a single-person household eliminates inefficient internal distribution in the prevailing households. In the presence of externalities this threat is not enough to prevent inefficient internal distribution. However, the threat to form a new household that is similar to the old one but makes better consumption decisions proves effective.

[3] Incidentally, a similar scenario is frequently invoked to counter the objection that individual consumers lack full rationality as postulated by neoclassical economic theory: Individual deviations from full rationality may offset each other and, thus, not affect aggregates.

In Section 10.6 we address the question of when inefficient net trades with the market are consistent with global efficiency.

10.2 Model of Competitive Exchange

To model competitive exchange among multi-member households, consider a pure exchange economy composed of finitely many households $h = 1, \ldots, H$. The commodity space is $\mathbb{R}^\ell$ with $\ell \geq 1$. Household h is endowed with a commodity bundle $\omega_h \in \mathbb{R}^\ell$, $\omega_h > 0$.

Each household h consists of finitely many members $i = hm$ with $m = 1, \ldots, m(h)$ and $m(h) \geq 1$. Put $I = \{hm : h = 1, \ldots, H; m = 1, \ldots, m(h)\}$. A generic individual $i = hm \in I$ has:

- consumption set $X_i = \mathbb{R}^\ell_+$;
- preferences $\succsim_i$ on the allocation space $\mathcal{X} \equiv \prod_{j \in I} X_j$ represented by a utility function $U_i : \mathcal{X} \longrightarrow \mathbb{R}$.

This general formulation allows for economy-wide externalities. The latter promises to be a fertile topic of research even in the traditional context of competitive exchange among individuals. But in accordance with the main focus of the current paper, we propose to restrict attention to externalities that are of particular interest for an inquiry into competitive exchange among households. In the sequel, condition (E1) will be imposed which requires that consumption externalities, if any, exist only between members of the same household. Some more notation is needed for an explicit formulation of such intra-household externalities.

Let $\mathbf{x} = (x_i)$, $\mathbf{y} = (y_i)$, $\mathbf{z} = (z_i)$ denote generic elements of $\mathcal{X}$. For $h = 1, \ldots, H$, define $\mathcal{X}_h = \prod_{n=1}^{m(h)} X_{hn}$ with generic elements $\mathbf{x_h} = (x_{h1}, \ldots, x_{hm(h)})$. If $\mathbf{x} \in \mathcal{X}$ is an allocation, then for $h = 1, \ldots, H$, household consumption is $\mathbf{x_h} = (x_{h1}, \ldots, x_{hm(h)}) \in \mathcal{X}_h$. Now we are ready to define the kind of intra-household externalities which will be assumed hereafter.

(E1) Intra-Household Externalities: $U_i(\mathbf{x}) = U_i(\mathbf{x_h})$
for $i = hm, \mathbf{x} \in \mathcal{X}$.

We shall also refer to the special case of no externalities, i.e.

(E2) Absence of Externalities: $U_i(\mathbf{x}) = u_i(x_i)$
for $i = hm, \mathbf{x} = (x_i) \in \mathcal{X}$.

The first theorem of welfare economics asserts that any competitive equilibrium allocation in the sense of Walras is Pareto-optimal. Here, like in Haller (2000), we want to allow for the possibility of a household composed of several members who arrive at a collective decision on household consumption. For the economy with social endowment $\omega = \sum_h \omega_h$ and consumers $i = hm$ $(h = 1, \ldots, H; m =$

$1, \ldots, m(h)$), an efficient or Pareto-optimal allocation is defined in the standard fashion based on individual preferences:

DEFINITION 1 *An allocation* $\mathbf{x} = (x_i) \in \mathcal{X}$ *is* **efficient or Pareto-optimal**, *if*

(i) $\sum_i x_i = \omega$, *i.e.* $\mathbf{x}$ *is feasible and*
(ii) *there does not exist a feasible allocation* $\mathbf{y} = (y_i) \in \mathcal{X}$ *with* $U_i(\mathbf{y}) \geq U_i(\mathbf{x})$ *for all* i *and* $U_i(\mathbf{y}) > U_i(\mathbf{x})$ *for some* i.

To complete the modeling of competitive exchange among households, one has to specify how households interact with the market. Haller (2000) assumes efficient bargaining within households. The latter means that a household h chooses an allocation at the Pareto frontier of its budget set, i.e. an element of its efficient budget set $EB_h(p)$ as defined below. In contrast, the present paper is aimed at investigating the impact of inefficient household decisions on market performance. This extended research agenda necessitates a more general definition of a competitive equilibrium among households than the one adopted in Haller (2000). To this end, consider a household h and a price system $p \in \mathbb{R}^\ell$. For $\mathbf{x_h} = (x_{h1}, \ldots, x_{hm(h)}) \in \mathcal{X}_h$, denote

$$p * \mathbf{x_h} = p \cdot \left(\sum_{m=1}^{m(h)} x_{hm} \right).$$

Then h's **budget set** is defined as $B_h(p) = \{\mathbf{x_h} \in \mathcal{X}_h \ : \ p * \mathbf{x_h} \leq p \cdot \omega_h\}$.

For future reference, we also define household h's **binding budget set** or **balanced budget set** as $BB_h(p) = \{\mathbf{x_h} \in \mathcal{X}_h \ : \ p * \mathbf{x_h} = p \cdot \omega_h\}$.

Demand correspondences describe the possible outcomes of collective household decision-making. A (possibly empty-valued) correspondence

$$D_h : \mathbb{R}^\ell \Longrightarrow \mathcal{X}_h$$

is called a **demand correspondence for household** h, if $D_h(p) \subseteq B_h(p)$ for all $p \in \mathbb{R}^\ell$. How households form their demands is a key component of the definition of a competitive equilibrium among households.

DEFINITION 2 *Given a profile* $D = (D_1, \ldots, D_H)$ *of demand correspondences for households, a* **competitive** D**-equilibrium** $(p; \mathbf{x})$ *is a price system* p *together with a feasible allocation* $\mathbf{x} = (x_i)$ *satisfying*

(iii) $\mathbf{x_h} \in D_h(p)$ *for* $h = 1, \ldots, H$.

Thus in a competitive equilibrium, each household makes a collective choice under its budget constraint and markets clear. At this general level, the concept of a competitive equilibrium among households is flexible enough to accommodate all conceivable collective decision criteria of households. Of course, additional restrictions on the profile D could and should be imposed whenever warranted by the objective of the research effort.

10.3 Efficient Household Decisions

Efficient choice by the household refers to the individual consumption and welfare of its members, not merely to the aggregate consumption bundle of the household. Such a notion of efficient household decision is captured by the concept of an efficient budget set.

Given a price system p, define consumer h's **efficient budget set** $EB_h(p)$ as the set of $\mathbf{x_h} \in B_h(p)$ with the property that there is no $\mathbf{y_h} \in B_h(p)$ such that

$U_{hm}(\mathbf{y_h}) \geq U_{hm}(\mathbf{x_h})$ for all $m = 1, \ldots, m(h)$;

$U_{hm}(\mathbf{y_h}) > U_{hm}(\mathbf{x_h})$ for some $m = 1, \ldots, m(h)$.

Classical versions of the first theorem of welfare economics are based on the crucial property that each consumer's demand lies on the consumer's budget line or hyperplane — which implies Walras' Law. This property follows from local non-satiation, for instance monotonicity of consumer preferences. With the possibility of multi-person households and intra-household externalities, the crucial property needs to be adapted. The modified property is called budget exhaustion and stipulates that each household's choice lies on the household's "budget line". For example, monotonicity in own consumption combined with non-negative externalities yields budget exhaustion. The formal definition is as follows.

(BE) Budget Exhaustion: For each household $h = 1, \ldots, H$,
and any price system $p \in \mathbb{R}^\ell$, $EB_h(p) \subseteq BB_h(p)$.

Notice that $EB \equiv (EB_1(\cdot), \ldots, EB_H(\cdot))$ is an example of a profile of demand correspondences for households. Therefore, a key result of Haller (2000) can be rephrased as follows.

Proposition 1 (First Welfare Theorem) *Suppose (E1) and (BE).*
If $(p; \mathbf{x})$ *is a competitive* EB*-equilibrium, then* $\mathbf{x}$ *is a Pareto-optimal allocation.*

In other words, efficiency at the household level implies efficiency at the economy level, if each household has to exhaust its budget in order to put into effect an efficient consumption decision for its members. For the existence of such an equilibrium see Gersbach and Haller (1999).

10.4 Inefficient Household Decisions

On purely analytical grounds, it is fruitful to treat the household decision as a two-step decision, although the household need not perceive it that way. In the first step, the household chooses an aggregate or total consumption bundle for the household subject to its budget constraint. In more technical terms, the household determines its net trade with the market. In a more graphic description, the household fixes the dimensions of an Edgeworth Box for the household. In the second step, the household distributes its total consumption bundle among its members. More graphically, the household picks a point (an allocation) within its previously chosen Edgeworth Box. To arrive at an efficient consumption decision under its budget constraint, the household has to first choose the right Edgeworth Box and then pick a point on the contract curve in that Edgeworth Box. Therefore, one can identify two sources of inefficiencies committed by the household:

a) inefficient net trade with the market;
b) inefficient internal distribution.

Of course, the two types of inefficient decision-making can be compounded. But it is analytically convenient to consider each of them separately. More importantly, this sort of piecemeal analysis renders interesting results already.

To formalize the two types of household inefficiency, it is convenient to introduce yet another distinguished subset of a household's budget set. For each household h and every price system p, we define the **potentially efficient budget set** $PEB_h(p)$ as the set of $\mathbf{x_h} = (x_{h1}, \ldots, x_{hm(h)}) \in B_h(p)$ for which there exists $\mathbf{y_h} = (y_{h1}, \ldots, y_{hm(h)}) \in EB_h(p)$ such that
$\sum_{m=1}^{m(h)} y_{hm} = \sum_{m=1}^{m(h)} x_{hm}$ and $U_{hm}(\mathbf{y_h}) \geq U_{hm}(\mathbf{x_h})$ for $m = 1, \ldots, m(h)$.
When choosing an element from its potentially efficient budget set, the household makes an efficient net trade, but may not achieve efficient internal distribution.

10.4.1 Inefficient Net Trades

Suppose that a household performs an inefficient net trade with the market which means that the household could improve (in a weak sense) the welfare of its members by making a different choice under its budget constraint, but in order to achieve that would have to change its net trade with the market. Formally, this means that the household chooses $\mathbf{x_h} \notin PEB_h(p)$. If the household wants to correct its mistake after market clearing, then the net trades of some other households would have to be altered as well, possibly to the detriment of the welfare of the other households' members. This line of argument suggests that inefficient net trades might lead to an efficient market allocation. In Section 10.6 we are going to investigate more systematically if and how inefficient net trades by one household can be compensated by inefficient net trades of other households so that an efficient equilibrium allocation results.

10.4.2 Inefficient Internal Distribution

Suppose that a household performs an efficient net trade with the market which means that the household can achieve an efficient choice under its budget constraint by suitably dividing its aggregate consumption bundle among its members. But the actually chosen internal distribution of commodities may be inefficient in the sense that redistribution within the household can improve the welfare of its members. Formally, this means $\mathbf{x_h} \in PEB_h(p) \backslash EB_h(p)$. If so, the mistake can be rectified simply by internal reallocation without affecting the welfare of members of other households. This leads to the conclusion that inefficient internal distribution, a particular type of inefficient household decision, always begets global inefficiency. Indeed, the following formal result holds true where $PEB \equiv (PEB_1(\cdot), \ldots, PEB_H(\cdot))$ denotes the profile of potentially efficient budget correspondences.

Proposition 2 (Anti-Welfare Theorem) *Suppose (E1). If $(p; \mathbf{x})$ is a competitive PEB-equilibrium and $\mathbf{x_h} \notin EB_h(p)$ for some household h, then the allocation $\mathbf{x}$ is not Pareto-optimal.*

PROOF. Assume (E1). Let $(p; \mathbf{x})$ be as hypothesized and h be a household with $\mathbf{x_h} \notin EB_h(p)$. Since $\mathbf{x_h} \in PEB_h(p)$, there exists $\mathbf{z_h} \in EB_h(p)$ with
$\sum_{m=1}^{m(h)} z_{hm} = \sum_{m=1}^{m(h)} x_{hm}$ and
$U_{hm}(\mathbf{z_h}) \geq U_{hm}(\mathbf{x_h})$ for all $m = 1, \ldots, m(h)$.
Since $\mathbf{z_h} \in EB_h(p)$, but $\mathbf{x_h} \notin EB_h(p)$, $U_{hm}(\mathbf{z_h}) > U_{hm}(\mathbf{x_h})$ has to hold for some $m = 1, \ldots, m(h)$. Now set $\mathbf{y_h} = \mathbf{z_h}$ and $\mathbf{y_k} = \mathbf{x_k}$ for households $k \neq h$. This defines a feasible allocation $\mathbf{y} = (y_i)_{i \in I}$. Because of (E1),
$U_i(\mathbf{y}) > U_i(\mathbf{x})$ for certain members i of household h and
$U_j(\mathbf{y}) = U_j(\mathbf{x})$ for all other consumers j.
Hence as asserted, $\mathbf{x}$ is not Pareto-optimal. Q.E.D.

10.5 When Outside Options Beget Efficiency

In Section 10.6 we find that inefficiency can beget efficiency, that inefficient individual or household consumption decisions can lead to Pareto-optimal equilibrium allocations. If an agent's mistake (inefficient net trade to be precise) is suitably compensated by the mistakes (inefficient net trades) of others, then the overall allocation can be efficient. In contrast, by the above Anti-Welfare Theorem, an equilibrium allocation will never be Pareto-optimal, if the sole source of an inefficient household decision is inefficient internal distribution. Elimination or reduction of inefficient internal distribution would improve welfare and obviously would be desirable.

Notice that inefficient internal distribution on the part of households constitutes the analogue of technological inefficiency in the production sector. It is a time honored theme in industrial economics that increased competition among producers reduces both allocative and technical inefficiencies.[4] Moreover, potential competition may suffice to further efficiency. To quote Schumpeter (1975, p. 85):

> *It is hardly necessary to point out that competition of the kind we now have in mind acts not only when in being but also when it is merely an ever-present threat. It disciplines before it attacks.*

In a similar vein, the concept of contestable markets forwarded by Baumol, Panzar and Willig (1986) postulates that potential hit-and-run competition has the same effect as actual competition.

In this section, we apply the idea that competitive forces can serve as a disciplinary device to the consumption sector. The hope is that competition will cause the elimination or reduction of inefficient internal distribution in households in a similar manner as it causes erosion of managerial slack in firms. Yet we know from the Anti-Welfare Theorem that competition for resources alone will be to no avail in this respect. But it turns out that if household stability is threatened by inefficient internal distribution, if in a sense households are competing for resources and members, then the households which exist in equilibrium must make efficient or not too inefficient decisions. This presumes that dissatisfied household members have the option to leave and that household stability (requiring that nobody wants to exercise the option) is an additional equilibrium condition. Accordingly, we are going to investigate whether and to what extent inefficient household decisions due to inefficient internal distribution are sustainable in equilibrium, if individuals have the option to form new and potentially more efficient households.

The outside options individuals have may vary: an individual may form a single-person household, join another household or form a new household with fractions of existing households. Individuals may leave a household because they dislike its composition. For this reason alone, certain households may not be viable if they are a total mismatch. However, even if a member is content with the household's composition, the member may be dissatisfied with the collective consumption decision and decide to leave. One reason could be that the endowment of the household is such that at the prevailing prices, the household can afford relatively little consumption compared to other households the individual might conceivably join. Another reason could be that the individual gets a bad deal because fellow members get a great deal at his expense. A final reason could be an inefficient consumption decision by the household.

The household cannot do anything about the first reason. It is bound to break up if it lacks sufficient resources to be attractive for its members. The household may be able to preempt the other two causes of a break-up by not exploiting some of its members for the benefit of others and by making efficient consumption decisions or at least not too inefficient ones. Then the question is how much inefficiency a household can afford without giving a member a reason to leave. Under certain circumstances the answer will be 'none': household stability requires efficiency. In

[4] Leibenstein's much acclaimed 1966 article has raised the awareness for technological inefficiencies or X-inefficiencies. Hart (1983) formalizes the idea that competition in the product market reduces managerial slack.

other words, the threat of desertion forces household efficiency. One can also ask how much exploitation a household can afford without giving a member a reason to leave. Under the very same circumstances the answer will be ‘none’ as well: household stability requires absence of exploitation.

So far the household structure has been fixed. The option to leave a household presupposes alternative households and a variable household structure. Imposing stability conditions familiar from the bilateral matching literature[5] makes the household structure endogenous. Consequently, we extend our analysis to a richer model with endogenous household structure. Inefficient consumption decisions in multi-member households may induce individuals to leave and form new households or join other households, if they have these options. We are going to show that competitive exchange across households combined with certain outside options may eliminate or mitigate inefficient internal distribution in the households prevailing in equilibrium.

10.5.1 Variable Household Structure

To elaborate on the theme of disciplinary capacity of competition, we consider a finite pure exchange economy with variable household structure. There exists a given finite and non-empty set of individuals or consumers, I. A (potential) household is any non-empty subset h of the population I. $\mathcal{H} = \{h \subseteq I | h \neq \emptyset\}$ denotes the set of all potential households. The households that actually form give rise to a **household structure** P, that is a partition of the population I into non-empty subsets. The commodity space, individual consumption sets, household consumption sets and commodity allocations are defined as before.

With a fixed household structure, household membership was part of an individual's identity. Individual $i = hm$ was the m's member of household h. With variable household structure, household membership is an endogenous outcome. An individual may care about household composition and household consumption. Different members may exert different consumption externalities upon others. We maintain the assumption of intra-household externalities. But instead of (E1) it now assumes the form

(HSP) Household-Specific Preferences: $U_i(\mathbf{x}; h) = U_i(\mathbf{x_h}; h)$
for $i \in h, h \in \mathcal{H}, \mathbf{x} \in \mathcal{X}$.

In the following, we are going to consider the special case of

(GSE) Group-Size Externalities: $U_i(\mathbf{x}; h) = V_i(x_i; |h|)$
for $i \in h, h \in \mathcal{H}, \mathbf{x} \in \mathcal{X}$.

In this case, individual i cares only about own consumption and household size. Still, preferences over own consumption may change with household size and, vice versa, preferences over household size can depend on own consumption. In the separable

[5] See the seminal contribution by Gale and Shapley (1962) and the monograph by Roth and Sotomayor (1990).

case, $U_i(\mathbf{x}; h) = u_i(x_i) + v_i(|h|)$ and preferences over own consumption and preferences over household size are independent. If $v_i \equiv 0$, then the separable case reduces to (E2), that is absence of externalities.

Every potential household h is endowed with a commodity bundle $\omega_h > 0$. In general, the aggregate or social endowment depends on the prevailing household structure P and equals $\omega_P = \sum_{h \in P} \omega_h$. The social endowment is independent of the household structure if (and only if) the following condition, called individual property rights, holds.

(IPR) Individual Property Rights: $\omega_h = \sum_{i \in h} \omega_i$ for all $h \in \mathcal{H}$.

IPR states that the endowment of each household equals the sum of the individual endowments of its members. IPR means that individuals can transfer their endowments at no cost when they consider leaving a household or forming new households. If the household endowment represents a rudimentary form of household production technology, then a violation of IPR can easily occur.

After having generalized preferences and endowments to allow for variable household structures, we can define budget sets, efficient budget sets, balanced budget sets, and demand correspondences for arbitrary households accordingly. Define an **allocation** of the economy with variable household structure as a pair $(\mathbf{x}; P)$ where $\mathbf{x} \in \mathcal{X}$ is an allocation of commodities and P is a household structure. The allocation is **feasible**, if $\sum_{i \in I} x_i = \omega_P$. Define a **state** of the economy as a triple $(p, \mathbf{x}; P)$ such that $p \in \mathbb{R}^\ell$ is a price system and $(\mathbf{x}; P) \in \mathcal{X} \times P$ is an allocation, i.e. $\mathbf{x} = (x_i)_{i \in I}$ is an allocation of commodities and P is an allocation of consumers (a household structure, a partition of the population into households). For a state $(p, \mathbf{x}; P)$ and an individual $i \in I$, let $P(i)$ denote the household in P (the element of P) to which i belongs. We say that in state $(p, \mathbf{x}; P)$,

(a) consumer i can benefit from exit, if $P(i) \neq \{i\}$ and there exists $y_i \in B_{\{i\}}(p)$ such that $U_i(y_i; \{i\}) > U_i(\mathbf{x_{P(i)}}; P(i))$;

(b) consumer i can benefit from joining another household g, if $g \in P$, $g \neq P(i)$ and there exists $\mathbf{y_{g \cup \{i\}}} \in B_{g \cup \{i\}}(p)$ such that $U_j(\mathbf{y_{g \cup \{i\}}}; \mathbf{g} \cup \{\mathbf{i}\}) > U_j(\mathbf{x_{P(j)}}; P(j))$ for all $j \in \mathbf{g} \cup \{\mathbf{i}\}$;

(c) a group of consumers h can benefit from forming a new household, if $h \notin P$ and there exists $\mathbf{y_h} \in B_h(p)$ such that $U_j(\mathbf{y_h}; h) > U_j(\mathbf{x_{P(j)}}; P(j))$ for all $j \in h$.

In the spirit of the matching literature (see e.g. Gale and Shapley (1962), Roth and Sotomayor (1990)), a household structure is a "matching" broadly defined and stability of the matching requires that no group of consumers can benefit from forming a new household. It is important to note that in our context stability of a matching depends on household decisions and market conditions, that is the prevailing price system. Next we generalize the notion of competitive equilibrium so that the household structure or matching becomes a constituent part of the equilibrium.

DEFINITION 3 *Let* $D = (D_h)_{h \in \mathcal{H}}$ *be a profile of demand correspondences for households and* $(p, \mathbf{x}; P)$ *be a state of the economy. The state* $(p, \mathbf{x}; P)$ *is a* **competitive** D**-equilibrium** *if the allocation* $(\mathbf{x}; P)$ *is feasible and*

(iv) $\mathbf{x_h} \in D_h(p)$ *for* $h \in P$.

Finally, we generalize the notion of efficient allocation to the current setting where the household structure forms an integral part of an allocation.

DEFINITION 4 *An allocation* $(\mathbf{x}; P)$ *is* **fully Pareto-optimal** *if it is feasible and there is no other feasible allocation which is weakly preferred to* $(\mathbf{x}; P)$ *by all consumers and strictly preferred to* $(\mathbf{x}; P)$ *by some consumer(s).*

10.5.2 Equilibrium Efficiency of Households

The following benchmark result is shown in Gersbach and Haller (2003):

Fact 1 (Household Efficiency) *Suppose IPR, absence of externalities, continuity and local non-satiation of consumer preferences. Let* D *be a profile of demand correspondences for households. Consider a* D*-equilibrium* $(p, \mathbf{x}; P)$ *at which* $p \gg 0$*, every household makes efficient net trades and no consumer benefits from exit. Then*

(i) $(p, \mathbf{x}; P)$ *is an* EB*-equilibrium at which no group of consumers can benefit from forming a new household.*
(ii) $(p, \mathbf{x})$ *is a traditional competitive equilibrium where each agent acts and trades individually.*

The fact depicts circumstances under which households cannot afford any inefficient distribution without giving a member a reason to leave. By assertion (ii) of the fact, household members cannot fare any better or worse than as single consumers, hence under the same circumstances, a household cannot afford to exploit any of its members without giving them a reason to leave. Thus household stability requires absence of inefficiencies and absence of exploitation. The assumptions also guarantee Pareto-optimality of the equilibrium allocation in addition to equilibrium efficiency of households. Moreover, under these circumstances, the condition that no consumer can benefit from exit implies that no consumer can benefit from joining another household. This implication need not hold in the case of externalities to which we turn next.

10.5.3 Externalities and (In)efficiency

In the absence of externalities, already the exit option induces efficient distribution of resources within households. When externalities are present, it is conceivable that inefficient distribution of resources within households persists despite the opportunities of individuals to explore alternative household affiliations and alternative commodity allocations at the going prices. As will become clear, the three possible outside options: to exit and go single, to join another household, and to form a new household without any restrictions, can differ considerably in their effectiveness in preventing inefficient household decisions when externalities are present.

Our fundamental result is that inefficiencies will be eliminated provided that each household has similar counterparts in society (in the prevailing household structure) with respect to the nature and strength of externalities and that consumers are free to form new households. In contrast, the subsequent example shows that the more restrictive options to exit and go single or to leave and join another household prove insufficient to eliminate all inefficiencies. As a rule, however, they are not without consequences as the example also demonstrates. Namely, excessive allocative distortions as well as excessive manifestations of power could cause exit of some household member. This, too, is illustrated by the example.

Proposition 3 *Suppose IPR and GSE and that for each $i \in I$, preferences are continuous, convex and strictly monotone in own consumption. Let D be a profile of demand correspondences for households and let $(p, \mathbf{x}; P)$ be a D-equilibrium at which no group benefits from forming a new household. If g and h are two multi-member households in P of equal size, that is $|g| = |h| > 1$, then $\mathbf{x_g} \in EB_g(p)$ and $\mathbf{x_h} \in EB_h(p)$.*

PROOF: Let g and h as hypothesized. We claim that $\mathbf{x_g} \in EB_g(p)$ and $\mathbf{x_h} \in EB_h(p)$. By symmetry, if suffices to show $\mathbf{x_h} \in EB_h(p)$. Suppose to the contrary that $\mathbf{x_h} \notin EB_h(p)$. Then there exists $\mathbf{x'_h} \in B_h(p)$ with $V_i(x'_i\,; |h|) \geq V_i(x_i\,; |h|)$ for all $i \in h$ and $V_i(x'_i\,; |h|) > V_i(x_i\,; |h|)$ for some $i \in h$. By continuity and strict monotonicity, there exists $\mathbf{x^*_h} \in B_h(p)$ with $\sum_{i \in h} x^*_i = \sum_{i \in h} x'_i$ and $V_i(x^*_i\,; |h|) > V_i(x_i\,; |h|)$ for all $i \in h$.

Let us turn to household g momentarily. Because of $\mathbf{x_g} \in B_g(p)$ and IPR, there exists a group f of $|g|-1$ members of g with $\sum_{j \in f} px_j \leq \sum_{j \in f} p\omega_j$. For otherwise, for each group f of $|g| - 1$ members of g, $\sum_{j \in f} px_j > \sum_{j \in f} p\omega_j$ and the one member j_f of g who does not belong to f would have to satisfy $px_{j_f} < p\omega_{j_f}$. Consequently, every member of j of g would satisfy $px_j < p\omega_j$ and each group f of $|g| - 1$ members of g satisfied $\sum_{j \in f} px_j < \sum_{j \in f} p\omega_j$, a contradiction. Choose a group f of $|g| - 1$ members of g with $\sum_{j \in f} px_j \leq \sum_{j \in f} p\omega_j$.

Let us now turn to household h again and proceed with an $\mathbf{x^*_h} \in B_h(p)$ such that $V_i(x^*_i\,; |h|) > V_i(x_i\,; |h|)$ for all $i \in h$. Because of $\mathbf{x^*_h} \in B_h(p)$ and IPR, there exists $i \in h$ with $px^*_i \leq p\omega_i$. Let us choose such an i and consider the new household $k = f \cup \{i\}$ and the household allocation $\mathbf{y_k} \in \mathcal{X}_k$ given by $y_j = x_j$ for $j \in f$ and $y_j = x^*_i$ for $j = i$. Then $k \notin P$, $|k| = |g| = |h|$ and $\mathbf{y_k} \in B_k(p)$ where g and h are the original households. Because of GSE, $V_j(y_j\,; |k|) = V_j(x_j\,; |g|)$ for $j \in f$ and $V_j(y_j\,; |k|) > V_j(x_j\,; |h|)$ for $j = i$. By continuity and strict monotonicity, there exists $\mathbf{z_k} \in B_k(p)$ with $\sum_{j \in k} z_j = \sum_{j \in k} y_j$ and $V_j(z_j\,; |k|) > V_j(x_j\,; |P(j)|)$ for all $j \in k$. But this contradicts the hypothesis that at the state $(p, \mathbf{x}; P)$ no group can benefit from forming a new household. Hence $\mathbf{x_h} \in EB_h(p)$ has to hold as asserted. Q.E.D.

What drives the argument in the foregoing proof is that in equilibrium, for example a member of a two-person household and a member of another two-person household should not benefit from forming a new two-person household, and likewise for larger households of equal size. If a household member can only leave and go single or join another household, then the argument does not hold. This can be seen in the following example which exhibits several other interesting features.

Example. We consider a society of $N = 2n$, $n > 1$ individuals so that $I = \{1, \dots, N\}$. Assume $\ell = 2$. All individuals are identical. Each $i \in I$ is endowed with the commodity bundle $\omega_i = (1, 1)$ and has preferences represented by the utility function U_i, given as follows for $i \in h \in \mathcal{H}$:

$$\begin{aligned} U_i(\mathbf{x_h}; h) &= (1 + x_i^1)(1 + x_i^2) && \text{if } h = \{i\}, \\ U_i(\mathbf{x_h}; h) &= (1 + x_i^1)(1 + x_i^2) + b && \text{if } |h| = 2, \\ U_i(\mathbf{x_h}; h) &= (1 + x_i^1)(1 + x_i^2) - c && \text{if } |h| > 2, \end{aligned}$$

with $b > 0$ and $c > 0$. Thus individuals would like to form two-person households but dislike larger households.

As a benchmark, let us consider the state $s^* = (\mathbf{x}^*, p; P)$ where $\mathbf{x}^* = (\omega_i)_{i \in I}$, $p = (1, 1)$, and $P = \{\{2\nu - 1, 2\nu\} : \nu = 1, \dots, n\}$. The state s^* is an EB-equilibrium at which no group of consumers can benefit from forming a new household. The allocation $(\mathbf{x}^*; P)$ is fully Pareto-optimal. By Corollary 1 (ii), $(p, \mathbf{x}^*)$ is a traditional competitive equilibrium where each agent acts and trades individually. Furthermore, local non-satiation of preferences implies that $(p, \mathbf{x}^*)$ belongs to the core of the traditional pure exchange economy.

Next, let us define for $0 < \epsilon < 1$ the feasible commodity allocation $\mathbf{x}(\epsilon) = (x_i(\epsilon))_{i \in I}$ by setting $x_i(\epsilon) = (1 + \epsilon, 1 - \epsilon)$ for i odd and $x_i(\epsilon) = (1 - \epsilon, 1 + \epsilon)$ for i even. Then one can define a profile of household demand correspondences D so that the state $(\mathbf{x}(\epsilon), p; P)$ is a D-equilibrium. Suppose $\epsilon^2 < b$. Then:

(a) $\mathbf{x_h}(\epsilon) \notin EB_h(p)$ for all $h \in P$.
(b) In state $(\mathbf{x}(\epsilon), p; P)$, no consumer can benefit from exit.
(c) In state $(\mathbf{x}(\epsilon), p; P)$, no consumer can benefit from joining another household.

Assertion (a) holds, since $\mathbf{x_h^*}$ strictly dominates $\mathbf{x_h}(\epsilon)$ for each household $h \in P$. Specifically, each of these households performs an efficient net trade with the market, but an inefficient internal distribution of resources. Assertion (b) holds, since for all $i \in h \in P$, $U_i(\mathbf{x_h}(\epsilon); h) = 4 - \epsilon^2 + b$ which exceeds the maximal utility level $u_i(x_i^*) = 4$ the consumer can achieve as a single person at the prevailing prices. To show assertion (c), suppose a consumer $i \in h \in P$ joins another two-person household $g \in P$ and each member of the three-person household $g \cup \{i\}$ fares better than before. Then because of (b) and the negative group externality $-c$, each $j \in g \cup \{i\}$ must consume a bundle y_j such that $u_j(y_j) > u_j(x_j^*)$. Therefore, $py_j > p\omega_j$ for $j \in g \cup \{i\}$, since $(p, \mathbf{x}^*)$ is a traditional competitive equilibrium where each agent acts and trades individually. Hence $\mathbf{y_{g \cup \{i\}}} \notin B_{g \cup \{i\}}(p)$ and consumer i cannot benefit from joining household g.

This specification of the model satisfies IPR, GSE and continuity, convexity, and strict monotonicity of preferences. At the D-equilibrium $(\mathbf{x}(\epsilon), p; P)$, there are

$n \geq 2$ households of size 2. No consumer can benefit from exit or joining another household. In contrast to the case without externalities, these two stability conditions alone do not require households to make efficient decisions. However, by the argument of Proposition 3, some of the existing households would break up, if fractions of existing households could combine into new households and make more efficient consumption decisions.

Instead of making efficient net trades with the market followed by inefficient internal distribution, households could be making different mistakes. For instance, let $n = 2r$ with $r \in \mathbb{N}$ so that $N = 4r$, and set for $0 < \epsilon < 1$ and $i \in I$: $y_i(\epsilon) = (1 + \epsilon, 1 - \epsilon)$ if $i \leq 2r$ and $y_i(\epsilon) = (1 - \epsilon, 1 + \epsilon)$ if $i > 2r$. Then $\mathbf{y}(\epsilon) = (y_i(\epsilon))_{i \in I}$ is a feasible commodity allocation and the state $(\mathbf{y}(\epsilon), p; P)$ has the same properties as the state $(\mathbf{x}(\epsilon), p; P)$, except that now households are making inefficient net trades with the market followed by efficient internal distribution.

Notice that even when inefficiencies within households cannot be ruled out, the weaker stability conditions can have some welfare implications. First, in order to prevent a consumer from leaving, the degree of inefficiency cannot be too large. In case $\epsilon^2 > b$, a consumer would benefit from exit. Therefore, $\epsilon^2 \leq b$ has to hold to prevent exit. Thus the exit option limits the degree of inefficiency a household can afford. Second, in order to prevent a consumer from exit, the gains and losses from internal redistribution cannot be too large. To illustrate this point, consider for $0 < \epsilon < 1$ the feasible commodity allocation $\mathbf{z}(\epsilon) = (z_i(\epsilon))_{i \in I}$ by setting $z_i(\epsilon) = (1 - \epsilon, 1 - \epsilon)$ for i odd and $z_i(\epsilon) = (1 + \epsilon, 1 + \epsilon)$ for i even. Then the allocation $(\mathbf{z}; P)$ is fully Pareto-optimal and the state $(\mathbf{z}, p; P)$ is an EB-equilibrium. But in each household $h \in P$, internal redistribution causes the even numbered member to gain at the expense of the odd numbered person. To prevent the odd numbered consumers to benefit from exit, it has to be the case that $\epsilon < b/3$. Hence excessive allocative distortions as well as excessive manifestations of power would cause exit of some household members. If externalities become smaller, then there is less and less leeway for distortions and exercise of power. In the limit, without externalities, household decisions have to be efficient and there cannot be any gains and losses from household membership.

Both the last proposition and the example can be reformulated in terms of type economies, at the cost of additional notation. Consumer preferences then depend on household profile (number of each type present) rather than household size. In the proposition, the condition of equal household size has to be replaced by equal household profile. In the alternative example, one obtains a simple model of bilateral matching or a "marriage model", if there are two types (male and female) and consumers prefer heterogeneous two-person households to other households.

Whether competitive forces can have an efficiency enhancing influence on household decisions depends on several factors. Above all, household members must not be locked into existing households by legal provisions or social conventions. For if household members are forced or feel obliged to stay, then the outside option is simply non-existent. Further, group preferences must not dominate consumption preferences. For if a household member finds the household extremely attractive relative to alternative households, then the member might stay regardless of consumption decisions. Similarly, if a member considers the household composition very unsatisfactory in comparison to other conceivable households, then the individual may want to leave irrespective of consumption decisions. In either case the outside option is ineffective. In contrast, the conclusion of the last proposition rests on the opportunity to form a new household of equal size or, more generally, of equal or similar composition so that the current and the alternative new household are equally attractive or comparable in terms of membership. Then the consumption decision becomes decisive for the choice to stay or to leave — which puts the less efficient household at a disadvantage. Still, the threat of outside options is empty if new potential households suffer from the same inefficiencies as existing ones. However, it is plausible that in the process of regrouping and reallocation, individuals search for and realize efficiency gains. If the threat of departure is credible, then household stability requires efficient decisions or at least avoidance of severe inefficiencies. The prevailing or, to be precise, stable households find ways to avoid grave mistakes or significant inefficiencies caused, for example, by strategic behavior, coordination failure, slackness or force of habit.

For a fixed household structure, efficient household decisions together with budget exhaustion guarantee Pareto-optimality of a competitive equilibrium allocation. When the household structure is variable and externalities are present, efficient household decisions together with budget exhaustion guarantee full Pareto-optimality in many but not all cases. It is possible that a competitive EB-equilibrium allocation is dominated by another feasible combination of a household structure and a commodity allocation. This can be the case even if at the equilibrium state no consumer can benefit from exit or joining another household — as exemplified in Gersbach and Haller (2002).

In contrast, the more demanding stability condition that no group of consumers can benefit from forming a new household has very strong welfare implications. First of all, it induces efficient household decisions under the hypothesis of Proposition 3. Second, a competitive EB-equilibrium $(p, \mathbf{x}; P)$ at which no group of consumers can benefit from forming a new household yields a full Pareto-optimum in the weak sense that it is impossible to make everybody better off by means of another feasible allocation. Third, weak Pareto-optimality can be replaced by a weak core inclusion property. To this end, consider a non-empty subset J of I. We say that coalition J can strictly improve upon the allocation $(\mathbf{x}; P)$, if there exists a partition Q of J into non-empty subsets and household consumption plans $\mathbf{y_h}, h \in Q$, such that $U_i(\mathbf{y_{Q(i)}}; Q(i)) > U_i(\mathbf{x_{P(i)}}; P(i))$ for all $i \in J$ and $\sum_{i \in J} y_i = \sum_{h \in Q} \omega_h$. In other words, a coalition can strictly improve upon the given allocation, if it can make each of its members better off by forming a subeconomy with its own household structure and allocation of available resources.

Proposition 4 (Weak Core Inclusion)
Let $(p, \mathbf{x}; P)$ be an EB-equilibrium at which no group benefits from forming a new household. Then no coalition of consumers can strictly improve upon the allocation $(\mathbf{x}; P)$.

PROOF: Let $(p, \mathbf{x}; P)$ be an EB-equilibrium at which no group benefits from forming a new household. Suppose coalition J can strictly improve upon the allocation $(\mathbf{x}; P)$ by means of a partition Q of J and household consumption plans $\mathbf{y_h}, h \in Q$. Now let $h \in Q$. Then $U_i(\mathbf{y_h}; h) > U_i(\mathbf{x_{P(i)}}; P(i))$ for all $i \in h$. If $h \in P$, then $p{*}\mathbf{y_h} > p\omega_h$, since $\mathbf{x_h} \in EB_h(p)$. If $h \notin P$, then $p{*}\mathbf{y_h} > p\omega_h$, since group h cannot benefit from forming a new household. But then $p\sum_{i \in J} y_i = \sum_{i \in J} py_i = \sum_{h \in Q}\sum_{i \in h} py_i = \sum_{h \in Q} p * \mathbf{y_h} > \sum_{h \in Q} p\omega_h = p\sum_{h \in Q}\omega_h$, contradicting $\sum_{i \in J} y_i = \sum_{h \in Q}\omega_h$. Hence no coalition J can strictly improve upon the allocation $(\mathbf{x}; P)$. Q.E.D.

If one assumes in addition the budget exhaustion property and the redistribution property of Gersbach and Haller (2001), then weak core inclusion can be replaced by strong core inclusion, that is the assertion that no coalition of consumers can weakly improve upon the allocation $(\mathbf{x}; P)$.

10.6 Inefficient Household Decisions and Equilibrium Efficiency

We begin this section by exploring the possibility that inefficient net trades under one price system constitute efficient net trades under another price system and, consequently, inefficient net trades are consistent with an efficient equilibrium allocation.

Fact 2 (Accidental Welfare Theorem) *There exist $\ell \geq 2$ commodities, a non-empty population partitioned into households, household endowments and consumer preferences satisfying intra-household externalities, a profile of demand correspondences D, prices systems $p^* \neq \widetilde{p}$, and a feasible allocation x^* such that the following properties hold for the associated exchange economy: with the following two properties:*

1. *$(p^*; \mathbf{x}^*)$ is a competitive D-equilibrium where each household h performs an inefficient net trade with the market in the sense that*
   $\mathbf{x}^*_\mathbf{h} \notin PEB_h(p^*)$.
2. *$(\widetilde{p}; \mathbf{x}^*)$ is a competitive EB-equilibrium and the allocation $\mathbf{x}^*$ is Pareto-optimal.*

SKETCH OF PROOF. It suffices to outline the argument for the simplest case of two commodities, $\ell = 2$, and a single household, $H = 1$, with a single member denoted i. Consequently, (E1) amounts to (E2). Let the consumer be endowed with the commodity bundle $\omega_i = (1, 1)$ and his preferences be represented by the Cobb-Douglas utility function

$$U_i(x_i) = x_{i1}^{1/2} x_{i2}^{1/2}$$

for $x_i = (x_{i1}, x_{i2}) \in \mathbb{R}^{\ell}_{+}$. For each price system $p = (p_1, p_2) \in \mathbb{R}^{\ell}_{++}$, this consumer has a Marshallian demand

$$x_i(p) = \left(\frac{p_1 + p_2}{2p_1}, \frac{p_1 + p_2}{2p_2} \right).$$

Conversely, at each consumption bundle $\mathbf{x} \in \mathbb{R}^{\ell}_{++}$, this consumer's inverse demand or supporting price system is given, up to normalization, by grad $U_i(\mathbf{x})$, the gradient of U_i at $\mathbf{x}$.

Let us assume that instead of realizing his Marshallian net trade $x_i(p) - \omega_i$ with the market, the consumer always chooses zero net trade with the market which corresponds to the constant demand function $D(p) = D_i(p) \equiv \omega_i$. Now consider the price system $p^* = (1, 2)$. Then $(p^*; \omega_i)$ is a competitive D-equilibrium and ω_i is a Pareto-optimal allocation for this economy. But under his budget constraint, the consumer performs an inefficient net trade with the market, because his actual demand $\omega_i = (1, 1)$ differs from his Marshallian demand $x_i(p^*) = (3/2, 3/4)$. However, the former is Pareto-optimal whereas the latter is socially infeasible. Finally, $\omega_i = (1, 1) = x_i(\widetilde{p})$. This proves the assertion. Q.E.D.

Obviously, this trivial example generalizes to arbitrary numbers of consumers ($|I| \geq 1$) and goods ($\ell \geq 2$), to arbitrary household structures and a wide variety of consumer characteristics including instances of competitive equilibria with active trade as in cases of self-inflicted rationing with net trades $\frac{1}{2}[x_i(p) - \omega_i]$. Why then the attribute "accidental"? The reason is that the phenomenon of inefficient household decisions consistent with market efficiency is frequent in some sense and rare in some other sense. In support of this assertion, let us revisit the case $\ell = 2$. Let there be $H \geq 2$ single-person households, with both households and consumers labelled $i = 1, 2, \ldots, H$. Furthermore, let each consumer i be endowed with a strictly positive commodity bundle $\omega_i = (\omega_{i1}, \omega_{i2}) \in \mathbb{R}^{\ell}_{++}$ and have preferences of the Cobb-Douglas type,

$$U_i(x_i) = x_{i1}^{\alpha_i} x_{i2}^{1-\alpha_i} \text{ for } x_i = (x_{i1}, x_{i2}) \in X_i, \text{ with } 0 < \alpha_i < 1.$$

Now fix $\omega_i, i \in I$, and some $\lambda > 0$. Then there exist unique exponents $\alpha_i, i \in I$, and coefficients $\mu_1 > 0, \ldots, \mu_H > 0$ such that

$$\mu_1 \cdot \text{grad } U_1(\omega_1) = \ldots = \mu_H \cdot \text{grad } U_H(\omega_H) = (\lambda, 1). \tag{10.1}$$

Namely, $\alpha_i = \frac{\omega_{i1}}{\omega_{i2}} \cdot \lambda \Big/ \left(1 + \frac{\omega_{i1}}{\omega_{i2}} \cdot \lambda\right), i \in I$, is necessary and sufficient for (10.1). Equation (10.1) in turn is necessary and sufficient for Pareto-optimality of the initial allocation of resources. Hence, whenever (10.1) holds, the essence of the above one-consumer example is preserved: Choose again $D_i(p) \equiv \omega_i$ for each i and set $p^* = (\lambda, 2)$. Then $(p^*; (\omega_1, \ldots, \omega_H))$ is a competitive D-equilibrium with inefficient net trades, but an efficient market outcome. This shows that in a specific sense, the phenomenon of inefficient household decisions consistent with market efficiency is a frequent one: Given the endowments ω_i, $i \in I$, variation of λ yields a continuum of corresponding examples. On the other hand, validity of (10.1) or, equivalently, Pareto-optimality of the initial allocation is not robust with respect to small perturbations of the preference parameters $\alpha_1, \dots, \alpha_H$. In fact, the no trade allocation given by the endowments ω_i, $i \in I$, is not Pareto-optimal for most choices of preference parameters. But if the initial allocation of resources is not Pareto-optimal, then the foregoing construction of inefficient net trades leading to an efficient market outcome easily collapses. This suggests that in a certain sense, the phenomenon of inefficient household decisions compatible with market efficiency is a rare one.

After having explored how isolated inefficient household decisions may or may not beget efficiency, we next consider (in)efficient decisions across households and ask whether inefficient net trades by one household can be compensated by inefficient net trades of other households so that an efficient equilibrium allocation obtains. This question is irrelevant in the case of the specific consumer characteristics we have used to derive and discuss the Accidental Welfare Theorem, because of special auto-corrective features of that case. Notice that in that case, consumers would always pick the right allocation, $\omega = (\omega_1, \dots, \omega_H)$, though possibly for the wrong reasons. Namely, let $D_i(p) \equiv \omega_i$ and $p^* = (1, 2)$ as before and further $p^0 = (1, 1)$. Then $(p^*; \omega)$ is a D-equilibrium and, up to price normalization, $(p^0; \omega)$ is the EB-equilibrium. In fact, if $\widehat{D}_i = D_i$ for some but not all i and $\widehat{D}_i = EB_i$ for all others, then $(p^0; \omega)$ is the $\widehat{D}$-equilibrium. Thus there are only two possibilities: If all consumers exhibit totally inelastic demands D_i, equilibrium prices may be distorted away from the Walrasian equilibrium prices, yet still the Walrasian equilibrium allocation obtains. If some consumers exhibit Marshallian demands and the rest exhibits totally inelastic demands, then both equilibrium prices and equilibrium quantities turn out to be Walrasian.

Now let us consider instead a situation where an inefficient net trade made by one household leads to an inefficient equilibrium allocation, unless it is compensated by an inefficient net trade of another household. It suffices to focus on the simplest case of two commodities, $\ell = 2$, and two one-person households $h_1 = \{i\}$ and $h_2 = \{j\}$. Consequently, $(E1)$ amounts to $(E2)$.

Let consumers be endowed with the strictly positive bundles $\omega_i = (\omega_{i1}, \omega_{i2})$ and $\omega_j = (\omega_{j1}, \omega_{j2})$. Preferences are represented by the respective Cobb-Douglas utility functions

$$U_i(x_i) = x_{i_1}^{\alpha_i}\, x_{i2}^{1-\alpha_i}, \;\; U_j(x_j) = x_{j_1}^{\alpha_j}\, x_{j2}^{1-\alpha_j}$$

for $x_i, x_j \in \mathbb{R}^2_+$. Finally, let us assume that the initial endowment allocation $\omega = (\omega_i, \omega_j)$ is not Pareto-optimal, contrary to our previous assumption, so that there are potential gains from trade. Let us normalize the price system p by setting $p_1 = 1$. Then the consumers' Marshallian demands are given by

$$x_i(p) = \Big(\alpha_i(\omega_{i1} + p_2\,\omega_{i2}), (1-\alpha_i)\frac{\omega_i + p_2\omega_{i2}}{p_2}\Big),$$
$$x_j(p) = \Big(\alpha_j(\omega_{j1} + p_2\,\omega_{j2}), (1-\alpha_j)\frac{\omega_j + p_2\omega_{j2}}{p_2}\Big).$$

With respect to these demand functions, there exists a competitive equilibrium $(p^0; \mathbf{x^0})$ with the price system $p^0 = (1, p_2^0)$ given by

$$p_2^0 = \frac{(1-\alpha_i)\omega_{i1} + (1-\alpha_j)\omega_{j1}}{\alpha_i\omega_{i2} + \alpha_j\omega_{j2}}.$$

Suppose now that consumer i chooses his demand according to

$$\widetilde{x}_i(p) = \Big(\alpha_i(\omega_{i1} + p_2\omega_{i2}) + \Delta_i, \frac{(1-\alpha_i)(\omega_{i1} + p_2\omega_{i2})}{p_2} - \frac{\Delta_i}{p_2}\Big)$$

with some mistake $\Delta_i \neq 0$ that is independent of p_2 and satisfies $|\Delta_i| < \min\{\alpha_i, 1-\alpha_i\} \cdot \omega_{i1}$. While consumer i now performs an inefficient net trade under her budget constraint, consumer j is assumed to behave according to his Marshallian demand. Consider the resulting competitive equilibrium $(p^*; \mathbf{x}^*)$ with

$$p_2^* = \frac{(1-\alpha_i)\omega_{i1} + (1-\alpha_j)\omega_{j1} - \Delta_i}{\alpha_i\omega_{i2} + \alpha_j\omega_{j2}}.$$

The allocation $\mathbf{x}^*$ will be inefficient regardless of the equilibrium price p_2^*.

Suppose now that consumer j makes an inefficient net trade as well according to

$$\widetilde{x}_j(p) = \Big(\alpha_j(\omega_{j1} + p_2\omega_{j2}) - \Delta_j, \frac{(1-\alpha_j)(\omega_{j1} + p_2\omega_{j2})}{p_2} + \frac{\Delta_j}{p_2}\Big) \tag{10.2}$$

for some $\Delta_j \neq 0$ independent of p_2 and such that $|\Delta_j| < \min\{\alpha_j, 1-\alpha_j\} \cdot \omega_{j1}$. The resulting competitive equilibrium is $(p^{**}; \mathbf{x}^{**})$ with prices $p_1^{**} = 1$ and

$$p_2^{**} = \tfrac{(1-\alpha_i)\omega_{i1} + (1-\alpha_j)\omega_{j1} - \Delta_1 - \Delta_2}{\alpha_i\omega_{i2} + \alpha_j omega_{j2}} = p_2^0 - \tfrac{\Delta_1 + \Delta_2}{\alpha_i\omega_{i2} + \alpha_j\omega_{j2}}\,. \tag{10.3}$$

The resulting allocation $\mathbf{x}^{**}$ is Pareto-optimal if and only if the marginal rates of substitution for the two consumers coincide. The latter condition amounts to

$$\frac{\omega_{i1} + p_2^{**}\omega_{i2} - \Delta_i}{\omega_{i1} + p_2^{**}\omega_{i2} + \Delta_i} = \frac{\omega_{j1} + p_2^{**}\omega_{j2} + \Delta_j}{\omega_{j1} + p_2^{**}\omega_{j2} - \Delta_j} \tag{10.4}$$

or

$$F(\Delta_i, \Delta_j) \equiv \frac{\omega_{i1} + p_2^{**}\omega_{i2} - \Delta_i}{\omega_{i1} + p_2^{**}\omega_{i2} + \Delta_i} - \frac{\omega_{j1} + p_2^{**}\omega_{j2} + \Delta_j}{\omega_{j1} + p_2^{**}\omega_{j2} - \Delta_j} = 0. \tag{10.5}$$

Now at the Walrasian outcome, $F(0,0) = 0$ and $\partial F/\partial\Delta_j(0,0) \neq 0$. Therefore, by the implicit function theorem, there exists an open neighborhood $N_i(0)$ such that for all $\Delta_i \in N_i(0)$, there is a unique $\Delta_j(\Delta_i)$ with $F(\Delta_i, \Delta_j(\Delta_i)) = 0$. That is, to each small "mistake" Δ_i corresponds exactly one "compensating mistake" $\Delta_j(\Delta_i)$ that leads to an optimal equilibrium allocation. $\Delta_j(\Delta_i)$ can be explicitly determined by solving the quadratic equation in Δ_j associated with (10.4).

The analysis of this example shows among other things:

Proposition 5 (Compensating Inefficient Net Trades)
There exist economies with intra-household externalities and at least two households, labelled h_1 and h_2, with two profiles of demand correspondences, $D^* = (D_1^*, \ldots, D_H^*)$ and $D^{**} = (D_1^{**}, \ldots, D_H^{**})$, with a competitive D^*–equilibrium $(p^*; \mathbf{x}^*)$ and a competitive D^{**}–equilibrium $(p^{**}; \mathbf{x}^{**})$ such that:

1. $D_{h_1}^* = D_{h_1}^{**}$ *and* $D_h^* = D_h^{**} = EB_h$ *for all* $h \notin \{h_1, h_2\}$.
2. $D_{h_1}^* \cap EB_{h_1} = \emptyset$, $D_{h_2}^* = EB_{h_2}$, *and the allocation* $\mathbf{x}^*$ *is Pareto inefficient.*
3. $D_{h_1}^{**} \cap EB_{h_1} = \emptyset$, $D_{h_2}^{**} \cap EB_{h_2} = \emptyset$, *and the allocation* $\mathbf{x}^{**}$ *is Pareto efficient (Pareto-optimal).*

Let us add a few more observations to the last example and the implied proposition. First, the example exhibits single-person households and absence of externalities. Second, given $\Delta_i \in N_i(0)$, consumer j has to make the "right mistake" $\Delta_j(\Delta_i)$ in order to achieve a Pareto-optimal outcome; otherwise, the ensuing equilibrium allocation is not Pareto-optimal. This observation parallels the "accidental" nature of the conclusion of the Accidental Welfare Theorem. But different parameters of the model are allowed to vary in the two situations. Now the mistakes have to match whereas before consumer characteristics had to match. Third, a suitable pair of mistakes, $\Delta = (\Delta_i, \Delta_j)$ with $\Delta_j = \Delta_j(\Delta_i)$ determines a unique Pareto-optimal allocation $\mathbf{x}^{**}(\Delta)$. Conversely, select any point $\mathbf{x}^{**}$ on the contract curve such that (in the Edgeworth Box) the straight line L through $\mathbf{x}^{**}$ and ω is negatively sloped. For instance, a core allocation will do. Choose $p_2^{**} > 0$ so that $p^{**} = (1, p_2^{**})$ is a normal vector to this line, i.e. L is the budget line with respect to the price system p^{**}. Set $\Delta_i = x_{i1}^{**} - x_{i1}(p^{**})$ and $\Delta_j = -[x_{j1}^{**} - x_{j1}(p^{**})]$. Then $\Delta_j = \Delta_j(\Delta_i)$ and $\mathbf{x}^{**} = \mathbf{x}^{**}(\Delta)$. Finally, observe that the Marshallian demands of Cobb-Douglas consumers exhibit fixed expenditure shares. Therefore, the "mistakes" Δ_i and Δ_j can be interpreted as mistakes in the determination of the expenditure shares.

Next, let us consider a different situation where an inefficient net trade made by one household cannot be compensated by an inefficient net trade of another household and necessarily leads to an inefficient equilibrium allocation. To this end, we focus again on the simplest case of two commodities, $\ell = 2$, and two one-person households $h_1 = \{i\}$ and $h_2 = \{j\}$.

Let consumer i be endowed with the commodity bundle $\omega_i = (\omega_{i1}, \omega_{i2}) = (2, 1)$ and consumer j be endowed with $\omega_j = (\omega_{j1}, \omega_{j2}) = (6, 12)$. Preferences are represented by the respective utility functions

$$U_i(x_i) = \min\{x_{i1}, x_{i2}\}, \ \ U_j(x_j) = x_{j_1}^{1/2} \, x_{j2}^{1/2}$$

for $x_i, x_j \in \mathbb{R}^2_+$. Then the consumers' Marshallian demands are given by

$$x_i(p) = \Big(\frac{2p_1 + p_2}{p_1 + p_2}, \frac{2p_1 + p_2}{p_1 + p_2}\Big),$$
$$x_j(p) = \Big(\frac{6p_1 + 12p_2}{2p_1}, \frac{6p_1 + 12p_2}{2p_2}\Big).$$

With respect to these demand functions, there exists a competitive equilibrium $(p^0; \mathbf{x^0})$ with the price system $p^0 = (1, p_2^0)$ given by $p_2^0 = (\sqrt{19} - 1)/6$. This shows that there are potential gains from trade and the initial endowment allocation is not Pareto-optimal.

Suppose now that consumer i chooses her demand according to

$$\widetilde{x}_i(p) = (x_i(p) + \omega_i)/2.$$

Then consumer i performs an inefficient net trade under her budget constraint, by means of self-inflicted rationing, realizing only half of her efficient net trade. The corresponding individual excess demand function is continuous, bounded, and satisfying Walras Law on the price simplex.

Suppose further that consumer j chooses his demand according to a demand function $\widetilde{x}_j(\cdot)$ such that the corresponding individual excess demand function is continuous and satisfying Walras Law in the interior of the price simplex; moreover, it satisfies the standard boundary condition. Then, by standard arguments, the economy with these demand functions has a competitive equilibrium $(p^*; \mathbf{x}^*)$ with $p^* \gg 0$. Furthermore, $x_i^* \gg 0$, $x_j^* \gg 0$ and $x_{i1}^* > x_{i2}^*$. Therefore, it is possible to find a feasible allocation that strictly Pareto dominates $\mathbf{x}^*$. The important point is that we do not impose any restrictions on the demand function $\widetilde{x}_j(\cdot)$ other than the standard assumptions that guarantee existence of equilibrium. Hence it can differ from Marshallian demand in almost arbitrary ways. Thus we obtain:

Proposition 6 (Lack of Compensation)
There exists an economy with several one-person households and the following property: If household 1 uses a particular inefficient demand function d_1 and each household $h = 2, \ldots, H$ uses any demand function d_h that satisfies standard conditions, then there exists an inefficient competitive d–equilibrium allocation.

Notice that the demand function given by (10.2) satisfies standard conditions and qualifies for the foregoing impossibility result. On the other hand, the particular demand function chosen for the Leontief consumer does not exhibit a constant shift $\Delta_i \neq 0$ of expenditure shares. The reason for this choice is purely technical: For the Leontief consumer, a constant shift $\Delta_i \neq 0$ would lead to a violation of the non-negativity of demand at certain prices. It is possible, albeit tedious, to recast the example with locally but not globally constant Δ_i and Δ_j.

10.7 Conclusion

The basic premise of this and our previous work is that the allocation of resources among consumers and the ensuing welfare properties are obviously affected by the specifics of a pre-existing partition of the population into households and by the way households make decisions. Conversely, the formation and dissolution of households can be driven in part by economic expectations. Becker (1978, 1993) has explored and popularized this idea. Gersbach and Haller (2001, 2002) study the simultaneous allocation of consumers and commodities in a general equilibrium context where households are assumed to make efficient consumption decisions. In the present paper, we also consider the simultaneous allocation of consumers and commodities, but allow for inefficient household decisions. We show that competition for resources and members may restore collective rationality. More specifically, we identify circumstances where household stability requires efficient internal distribution, although in principle households could make inefficient choices. Furthermore, we investigate in some detail when and how inefficient net trades by one household can be compensated by inefficient net trades of other households so that an efficient equilibrium allocation occurs.

Regarding the original, broader question whether distinguishing between a household and its members makes any difference, Haller (2000) compares the case of efficient collective household decisions and the case where each household member shops on her own with her own interest in mind — after being allotted suitable income or endowment shares. He finds that as a rule, individual market participants do not fully internalize intra-household externalities. Thus individual behavior can impede collective rationality. In the club models of Gilles and Scotchmer (1997) and Ellickson, Grodal, Scotchmer and Zame (1999, 2001), individual market participation does not conflict with collective rationality of groups (clubs, households) because of the absence of consumption externalities. It is possible to design a general equilibrium model that encompasses both the household model and the club model. It remains to be seen whether the analysis of such a grand model can provide additional insights.

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11

Equilibrium with Arbitrary Market Structure

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Summary. Fifty years ago Arrow [1] introduced contingent commodities and Debreu [4] observed that this reinterpretation of a commodity was enough to apply the existing general equilibrium theory to uncertainty and time. This interpretation of general equilibrium theory is the Arrow-Debreu model. The complete market predicted by this theory is clearly unrealistic, and Radner [10] formulated and proved existence of equilibrium in a multiperiod model with incomplete markets.
In this paper the Radner result is extended. Radner assumed a specific structure of markets, independence of preferences, indifference of preferences, and total and transitive preferences. All of these assumptions are dropped here. We – like Radner – keep assumptions implying compactness.

Key words: Incomplete markets, Coordination.

JEL Classification Numbers: D52, D40.

11.1 Introduction

Arrow [1] introduced contingent commodities and Debreu [4] observed that this reinterpretation of a commodity was enough to apply the existing general equilibrium theory to uncertainty and time. This interpretation of general equilibrium theory is the Arrow-Debreu model. The complete market predicted by this theory is clearly unrealistic, and Radner [10] formulated and proved existence of equilibrium in a multiperiod model with incomplete markets. Hart [8] showed that the lower bound on consumption sets assumed by Radner was an essential limitation, but needed for the result in order to get compactness.

The problem raised by the Hart examples was eventually solved by Duffie and Shafer [6]. In these papers are shown *generic* existence of equilibria in economies with incomplete markets.

In this paper the Radner result is extended in other directions. Radner assumed a specific structure of markets, independence of preferences, indifference of preferences, and total and transitive preferences. All of these assumptions are dropped here. Assumption implying compactness are maintained and the existence proved is not generic.

A *premarket* can be any linear subspace of the commodity space, and a *market* is then what can be bought at the premarket at a *price*. A *market structure* is any set of premarkets, and there are no assumptions needed for the market structure.

The preferences are defined on the state space, so they may depend on the trades on the markets – and not just on the sum of the trades over all markets -, on the trades of the other consumers, and on the prices. The preferences are not assumed to be total or transitive. The possibility that the preferences depend on how the consumer obtains his consumption, is needed if this type of theory is used to understand what the market structure will be, and it is useful if we want to understand the role of money.

The way the existence theorem is proved may be as important as the theorem. General equilibrium theory, the Arrow Debreu model, Radner etc. has as the basic equilibrium concept Nash equilibrium – all agent choose what is best for them, taking as given what the other agents are doing. Here we will need a more general equilibrium concept. The idea in this paper is that an economy with incomplete markets is translated into an economy with a complete market. The translation is that each *old* consumer is made into a set of *new* consumers, one for each market, and each commodity is made into a set of commodities, one for each market.

A *new* consumer will however not take as given the trades of the other copies of the same *old* consumer, so a new equilibrium concept *Walras equilibrium with coordination* is needed. [7] and Appendix 2 page 221 gives an existence theorem for this equilibrium, and this theorem is applied here to give existence of first Walras equilibrium with coordination in the new economy and then by a simple translation an equilibrium with incomplete markets. This equilibrium concept is again a special case of equilibrium with coordination. [12] gives an existence theorem and explains the concepts. The definition of equilibrium with coordination is repeated in Appendix 1 page 220. Keiding [9] gives an improved existence result. So Walras equilibrium with coordination is an interpretation of equilibrium with coordination – just like the existence result in Arrow Debreu [2] can be regarded as a translation of the result in Debreu [3]. This paper is then just another translation from Walras equilibrium with coordination to an economy with incomplete markets. The fact that the results are ”just translations” means that there is almost nothing to prove.

Letting the set of premarkets be very large (for example including subspaces spanned by coordinates and diagonals) we would with realistic assumptions on preferences get that only very few markets will be used in an equilibrium, and only under special assumptions would the market structure used in equilibrium be complete.

The way the theorem is proved shows that incomplete markets can be regarded both as a generalization of a complete market and – by a reinterpretation of commodities and consumers – as a special case of an economy with a complete market. This means that economies with incomplete markets can be regarded as special cases of economies with externalities, and more generally that the existing theory of Walras equilibria, optimality, fairness, etc. can be used on the special case corresponding to an economy with incomplete markets.

11.2 The economy

$$E = \left(L, C, I, (M_i)_{i \in I}, (X_c, \mathcal{P}_c, I_c)_{c \in C}\right)$$

is the economy studied.

Notation	Interpretation
L, finite set, $h \in L$	Commodities
C, finite set, $c \in C$	Consumers
I, finite set, $i \in I$	Market institutions
$M_i \subset \mathbb{R}^L$, linear subspace, $i \in I$	Premarket i
$X_i = \left\{x \in \mathbb{R}^{LIC} \left\| \sum_c x(i,c) = 0\right.\right\}$	Feasible set for market i
$X_c \subset \mathbb{R}^{LIC}$	Feasible trade set for consumer c
$\mathcal{P}_c : X_c \to 2^{X_c}$	Preferences of consumer c
$I_c \subset I$	Premarkets available to consumer c
$T = \left(\mathbb{R}^L \setminus \{0\}\right)^I$	Prices

The only new concept here is a premarket. A market will be a premarket and a price. A premarket could determined by a subset of the commodities. In general it can be any linear subspace of the commodity space. This definition of a market comes from [11] where a market can be any subset of the commodity space.

11.3 Definitions

Definition 1 (Natural premarket). *A premarket is* natural *if it is spanned by points from* $\mathbb{R}^L_+$

We need a restriction to natural premarkets in order to avoid that the price hyperplane will contain the premarket.

Definition 2 (Premarket structure). *A* premarket structure $(M_i)_{i \in I}$ *is any finite family of natural premarkets.*

Definition 3 (Market i). *Given a* $p = (p_i)_{i \in I} \in T = \left(\mathbb{R}^L \setminus \{0\}\right)^I$ *we define* market i *as*

$$M_i(p) = \{z \in M_i \, | p_i z = 0\}$$

Definition 4 (Market structure). *The* market structure *is then*

$$(M_i(p))_{i \in I}$$

Definition 5 (Complete). *A premarket is* complete *if*

$$M = \mathbb{R}^L,$$

a market structure with a price p *is* complete *if*

$$\dim \sum_{i \in I} M_i (p) \geq L - 1$$

and a premarket structure is complete with respect to $T_0 \subset T$ *if the market structure is complete with respect to all* $p \in T_0$ *i.e.*

$$\dim \sum_{i \in I} M_i (p) \geq L - 1 \textit{ for all } p \in T_0 \textit{ with } p_i M_i \neq \{0\}$$

If the premarket or the market structure is complete we are back in the usual general equilibrium theory.

11.4 The state space

Definition 6 (State space). *The state space for* E *is now* $\mathbb{R}^{LIC}$

The notation

$$\begin{aligned} x = (x(i))_{i \in I} = (x(c))_{c \in C} = (x(h))_{h \in H} = \\ = (x(i,c))_{(i,c) \in IC} = (x(h,i,c))_{(h,i,c) \in LIC} \in \mathbb{R}^{LIC} \end{aligned}$$

will be used for elements in the state space.

For example $x(c) \in \mathbb{R}^{LI}$ gives the amount of the L commodities obtained in the I premarkets by consumer c.

Definition 7 (Market feasible). *A state* x *is* market feasible for c given the price p *if*

$$x \in V_c (p) = \left\{ x \in \mathbb{R}^{LIC} \middle| \begin{array}{ll} x(i,c) \in M_i (p), & \forall i \in I_c, \\ x(i,c) = 0, & \forall i \notin I_c \end{array} \right\}$$

This is the usual budget constraint, but one for each premarket.

Definition 8 (Feasible state-price). *A* state and a price (x,p) is feasible *if*

$$x \in F(E) = \bigcap_{c \in C} X_c \cap \bigcap_{i \in I} X_i \cap \bigcap_{c \in C} V_c (p) = X_C \cap X_I \cap V_C (p)$$

The feasibility for the whole economy is now the budget constraints, $X_i = \left\{ x \in \mathbb{R}^{LIC} \middle| \sum_c x(i,c) = 0 \right\}$ for each premarket – no commodities disappear, or appear in the market – and the state is in the feasible trade space for each consumer.

11.5 Equilibrium

Definition 9 (Equilibrium). *A* state and a price (x, p) is equilibrium in E *if it is feasible and there does not exist a* $y \in \mathbb{R}^{LIC}$ *and a* $c \in C$ *such that*

$$y \in X_c \cap V_c(p) \cap \mathcal{P}_c(x) \quad \textit{where}$$
$$y(i, d) = x(i, d) \textit{ for } d \neq c.$$

This is the usual Nash equilibrium, each consumer takes as given the prices and what the other consumers are getting. Given that, they can not find a preferred trade in the available markets.

Definition 10 ($Eq(E)$). *The* equilibrium set *will be denoted* $Eq(E) \subset \mathbb{R}^{LIC} \times T$

In the general definition there it not a condition that the prices in the different market are the same. If this is the case in an equilibrium we have Walras equilibrium.

Definition 11 (Walras equlibrium). *A state and a price* (x, p) *is Walras equilibrium in* E *if it is equilibrium in* E *and*

$$M_i(p) = \{z \in M_i | qz = 0\}$$

for some $q \in \mathbb{R}^{L \setminus \{0\}}$ *for all* $i \in I$.

So with this definition of Walras equilibrium we may have Walras equilibrium with incomplete market if only the prices in all markets are the same.

Definition 12 ($W(E)$). *The set of* Walras equilibria *will be denoted*

$$W(E) \subset \mathbb{R}^{LIC} \times T$$

Definition 13 (Complete Walras equilibrium). *A* state and a price (x, p) is a complete Walras equilibrium in E *if it is Walras equilibrium in* E *and the market structure is complete (with respect to* p*)*

This finally brings us back to the usual definition in general equilibrium theory.

Definition 14 ($CW(E)$). *The set of* complete Walras equilibria *will be denoted*

$$CW(E) \subset \mathbb{R}^{LIC} \times T$$

The relations

$$CW(E) \subset W(E) \subset Eq(E) \subset F(E) \times T \subset \mathbb{R}^{LIC} \times T$$

are obvious.

11.6 Examples

Example 1 (Complete Walras equilibrium with one market). Let $I = \{1\}$ and $M_1 = \mathbb{R}^L$, and let $I_c = I$ for all $c \in C$, then *equilibrium in* E is obviously *a complete Walras equilibrium (with one market).*

Example 2 (The prototype financial market). See [6]. Let $t = 0, 1$ be two points in time. At time $t = 1$ there are $\{1, 2, \cdots, S\} = \mathbf{S}$ states of nature. At time 0 and at each of the states there are ℓ commodities, so the commodity space is $\mathbb{R}^L$ where $L = \ell(1 + S)$. The market structure consists of spot markets for each $s \in \{1, 2, \cdots, S\}$ and for $t = 0$ a spot market for the commodities and for k real assets $a^1, a^2, \cdots, a^k \in \mathbb{R}^L$. Denote by A the $L \times k$ matrix with a^j as the j'column.
In our terminology we have in the economy a premarket structure $(M_s)_{s=0}^S$, which is available to all consumers. For $s \in \mathbf{S}$ the premarket is defined by

$$M_s = \left\{ z \in \mathbb{R}^L \, | z(h, s') = 0, \text{ for } \forall h \text{ and } s \neq s' \right\}$$

for $t = 0$ the premarket M_0 is defined by

$$M_0 = \left\{ z \in \mathbb{R}^L \, \middle| y \in R^{\ell+k} \text{ with } \widetilde{A}y = z \right\}$$

where

$$\widetilde{A} = \underset{L\times(\ell+k)}{\begin{pmatrix} E & \\ 0 & A \end{pmatrix}}$$

where $\underset{\ell\times\ell}{E}$ and $\underset{s\ell\times\ell}{0}$ are unit and zero matrices.

11.7 The existence theorem

Theorem 1. *Let E be the economy as described above.*
Assume:
For each $c \in C$ that $(X_c, \mathcal{P}_c)$satisfies
(i) X_c is convex, $0 \in intX_c$ (interior relative to $spanX_c$), and

$$\left\{ x(c) \in \mathbb{R}^{LI} \, | x \in X_c \right\}$$

is bounded below.
(ii) $\mathcal{P}_c$ is irreflexive and has open graph and convex values
For all $x \in X_c$

$$\mathcal{P}_c(x) \supset \{x\} + \overline{\mathbb{R}}_+^{LI} \setminus \{0\}$$

There exist $(X^c, \mathcal{P}^c)$ such that $intX^c \supset X_c$, $\mathcal{P}_c(x) = \mathcal{P}^c(x) \cap X_c$ for all $x \in X_0$, and $(X^c, \mathcal{P}^c)$ satisfies (i) and (ii)
Then $Eq(E) \neq \emptyset$

Proof.. Define the economy

$$\mathsf{E} = \left(\mathsf{L}, \mathsf{C}, (\mathsf{X_c}, \mathsf{P_c}, e_\mathsf{c})_{\mathsf{c}\in\mathsf{C}}, g\right)$$

from the economy

$$E = \left(L, C, I, (M_i)_{i\in I}, (X_c, \mathcal{P}_c, I_c)_{c\in C}\right)$$

by

$L \times I = \mathsf{L}$	A finite set of $\nu-$commodities
$C \times I = \mathsf{C}$	A finite set of $\nu-$consumers
$\nu : \mathbb{R}^{LI} \to \mathbb{R}^{\mathsf{LC}}$	$\nu(x)(\mathsf{h}, \mathsf{c}) = \nu(x)(h, i, c, j) = x(h, i, c), i = j$
$\nu\left(X_c \cap \prod_{i\in I_c} M_i\right) = \mathsf{X_c} \subset \mathbb{R}^{\mathsf{LC}}$	The feasible set for consumer c
$\mathsf{P_c} : \mathsf{X_c} \to 2^{\mathsf{X_c}},$	The preferences of consumer c
$e_\mathsf{c} : \mathsf{Y} \times \mathsf{Y} \to \mathsf{Y}$	Coordination function
$g : \mathsf{Y} \times \mathsf{Y} \to 2^{\mathsf{C}}$	Approval function
$T = \left(\mathbb{R}^L \setminus \{0\}\right)^I \subset \mathbb{R}^{IL} \setminus \{0\}$	Prices

Most of the notation explains itself.
ν maps $\left(\mathbb{R}^{LC}\right)^I$ into the diagonal of $\left(\mathbb{R}^{LC}\right)^{2I}$.

Definition 15. $(y) \in \mathsf{P_c}(\nu(x))$ *if and only if* $y \in \mathcal{P}_c(x), \mathsf{c} = (c, i)$ *(so all* c *coming from the same* c *have the same preferences).*

Definition 16 (e_c)**.** $e_\mathsf{c}(x, y) = \left(\left(x(d)_{d\neq c}\right), y(c)\right)$ *for* $\mathsf{c} = (c, i)$. *All new consumers coming from the same old consumer coordinates their trades, and no one else coordinate.*

Definition 17 (g)**.** $g(x, y) = \{\mathsf{c} \in \mathsf{C} \,|\, x(c) \neq y(c), \mathsf{c} = (c, i)\}$

E is now an economy with coordination with a feasible set

$$F(\mathsf{E}) = \bigcap_{\mathsf{c}\in\mathsf{C}} \mathsf{X_c} \cap \left\{\mathsf{x} \in \mathbb{R}^{\mathsf{LC}} \,\middle|\, \sum_{\mathsf{c}\in\mathsf{C}} \mathsf{x}(\mathsf{c}) = 0\right\}$$

it is easy to check that

$$\nu(F(E)) = F(\mathsf{E}).$$

A feasible state x and a price $p \in \mathbb{R}^{IL} \setminus \{0\}$ is a Walras equilibrium with coordination in E if there does not exist a

$$\text{y and a c such that } e_{\mathsf{c}}(\mathsf{x},\mathsf{y}) \in \mathsf{P}_{\mathsf{c}}(\mathsf{x}) \text{ and } p\mathsf{y}(\mathsf{c}) = 0$$

The set of equilibria will denoted $Eq(\mathsf{E})$. It is clear that

$$(x,p) \in Eq(E) \Rightarrow (\nu(x),p) \in Eq(\mathsf{E})$$

and conversely if $p_i \neq 0$ for all $i \in I$. The function ν is linear and the sets $\sum_{i \in I_{\mathsf{c}}} \nu(M_i)$ are closed and convex (linear subspaces) so all the properties of $(X_c, \mathcal{P}_c)_{c \in C}$ are inherited by $(\mathsf{X}_{\mathsf{c}}, \mathsf{P}_{\mathsf{c}})_{\mathsf{c} \in \mathsf{C}}$. This again means that all the properties assumed in Theorem 5 page 222 will hold . So the economy E with the assumptions inherited from E is a special case of the E economy with the assumptions from Theorem 5 page 222. The assumption that there is local non-satiation in all markets means that no p_i can be zero, so the existence of equilibria in E gives equilibria in E.

11.8 Indifference

The special case of indifference gives back Walras equilibrium if the market structure is complete.

In order to define indifference we define for all B the map

$$\sum\nolimits_I : \mathbb{R}^{BI} \rightarrow \mathbb{R}^B \text{ by } \sum\nolimits_I (x) = \sum_{i \in I} x(i) \in \mathbb{R}^B \qquad (\textstyle\sum_I)$$

Definition 18 (Indifference). *A consumer i in the economy E is indifferent (between the institutions in I) if*

$$\sum\nolimits_I^{-1} \sum\nolimits_I (X_c \times gr\mathcal{P}_c) = X_c \times gr\mathcal{P}_c \subset \mathbb{R}^{3LIC} \qquad (Indifference)$$

The interpretation is that only the sum over all institutions matters for whether a trade is in the feasible set for a consumer or for whether a trade is preferred to another. $\sum_I X_c \subset \mathbb{R}^{LIC}$ will be the net trade space for consumer c and $\sum_I gr\mathcal{P}_c = gr \sum_I \mathcal{P}_c \subset \mathbb{R}^{2LIC}$ will be the preferences $\sum_I \mathcal{P}_c$ for net trades. With the indifference assumption

$$y \in \mathcal{P}_c(x) \Leftrightarrow \sum\nolimits_I y \in \sum\nolimits_I \mathcal{P}_c\left(\sum\nolimits_I x\right)$$

All properties of $(X_c, \mathcal{P}_c)$ in Theorem 1 page 216 will be inherited by

$$\left(\sum\nolimits_I X_c, \sum\nolimits_I \mathcal{P}_c\right)$$

If $\mathcal{P}_c$ is monotone, $\sum_I \mathcal{P}_c$ is monotone.

Theorem 2. *Assume in the economy E that all consumers are indifferent, and that preferences are monotone and convex valued. Then any equilibrium will be Walras equilibrium, i.e.*

$$W(E) = Eq(E)$$

Proof.. Define the economy

$$\sum_I E = \left(L, C, \left(\sum_I M_i\right), \left(\sum_I X_c, \sum_I \mathcal{P}_c\right)_{c \in C}\right) \qquad (\textstyle\sum_I E)$$

as the same economy, except that there is only one premarket, namely the sum in $\mathbb{R}^L$ over all the premarkets in E. Because of the indifference assumption, for any equilibrium (x, p) in $E, \sum_I x$ will be equilibrium in $\sum_I E$ in the market $\sum_I M_i(p)$. $\sum_I M_i(p)$ is a linear subspace containing $\sum_I x(c)$ for all $c \in C, \sum_I \mathcal{P}_c\left(\sum_I x\right)$ is convex and disjoint from $\sum_I M_i(p)$, so a separating hyperplane argument implies that $\sum_I M_i(p)$ is contained in a hyperplane in $\mathbb{R}^L$, so there exist a $q \in \mathbb{R}^L_+$ such that

$$M_i(p) \subset \{z \in M_i \,|qz = 0\}$$

for all $i \in I$

Theorem 3. *Assume in the economy E that all consumers are indifferent, and that preferences are monotone and convex valued. Then any equilibrium in an economy with a complete market structure will be a complete Walras equilibrium, i.e*

$$CW(E) = Eq(E)$$

Proof.. Trivial from the preceding theorem and the definition of a complete market.

11.9 Conclusion

Transaction costs for the consumers are not explicit in this paper, but are of course implicit in the preferences on the state space. Transaction costs could be one reason for not making the indifference assumption. Transaction costs can also be used to justify the compactness assumption.

Letting the set of premarkets be very large (for example including subspaces spanned by coordinates and diagonals) we would with realistic assumptions on preferences get that only very few markets will be used in an equilibrium, and only under special assumptions would the market structure used in equilibrium be complete.

The most serious limitation in the results in this paper, as an explanation of which markets will exist, is the assumption of no transaction costs for the market, and no surplus for a market agent

$$X_i = \left\{ x \in \mathbb{R}^{LIC} \,\middle|\, \sum_c x(i, c) = 0 \right\}$$

If there are no transaction costs for the market agent and the possibility of creating any market is open to more than one agent, competition would imply $\sum_c x(i, c) = 0$. But with transaction costs or cooperation between market agents and producers, the theory of which markets would exist, will be much more complicated.

The way the theorem is proved shows that incomplete markets can be regarded both as a generalization of a complete market and – by a reinterpretation of commodities and consumers – as a special case of an economy with a complete market. This means that economies with incomplete markets can be regarded as special cases of economies with externalities, and more generally that the existing theory of Walras equilibria, optimality, fairness etc. can be used on the special case corresponding to an economy with incomplete markets.

A 1. Equilibrium with coordination

This section just repeats concepts and the theorem from [12] with the assumptions in the improved version from Keiding [9]

Let

$$\Gamma = \left(A, X, (\beta_a, \mathcal{P}_a, e_a)_{a\in A}, g\right)$$

be a social system with coordination

A	Finite set of agents	
X	The state space, arbitrary	
$\beta_a : X \to 2^X$	Attainable states for $a \in A$	
$\beta : X \to 2^X, \bigcap_{a\in A} \beta_a(x) = \beta(x)$	Attainable states	(Γ)
$\mathcal{P}_a : X \to 2^X$	Preferences of $a \in A$	
$e_a : X \times X \to X$	Coordination function of $a \in A$	
$g : X \times X \to 2^A$	Approval function	

This equilibrium concept has as one special case Nash equilibrium and at the other extreme optimality. The two concepts introduced in addition to the known concepts from social systems are

1. The coordination function $e_a : X \times X \to X, e_a(x, y)$ is the state expected by a to be the result of a change from x to y. For the case of Nash equilibrium $e_a(x, y) = \left(y_a, (x_b)_{b\neq a}\right)$. (Each agent takes as given the actions of the other agents). For the case corresponding to optimality $e_a(x, y) = y$. Many intermediate cases are of course possible.
2. The approval function $g : X \times X \to 2^A$. $g(x, y) \subset A$ is the subset which have to approve of the change from x to y. In many applications $g(x, y)$ would be the set of agents for which $e_a(x, y) \neq x$. In the special case where the concept is used to express conjectural equilibrium $g(x, y)$ would be the set of agents for which $y_a \neq x_a$

Definition 19 (Equilibrium). *An equilibrium in the social system with coordination is a state $x \in X$ such that*

$$x \in \beta(x)$$

$$\nexists y \in X \text{ such that } e_a(x,y) \neq x \text{ for some } a \in A, \text{ and}$$
$$e_a(x,y) \in \mathcal{P}_a(x) \cap \beta_a(x) \text{ for } a \in g(x,y)$$

Theorem 4. *Let* $\left(A, X, (\beta_a, \mathcal{P}_a, e_a)_{a\in A}, g\right)$ *be a social system with coordination such that for all* $a \in A$ *and for all* $x \in X$

(a) X is a non-empty, convex, compact subset of $\mathbb{R}^\ell$
(b) β is continuous with closed, convex and non-empty values
(c) $\mathcal{P}_a$ has open graph, convex values and $x \notin \mathcal{P}_a(x)$
(d) e_a is continuous, $e_a(x,\cdot)$ is affine, and $e_a(x,x) = x$
(e) $e_a(x,y) \in \beta(x) \Rightarrow y \in \beta(x)$
(f) $\beta(x) \subset intX$
(g) $g(x,y) = \{a \in A \,|e_a(x,y) \neq x\}$

Then there exists an equilibrium

Proof.. See [9, Theorem 2 page 107] (The assumption that β has convex values is used in the proof but forgotten in the statement of the theorem in [9])

A 2. Walras equilibrium with coordination [7]

E is the economy in which equilibrium with coordination is defined and existence proved

$$\mathsf{E} = \left(\mathsf{L}, \mathsf{C}, (\mathsf{X_c}, \mathsf{P_c}, e_\mathsf{c})_{\mathsf{c}\in\mathsf{C}}, \varDelta, g\right)$$

with

L	A finite set of commodities	
C	A finite set of consumers	
$\mathsf{X_c} \subset \mathbb{R}^{\mathsf{LC}} = \mathsf{Y}$	The feasible set for consumer c	
$\mathsf{P_c} : \mathsf{X_c} \to 2^{\mathsf{X_c}}$,	The preferences of consumer c	(E)
$e_\mathsf{c} : \mathsf{Y} \times \mathsf{Y} \to \mathsf{Y}$	Coordination function	
$g : \mathsf{Y} \times \mathsf{Y} \to 2^{\mathsf{C}}$	Approval function	
$\varDelta = \left\{\mathsf{p} \in \mathbb{R}^{\mathsf{L}}_+ \,|\sum \mathsf{p_h} = 1\right\}$	Prices	

The state space is $\mathbb{R}^{\mathsf{LC}} = \mathsf{Y}$, the *feasible set* is

$$F(E) = \bigcap_{\mathsf{c}\in\mathsf{C}} \mathsf{X_c} \cap \left\{\mathsf{x} \in \mathsf{Y} \,\middle|\, \sum_{\mathsf{c}\in\mathsf{C}} \mathsf{x}(\mathsf{c}) = 0\right\}$$

A pair (x, p) where $\mathsf{x} \in \mathsf{X}_0$ and $\mathsf{p} \in \varDelta$ is *equilibrium with coordination* if there does not exist a $\mathsf{y} \in \mathsf{Y}$ such that

$$e_\mathsf{c}(\mathsf{x},\mathsf{y}) \in \mathsf{X_c} \cap \mathsf{P_c}(\mathsf{x}), \mathsf{py}(\mathsf{c}) = 0 \text{ for } \mathsf{c} \in g(\mathsf{x},\mathsf{y})$$

Theorem 5 (Equilibrium). *Let* E *be the economy as described above. Assume for each* $c \in C$ *that* (X_c, P_c, e_c) *satisfies*

X_c *is convex,* $0 \in int X_c$ *(interior relative to* $span X_c$*), and*

$$\left\{x(c) \in \mathbb{R}^L \,|x \in X_c\right\}$$

is bounded below.

P_c *is irreflexive and has open graph and convex values*

$P_c(x) \supset \mathbb{R}_+^L \times \overline{\mathbb{R}}_+^{(C\setminus\{c\})L} + x$

$e_c(x, y) \in \beta(x, y) \Rightarrow y \in \beta(x, y)$

There exist (X^c, P^c) *such that*

$$int X^c \supset X_c, X^C = \bigcap_{c \in C} X^c,$$
$$P_c(x) = P^c(x) \cap X_c \text{ for all } x \in X_0$$
$$\text{and } (X^c, P^c) \text{ satisfies (i) and (ii)}$$

Then $Eq(E) \neq \emptyset$

Proof.. See [7]

A special case of this is *Walras equilibrium with coordination in a partition.* Coordination in a partition $(C_j)_{j \in J}$ means that

$$e_c(x, y) = \left((x(b))_{b \notin C_j}, (y(b))_{b \in C_j}\right),$$
$$\text{where } (C_j)_{j \in J} \text{ is a partition of } C \text{ and } c \in C_j$$
$$\text{and } g(x, y) = \left\{c \in C \,\middle|\, (x(d))_{d \in C_j} \neq (y(d))_{d \in C_j}, \text{ for } c \in C_j\right\}$$

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12

Pareto Improving Price Regulation when the Asset Market is Incomplete

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Summary. Incomplete asset markets cause competitive equilibria to be constrained suboptimal and provides scope for Pareto improving interventions. In this paper, we examine how intervention in prices in asset or spot commodity markets serves this purpose. We show that, if fix-price equilibria behave sufficiently regularly near Walrasian equilibria, Pareto improving price regulation is generically possible. An advantage of price regulation, contrasted with interventions in individuals' asset portfolios, is that it operates anonymously, on market variables.

Key words: Incomplete asset market, Fix-price equilibria, Pareto improvement.

JEL Classification Numbers: D45, D52, D60.

12.1 Introduction

When asset markets are incomplete, competitive equilibrium allocations generically fail to satisfy the criterion of constrained Pareto optimality which recognizes the incompleteness of the asset market. Geanakoplos and Polemarchakis (1986) showed that, generically, there exist reallocations of portfolios that yield Pareto improvements in welfare after prices in spot commodity markets adjust to attain equilibrium; Citanna, Kajii and Villanacci (1998) developed further and generalized the argument. The failure of constrained optimality casts doubt on non-intervention with competitive markets. Expansions of the asset market do not necessarily lead to Pareto improvements: Hart (1975) gave an example of financial innovation that leads to a Pareto deterioration; Cass and Citanna (1998), Elul (1995) and Hara (1997) identified conditions for Pareto improving financial innovation. The taxation of trades in assets, which is anonymous, can generically implement a Pareto improvement. Citanna, Polemarchakis and Tirelli (2001) demonstrated the result, which requires that the number of individuals not exceed the number of traded assets; it provides only a partial answer to Kajii (1994), who pointed out that, apart from informational requirements, the heterogeneity of individuals and the requirement of anonymity may interfere with improving interventions. The direct regulation of prices in spot commodity markets is an alternative to the reallocation of portfolios or the taxation of trades in assets. Importantly, this is not an intervention in individual choice variables but in market variables, and, as such, it satisfies the requirement of anonymity. An extension of the fix-price equilibrium of Drèze (1975) provides a notion of equilibrium that allows for trading at non-competitive prices; alternative specifications, in Barro and Grossman (1971), Bénassy (1975) or Younès (1975) should not affect the argument. The results of Laroque (1978, 1981), nevertheless, point out a stumbling block: the behavior of fix-price equilibria in the neighborhood of competitive equilibria is particularly complicated. There are robust examples for which, at regulated prices close to competitive prices, there are no fix-price equilibria close to competitive equilibria. Here, we restrict attention to the class of economies, evidently restrictive, that satisfy conditions sufficient for the local uniqueness of fix-price equilibria. In Herings and Polemarchakis (2003), a robust example illustrates the approach as well as the results. The conditions under which the result holds, that the number of instruments (contingent commodities) exceed the number of objectives (individuals), imply that the result complements the one of Geanakoplos and Polemarchakis (1986) and Citanna, Polemarchakis and Tirelli (2001). Antecedents of this result are the argument in Polemarchakis (1979), where fixed wages that need not match shocks in productivity may yield higher expected utility in spite of the loss of output in an economy of overlapping generations; and the argument in Drèze and Gollier (1993), which employs the capital asset pricing model to determine optimal schedules of wages that differ from the marginal productivity of labor. Kalmus (1997) gave a heuristic example of Pareto improving price regulation. Of serious concern are the informational requirements needed to determine, even compute, improving interventions. In the case of price regulation they involve knowledge of marginal utilities of income and excess demands for commodities across states. Geanakoplos and Polemarchakis (1990) and Kübler, Chiappori, Ekeland and Polemarchakis (2002) are only first steps towards an analysis of the informational requirements of active policy.

12.2 The economy

The economy is that of the standard two-period general equilibrium model with numéraire assets and incomplete asset markets. Assets exchange before and commodities after the state of the world realizes. States of the world are $s \in \mathcal{S} = \{1, \ldots, S\}$ and commodities are $l \in \mathcal{L} = \{1, \ldots, L+1\}$. At state s, commodity $(L+1, s)$ is numéraire. Assets are $a \in \mathcal{A} = \{1, \ldots, A+1\}$. Asset $A+1$ is numéraire. The payoffs of assets are denominated in the numéraire commodity, $(L+1, s)$, at every state of the world. A bundle of commodities at a state of the world is $x_s = (\ldots, x_{l,s}, \ldots, x_{(L+1),s})$; across states of the world, $x = (\ldots, x_s, \ldots)$. A portfolio of assets is $y = (\ldots, y_a, \ldots, y_{A+1})$. The payoffs of an asset across states of the world are $R_{a\cdot} = (\ldots, R_{a,s}, \ldots)'$; at a state of the world, payoffs of assets are $R_{\cdot s} = (\ldots, R_{a,s}, \ldots)$; across states of the world, the asset payoff matrix is

$R = (\ldots, R_{a\cdot}, \ldots)$. The asset payoff matrix has full column rank, and the numéraire asset has positive payoffs: $R_{A+1\cdot} > 0$. Individuals are $i \in \mathcal{I} = \{1, \ldots, I\}$. A utility function, u^i, that satisfies standard conditions of continuity, monotonicity, quasi-concavity and, when required, smoothness and boundary behavior, and the endowment, e^i, a strictly positive bundle, describe an individual — the boundary behavior of the utility function, together with the strict positivity of the endowment guarantee that consumption bundles demanded by the individual lie in the interior of the consumption set, as in see Debreu (1972). The utility functions and consumption sets of individuals as well as the matrix of asset payoffs do not vary. The allocation of endowments, $\omega = (\ldots, e^i, \ldots)$, identifies an economy, and the set of economies coincides with the strictly positive orthant of the commodity space; a property holds generically if it holds for an open set of economies of full Lebesgue measure. Prices of commodities at a state of the world are $(\ldots, p_{l,s}, \ldots, 1)$; across states of the world, $p = (\ldots, p_s, \ldots) \gg 0$; the price of the numéraire commodity at a state of the world is $p_{L+1,s} = 1$; the domain of prices of commodities is $\mathcal{P}$. Prices of assets are $q = (\ldots, q_a, \ldots, 1)$; The price of the numéraire asset is $q_{A+1} = 1$. The domain of prices of assets is $\mathcal{Q}$. It is often convenient to truncate prices of commodities and prices of assets by deleting the prices of the numéraires. Commodities or assets other than the numéraire are $\check{\mathcal{L}}$ or $\check{\mathcal{A}}$; the domain of prices of commodities or assets other than the numéraires is $\check{\mathcal{P}}$ or $\check{\mathcal{Q}}$. At arbitrary terms of trade, a competitive equilibrium, typically, does not exist. In commodities and assets other than the numéraire, rationing on net trades, uniform across individuals, serves to attain market clearing. Rationing in the supply (demand) of commodities other than the numéraire is $\underline{z} \leq 0$ $(\overline{z} \geq 0)$. Rationing in the supply (demand) of assets other than the numéraire is $\underline{y} \leq 0$ $(\overline{y} \geq 0)$. Without appropriate rationing constraints, demand and supply of commodities will not match, which leads to inconsistencies. Equilibrium rationing constraints are exactly such that markets clear. At prices and rationing $(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y})$, the budget set of an individual is

$$\beta^i(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y}) = \left\{ (x, y) : \begin{array}{l} qy \leq 0, \\ p_s(x_s - e^i_s) \leq R_{s\cdot}y, \quad s \in \mathcal{S}, \\ \underline{z}_{l,s} \leq x_{l,s} - e^i_{l,s} \leq \overline{z}_{l,s}, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S}, \\ \underline{y}_a \leq y \leq \overline{y}_a, \quad a \in \check{\mathcal{A}} \end{array} \right\};$$

his optimization problem is to choose a utility maximizing consumption bundle and asset portfolio in his budget set. The set of all optimal consumption bundles and asset portfolios is denoted $d^i(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y})$. An individual is effectively rationed in his supply (demand) for a commodity or an asset if he could increase his utility when the rationing constraint in the supply (demand) of that commodity or asset is removed. There is effective supply (demand) rationing in the market for a commodity or an asset if at least one individual is effectively rationed in his supply (demand) for this commodity or asset. At a competitive equilibrium, there is neither effective supply rationing nor effective demand rationing in the market for any commodity or asset. In this sense, a competitive equilibrium is a special case of a fix-price equilibrium.

Definition 1 (Fix-price equilibrium). *A fix-price equilibrium at prices (p, q) is a pair $((x^*, y^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*))$, such that*

1. *for every individual, $(x^{i*}, y^{i*}) \in d^i(p, q, \underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*)$,*
2. *$\sum_{i=1}^{I} x^{i*} = \sum_{i=1}^{I} e^i$ and $\sum_{i=1}^{I} y^{i*} = 0$,*
3. *for every $l \in \check{\mathcal{L}}$, if for some i' $x_{l,s}^{i'*} - e_{l,s}^{i'} = \underline{z}_{l,s}^*$, then for all $i \in \mathcal{I}$ $x_{l,s}^{i*} - e_{l,s}^{i} < \overline{z}_{l,s}^*$, while if for some i' $x_{l,s}^{i'*} - e_{l,s}^{i'} = \overline{z}_{l,s}^*$, then for all $i \in \mathcal{I}$ $x_{l,s}^{i*} - e_{l,s}^{i} > \underline{z}_{l,s}^*$,*
4. *for every $a \in \check{\mathcal{A}}$, if for some i' $y_a^{i'*} = \underline{y}_a^*$, then for all $i \in \mathcal{I}$ $y_a^{i*} < \overline{y}_a^*$, while if for some i' $y_a^{i'*} = \overline{y}_a^*$, then for all $i \in \mathcal{I}$ $y_a^{i*} > \underline{y}_a^*$.*

At a fix-price equilibrium, only one side of the market is effectively rationed; this is expressed by Conditions 3 and 4. At prices (p, q), fix-price equilibria exist. Appendix 1 spells out the arguments in detail. Herings and Polemarchakis (2002) provide a (more complicated) proof that requires weaker assumptions on the primitives. A sign vector,

$$r = (r_{1,1}, \ldots, r_{L,S}, r_1, \ldots, r_A),$$

describes the state of markets at a fix-price equilibrium. If there is effective supply rationing in the market for a commodity or an asset, the associated component of the sign vector is -1, if there is effective demand rationing it is +1, and if there is no effective rationing it is 0. For a sign vector r, the set $\mathcal{PQ}(r)$ is the set of prices $(p, q) \in \mathcal{P} \times \mathcal{Q}$, for which there exists a fix-price equilibrium at prices (p, q) with state of the markets r. For prices $(p, q) \in \mathcal{P} \times \mathcal{Q}$, the set of fix-price equilibrium allocations is $\mathcal{D}(p, q)$, and, for a sign vector r, the set of fix-price equilibrium allocations with state of the markets r is $\mathcal{D}(p, q, r)$. A neighborhood of α is $\mathcal{N}_\alpha$. If $((p^*, q^*), (x^*, y^*))$ is a competitive equilibrium, the allocation (x^*, y^*) is locally unique as a fix-price equilibrium allocation if there exists a neighborhood $\overline{\mathcal{N}}_{x^*,y^*}$ such that for every $\mathcal{N}_{x^*,y^*} \subset \overline{\mathcal{N}}_{x^*,y^*}$ there exists a neighborhood $\mathcal{N}_{p^*,q^*}$ with $\mathcal{D}(p, q) \cap \mathcal{N}_{x^*,y^*}$ a singleton for every $(p, q) \in \mathcal{N}_{p^*,q^*}$. If a competitive equilibrium allocation is locally unique as a fix-price equilibrium allocation, then, for prices close to competitive equilibrium prices, there is exactly one fix-price equilibrium allocation close to the competitive allocation. For a locally unique competitive equilibrium allocation, for each sign vector r, the function $(\widehat{x}^r, \widehat{y}^r) : \overline{\mathcal{N}}_{p^*,q^*} \cap \mathcal{PQ}(r) \to \mathbb{R}^{I(L+1)S+I(A+1)}$ associates the unique fix-price equilibrium allocation in $\overline{\mathcal{N}}_{x^*,y^*} \cap \mathcal{D}(p, q, r)$ to (p, q). Comparative statics require a differentiable form of local uniqueness.

Definition 2 (Differentiable local uniqueness). *If $((p^*, q^*), (x^*, y^*))$ is a competitive equilibrium, the allocation (x^*, y^*) is differentiably locally unique as a fix-price equilibrium allocation if it is locally unique and there is a neighborhood $\mathcal{N}_{p^*,q^*}$ such that, for every sign vector r, the function $(\widehat{x}^r, \widehat{y}^r)_{|\mathcal{N}_{p^*,q^*} \cap\ \mathcal{PQ}(r)}$ is differentiable.*

Laroque and Polemarchakis (1978) proved, for a complete asset market, that, generically, the set of fix-price equilibrium allocations can be represented by a finite number of continuously differentiable functions of prices. Nevertheless, the results in Laroque (1978) and the examples in Madden (1982) show that competitive equilibria need not be locally unique as fix-price equilibria. Even though fix-price equilibrium allocations exist for all prices, there may be robust local non-existence, and therefore local non-uniqueness as a fix-price equilibrium, at competitive prices.

Assumption 1. *For endowments in Ω^*, an open set of full Lebesgue measure, if $((p^*, q^*),\ (x^*, y^*))$ is a competitive equilibrium, then the competitive equilibrium allocation is differentiably locally unique as a fix-price equilibrium allocation.* By an argument similar to the one in the proof of Theorem 1, Laroque

(1981), Appendix 2 characterizes economies where competitive equilibrium allocation are differentiably locally unique as a fix-price equilibrium allocation. Local uniqueness of fix-price equilibrium allocations at competitive equilibria is not too strong a requirement; it is less demanding than the requirement of uniqueness of fix-price equilibrium allocations at prices in a neighborhood of competitive prices. This guarantees a certain degree of generality of the results. The function $(\widehat{x}, \widehat{y})$ associates the unique fix-price equilibrium allocation in $\mathcal{N}_{x^*,y^*}$ to $(p, q) \in \mathcal{N}_{p^*,q^*}$. At a locally unique fix-price equilibrium,

$$v^i(p, q) = u^i(\widehat{x}^i(p, q)), \quad (p, q) \in \mathcal{N}_{p^*,q^*}$$

defines the indirect utility function of an individual. Lemma 2 in Appendix 2 implies that it is differentiable when the fix-price equilibrium is differentiably locally unique, with partial derivatives given by

$$\partial_{p_{l,s}} v^i(p^*, q^*) = -\partial_{x_{L+1,s}} u^i(x^{i*})(x^{i*}_{l,s} - e^i_{l,s}), \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S}.$$

The effect of a change in the spot market price of commodity $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$ is equal to minus the marginal utility of the numéraire commodity in state s multiplied by the excess demand of commodity (l, s) at the competitive equilibrium. Lemma 2 in Appendix 2 therefore implies that the indirect welfare effects of a change in prices, generated by the induced change in the rationing constraints and individuals' choices, equal zero.

Pareto improving price regulation

If the asset market is incomplete, $A + 1 < S$, generically, competitive equilibrium allocations are not Pareto optimal. Price regulation can Pareto improve on a competitive equilibrium $((p^*, q^*), (x^*, y^*))$ if there exist prices of commodities p such that a fix-price equilibrium of commodities at prices of commodities and assets (p, q^*) Pareto dominates the allocation x^*. The ambiguity introduced by the possibility of multiple fix-price equilibrium allocations at prices (p, q^*) is circumvented by considering local variations at competitive equilibrium allocations that are differentiably locally unique as fix-price equilibria.

Definition 3 (Pareto improving price regulation). *Price regulation can Pareto improve upon a competitive equilibrium, $((p^*, q^*), (x^*, y^*))$, that is locally unique as a fix-price equilibrium if there exists $d\check{p} \in \mathbb{R}^{LS}$ such that*

$$\sum_{(l,s)\in\check{\mathcal{L}}\times\mathcal{S}} \partial_{p_{l,s}} v^i(p^*, q^*) dp_{l,s} > 0, \quad \textit{for every } i \in \mathcal{I}.$$

Uniform price regulation can improve upon a competitive equilibrium if

$$d\check{p}_s = d\check{p}_{s'}, \quad \textit{for all } s, s' \in \mathcal{S}.$$

If (uniform) price regulation can improve upon a competitive equilibrium, there is $\overline{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \overline{\varepsilon}]$, the fix-price equilibrium in the neighborhood of the competitive equilibrium $((p^*, q^*), (x^*, y^*))$ at commodity prices $\check{p}^* + \varepsilon d\check{p}$ and asset prices q^* Pareto dominates the competitive equilibrium. Uniform price regulations are of interest because they imply a state-independent change in prices. It shows that relatively simple policies suffice for the realization of a Pareto improvement [3]. Generically, it is possible to make every individual better off by choosing appropriate price regulations on the spot markets when asset markets are incomplete. One needs at least as many instruments, LS, as individuals, I. Uniform price regulation is effective when $L \geq I$, which reflects again that the number of instruments has to exceed the number of objectives. This complements the constrained suboptimality result of Geanakoplos and Polemarchakis (1986), which applies when $2L \leq I \leq L(S-1)+1$.

Proposition 1. *Suppose that*

1. *For every individual, the consumption set is* $\mathcal{X}^i = \{x : x \geq 0\}$; *the utility function is continuous and quasi - concave; in the interior of the consumption set,*[4] $\text{Int } \mathcal{X}^i$, *it is twice continuously differentiable,* $\partial u^i \gg 0$ *and* $\partial^2 u^i$ *is negative definite on*[5] $(\partial u^i)^{\perp}$; *the endowment is strictly positive:* $e^i \in \text{Int } \mathcal{X}^i$, *and* $u^i(e^i) > u^i(x)$, *for every* $x \in \text{Bd } \mathcal{X}^i$.
2. *The matrix of payoffs of assets has full column rank. The numeraire asset, has positive payoff:* $r_{A+1} > 0$.

If Assumption 1 holds, the asset market is incomplete ($A+1 < S$)*, and* $LS \geq I > 1$*, then, generically, price regulation can improve upon any competitive equilibrium. If* $L \geq I > 1$*, then, generically, uniform price regulation can improve upon any competitive equilibrium.*

Appendix 3 gives the proof, which follows Geanakoplos and Polemarchakis (1986) and Citanna, Kajii and Villanacci (1998). In the paper we focus on the adjustment of prices of state-contingent commodities, rather than asset prices. One reason for this is that government interventions in commodity prices seem to occur much more frequently than government control of asset prices. From a purely normative point of view, the case where the central planner is limited to adjustments of asset prices only, with competitive spot commodity markets in future periods, is interesting as well. It provides an anonymous alternative to the adjustment of individual asset portfolios as proposed in Geanakoplos and Polemarchakis (1986). A final issue of interest is the

[3] John Geanakoplos and Hamid Sabourian insisted on this point.
[4] "Int" denotes the interior of a set and "Bd" the boundary.
[5] "$\perp$" denotes the orthogonal complement.

case where asset prices adjust to clear markets after the central planner intervened in spot commodity markets. The current analysis fixes asset prices at q^*, which allows us to derive a simple expression for the derivative with respect to spot commodity prices of the indirect utility function in Lemma 2. The corresponding expression for an indirect utility function where asset prices adjust to clear the asset markets is more complicated, and the question whether Pareto improving price regulation is possible in that sense is open.

Appendix 1: existence of fix-price equilibria

A compact, convex subset of the consumption set that contains the aggregate initial endowment in its interior is $\widetilde{\mathcal{X}}^i$ The assumptions on utility functions and on the asset return matrix imply that all $S+1$ budget inequalities in the definition of the budget set hold with equality at the optimal choice of an individual. The rationing inequalities do not necessarily hold with equality. The budget set related to $\widetilde{\mathcal{X}}^i$ with all budget inequalities required to hold with equality is $\widetilde{\beta}^i$ and the corresponding demand function $\widetilde{d}^i$. Since prices are fixed at (p,q), they are omitted in the notation. The demand functions $\widetilde{d}^i$, $i \in \mathcal{I}$, are continuous. If $(\underline{z}_n, \overline{z}_n, \underline{y}_n, \overline{y}_n)$ is a sequence that converges to $(\underline{z}, \overline{z}, \underline{y}, \overline{y})$, then the sequence $(\widetilde{d}^i(\underline{z}_n, \overline{z}_n, \underline{y}_n, \overline{y}_n))$ has a convergent subsequence, with limit $(\widehat{x}, \widehat{y}) \in \widetilde{\beta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$. If there exists $(\widetilde{x}, \widetilde{y}) \in \widetilde{\beta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$, such that $u^i(\widetilde{x}) > u^i(\widehat{x})$, and $\widetilde{\mathcal{L}}_-$, $\widetilde{\mathcal{L}}_+$, $\widetilde{\mathcal{A}}_-$, and $\widetilde{\mathcal{A}}_+$, is the sets of non-numéraire commodities and non-numéraire assets for which $\widetilde{x}_{l,s} - e^i_{l,s}$ is negative, positive, $\widetilde{y}_a$ is negative, and positive, respectively, then, for

$$\lambda^n = \min\left\{1, \frac{\underline{z}^n_{l,s}}{\widetilde{x}_{l,s} - e^i_{l,s}}, (l,s) \in \widetilde{\mathcal{L}}_-, \frac{\overline{z}^n_{l,s}}{\widetilde{x}_{l,s} - e^i_{l,s}}, (l,s) \in \widetilde{\mathcal{L}}_+, \frac{\underline{y}^n_a}{\widetilde{y}_a}, a \in \widetilde{\mathcal{A}}_-, \frac{\overline{y}^n_a}{\widetilde{y}_a}, a \in \widetilde{\mathcal{A}}_+\right\},$$

$\widetilde{x}^n = e^i + \lambda^n(\widetilde{x} - e^i)$, and $\widetilde{y}^n = \lambda^n \widetilde{y}$, $(\widetilde{x}^n, \widetilde{y}^n) \in \widetilde{\beta}^i(\underline{z}^n, \overline{z}^n, \underline{y}^n, \overline{y}^n)$. Evidently, $\lim_{n\to\infty} \lambda^n = 1$, and $\lim_{n\to\infty}(\widetilde{x}^n, \widetilde{y}^n) = (\widetilde{x}, \widetilde{y})$. By the continuity of u^i, $\widetilde{x}^n$ is strictly preferred to the consumption bundle in $\widetilde{d}^i(\underline{z}_n, \overline{z}_n, \underline{y}_n, \overline{y}_n)$, a contradiction. Since there is no rationing in the market of the numéraire asset nor in the market of the numéraire commodities, the argument for equilibrium existence is not trivial. If $((x^*, y^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*))$ is a fix-price equilibrium at prices (p,q), then $x^{*i'}_{l,s} < \sum_{i=1}^I e^i_{l,s} + \varepsilon$, with ε some fixed positive number. Since R has full column rank, this implies that there is $\alpha > 0$ such that $\|y^{*i}\|_\infty < \alpha$ for any y^{*i} consistent with a fix-price equilibrium at prices (p,q). The functions $(\underline{z}, \overline{z}) : \mathcal{C}^{LS} \to -\mathbb{R}^{LS}_+ \times \mathbb{R}^{LS}_+$ and $(\underline{y}, \overline{y}) : \mathcal{C}^A \to -\mathbb{R}^A_+ \times \mathbb{R}^A_+$, where $\mathcal{C}^K = \{r \in \mathbb{R}^K : 0 \leq r_k \leq 1\}$ is the unit cube of dimension K, are defined by

$$\underline{z}_{l,s}(r) = -\min\left\{2r_{l,s}\left(\sum_{i=1}^{I} e^i_{l,s} + \varepsilon\right), \sum_{i=1}^{I} e^i_{l,s} + \varepsilon\right\}, \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$\overline{z}_{l,s}(r) = \min\left\{(2 - 2r_{l,s})\left(\sum_{i=1}^{I} e^i_{l,s} + \varepsilon\right), \sum_{i=1}^{I} e^i_{l,s} + \varepsilon\right\}, \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$\underline{y}_a(\rho) = -\min\{2\rho_a\alpha, \alpha\}, \quad a \in \check{\mathcal{A}},$$

$$\overline{y}_a(\rho) = \min\{(2 - 2\rho_a)\alpha, \alpha\}, \quad a \in \check{\mathcal{A}}.$$

The excess demand function $\widetilde{z} : \mathcal{C}^{LS} \times \mathcal{C}^A \rightarrow \mathbb{R}^{LS} \times \mathbb{R}^A$ is

$$\widetilde{z}_{l,s}(r,\rho) = \sum_{i=1}^{I} \widetilde{d}^i_{l,s}(\underline{z}(r), \overline{z}(r), \underline{y}(\rho), \overline{y}(\rho)) - \sum_{i=1}^{I} e^i_{l,s}, \quad (l,s) \in \check{\mathcal{L}} \times S$$

$$\widetilde{z}_a(r,\rho) = \sum_{i=1}^{I} \widetilde{d}^i_a(\underline{z}(r), \overline{z}(r), \underline{y}(\rho), \overline{y}(\rho)), \quad a \in \check{\mathcal{A}}.$$

If $(r^*, \rho^*) \in \mathcal{C}^{LS} \times \mathcal{C}^A$ is such that $\widetilde{z}(r^*, \rho^*) = 0$, then $((x^*, y^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*))$, where $(x^{*i}, y^{*i}) = \widetilde{d}^i(\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), i \in \mathcal{I}$, $(\underline{z}^*, \overline{z}^*) = (\underline{z}(r^*), \overline{z}(r^*))$, $(\underline{y}^*, \overline{y}^*) = (\underline{y}(r^*), \overline{y}(r^*))$, is a fix-price equilibrium. Conditions 1 and 2 of Definition 1 are satisfied for non-numeraire commodities and assets. The construction of the functions $(\underline{z}, \overline{z})$ and $(\underline{y}, \overline{y})$ takes care of Conditions 3 and 4. The set $\widetilde{z}(\mathcal{C}^{LS} \times \mathcal{C}^A)$ is compact. Let the set $\mathcal{ZY}$ be a compact, convex set that contains $\widetilde{z}(\mathcal{C}^{LS} \times \mathcal{C}^A)$. The correspondence $\mu : \mathcal{ZY} \rightarrow \mathcal{C}^{LS} \times \mathcal{C}^A$ is defined by

$$\mu(z,y) = \arg\max\left\{\sum_{(l,s)\in\check{\mathcal{L}}\times\mathcal{S}} r_{l,s} z_{l,s} + \sum_{a\in\check{\mathcal{A}}} \rho_a y_a : r \in \mathcal{C}^{LS}, \rho \in \mathcal{C}^A\right\}.$$

The correspondence $\varphi : \mathcal{ZY} \times \mathcal{C}^{LS} \times \mathcal{C}^A \rightarrow \mathcal{ZY} \times \mathcal{C}^{LS} \times \mathcal{C}^A$ is defined by $\varphi(z,y,r,\rho) = \{\widetilde{z}(r,\rho)\} \times \mu(z,y)$. It is a non-empty, compact, convex valued, upper hemi-continuous correspondence defined on a non-empty, compact, convex set. By Kakutani's fixed point theorem, φ has a fixed point, (z^*, y^*, r^*, ρ^*). If, for some $a \in \check{\mathcal{A}}$, $y^*_a < 0$, then, by the definition of μ, $\rho^*_a = 0$, so $y^*_a \geq 0$, a contradiction. If, for some $a \in \check{\mathcal{A}}$, $y^*_a > 0$, then, by the definition of μ, $\rho^*_a = 1$, so $y^*_a \leq 0$, a contradiction. Consequently, $y^*_a = 0$, for all $a \in \check{\mathcal{A}}$. Moreover, $y^*_{A+1} = -\sum_{a\in\check{\mathcal{A}}} q_a y^*_a = 0$. If, for some $(l,s) \in \check{\mathcal{L}} \times \mathcal{S}$, $z^*_{l,s} < 0$, then, by the definition of μ, $r^*_{l,s} = 0$, so $z^*_{l,s} \geq 0$, a contradiction. If, for some $(l,s) \in \check{\mathcal{L}} \times \mathcal{S}$, $z^*_{l,s} > 0$, then, by the definition of μ, $r^*_{l,s} = 1$, so $z^*_{l,s} \leq 0$, a contradiction. Consequently, $z^*_{l,s} = 0$, for all $(l,s) \in \check{\mathcal{L}} \times \mathcal{S}$. Moreover, for every $s \in \mathcal{S}$, $z^*_{L+1,s} = -\sum_{(l,s)\in\check{\mathcal{L}}\times\mathcal{S}} p_{l,s} z^*_{l,s} + R_s.y^* = 0$. It follows that $0 \in \widetilde{z}(r^*, \rho^*)$, so a fix-price equilibrium at prices (p, q) exists.

Appendix 2: local comparative statics

In the optimization problem an individual faces when determining his demand, the Lagrange multipliers corresponding to the rationing constraints in the markets for commodities (assets) are π (ρ). The individual optimization problem leads to the study of a modified demand function, $\widehat{d^i}$. At prices and Lagrange multipliers (p,q,π,ρ), $\widehat{d^i}$ is defined by the solution to the optimization problem

$$\max u^i(x) - \sum_{(l,s)\in\check{\mathcal{L}}\times\mathcal{S}} \pi_{l,s}x_{l,s} - \sum_{a\in\check{\mathcal{A}}} \rho_a y_a,$$

$$\text{s.t. } qy \leq 0,$$

$$p_s(x_s - e^i_s) \leq R_s.y, \quad s \in \mathcal{S}.$$

The set of (p,q,π,ρ) on which each individual optimization problem has a solution is $\mathcal{N}$. Whenever (p^*,q^*) are competitive equilibrium prices, $\mathcal{N}$ is a neighborhood of $(p^*,q^*,0,0)$. The modified demand function, $\widehat{d^i}$, $i \in \mathcal{I}$, is continuously differentiable on $\mathcal{N}$. At a competitive equilibrium, $((p^*,q^*),(x^*,y^*))$, $z^-_{l,s}$, $z^+_{l,s}$, y^-_a and y^+_a, defined by

$$z^-_{l,s} = \min_{i\in\mathcal{I}} x^{i*}_{l,s} - e^i_{l,s},\; z^+_{l,s} = \max_{i\in\mathcal{I}} x^{i*}_{l,s} - e^i_{l,s}, \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$y^-_a = \min_{i\in\mathcal{I}} y^{i*}_a, \qquad y^+_a = \max_{i\in\mathcal{I}} y^{i*}_a, \qquad a \in \check{\mathcal{A}},$$

determine the minimal and the maximal excess demands on both the spot and the asset markets. If

$$\underline{\mathcal{I}}_{l,s} = \left\{ i \in \mathcal{I} : x^{i*}_{l,s} - e^i_{l,s} = z^-_{l,s} \right\}, \; (l,s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$\overline{\mathcal{I}}_{l,s} = \left\{ i \in \mathcal{I} : x^{i*}_{l,s} - e^i_{l,s} = z^+_{l,s} \right\}, \; (l,s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$\underline{\mathcal{I}}_a = \left\{ i \in \mathcal{I} : y^{i*}_a = y^-_a \right\}, \qquad a \in \check{\mathcal{A}},$$

$$\overline{\mathcal{I}}_a = \left\{ i \in \mathcal{I} : y^{i*}_a = y^+_a \right\}, \qquad a \in \check{\mathcal{A}},$$

then, in a neighborhood of the competitive equilibrium, only individuals in $\underline{\mathcal{I}}_{l,s}$ $(\overline{\mathcal{I}}_{l,s})$ may be rationed on supply (demand) in the spot market (l,s), and only individuals in $\underline{\mathcal{I}}_a$ $(\overline{\mathcal{I}}_a)$ on supply (demand) in the asset market a. For an open set of endowments with full Lebesgue measure $\Omega \subset \mathbb{R}^{I(L+1)S}_{++}$, for any competitive equilibrium $((p^*,q^*),(x^*,y^*))$ of $\mathcal{E}$, $\mid \underline{\mathcal{I}}_{l,s} \mid = \mid \overline{\mathcal{I}}_{l,s} \mid = 1$, $(l,s) \in \check{\mathcal{L}} \times \mathcal{S}$, and $\mid \underline{\mathcal{I}}_a \mid = \mid \overline{\mathcal{I}}_a \mid = 1$, $a \in \check{\mathcal{A}}$. For a generic set of economies, there is exactly one individual in each market with the minimal excess demand and exactly one individual with the maximal excess demand. For the remainder, the allocation of endowments in the economy $\mathcal{E}$ belong to the set Ω, which permits the study of the local structure of the set fix-price equilibria in the neighborhood of a competitive equilibrium $((p^*,q^*),(x^*,y^*))$ of $\mathcal{E}$. For every individual, the function $c^i : \mathbb{R}^{LS} \times \mathbb{R}^A \to \mathbb{R}^{LS} \times \mathbb{R}^A$ is defined by

$$c^i_{l,s}(\pi,\rho) = \begin{cases} \pi_{l,s}, \text{ if } \pi_{l,s} \leq 0 \text{ and } \{i\} = \underline{\mathcal{I}}_{l,s} \\ \qquad \text{or } \pi_{l,s} \geq 0 \text{ and } \{i\} = \overline{\mathcal{I}}_{l,s}, \\ 0, \quad \text{otherwise,} \end{cases}$$

$$c^i_a(\pi,\rho) \;\; = \begin{cases} \rho_a, \text{ if } \rho_a \leq 0 \text{ and } \{i\} = \underline{\mathcal{I}}_a \\ \qquad \text{or } \rho_a \geq 0 \text{ and } \{i\} = \overline{\mathcal{I}}_a, \\ 0, \quad \text{otherwise.} \end{cases}$$

The function c associates with Lagrange multipliers, (π,ρ), fix-price equilibria in the neighborhood of the competitive equilibrium. The aggregate modified excess demand function for commodities and assets other than the numéraire is $\widehat{z} : \mathcal{N} \rightarrow \mathbb{R}^{LS+A}$ defined by

$$\widehat{z}_{l,s}(p,q,\pi,\rho) = \textstyle\sum_{i\in\mathcal{I}} \widehat{d^i_{l,s}}(p,q,c^i(\pi,\rho)) - \sum_{i\in\mathcal{I}} e^i, \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$\widehat{z}_a(p,q,\pi,\rho) \;\; = \textstyle\sum_{i\in\mathcal{I}} \widehat{d^i_a}(p,q,c^i(\pi,\rho)), \quad a \in \check{\mathcal{A}}.$$

For the study of fix-price equilibria in the neighborhood of the competitive equilibrium, it is sufficient to restrict attention to the zero points of $\widehat{z}$. Neighborhoods $\mathcal{N}^i_{x^{i*},y^{i*}}$ are such that, for every $(x,y) \in \mathcal{N}_{x^*,y^*} = \times_{i\in\mathcal{I}} \mathcal{N}^i_{x^{i*},y^{i*}}$, for all $(l,s) \in \check{\mathcal{L}} \times \mathcal{S}$, for all $a \in \check{\mathcal{A}}$,

$$x^{i'}_{l,s} - e^{i'}_{l,s} < 0 \text{ and } x^{i'}_{l,s} - e^{i'}_{l,s} < x^i_{l,s} - e^i_{l,s},\; i \neq i', i' \in \underline{\mathcal{I}}_{l,s}$$

$$x^{i'}_{l,s} - e^{i'}_{l,s} > 0 \text{ and } x^{i'}_{l,s} - e^{i'}_{l,s} > x^i_{l,s} - e^i_{l,s},\; i \neq i', i' \in \overline{\mathcal{I}}_{l,s}$$

$$y^{i'}_a < 0 \text{ and } y^{i'}_a < y^i_a, \qquad i \neq i', i' \in \underline{\mathcal{I}}_a$$

$$y^{i'}_a > 0 \text{ and } y^{i'}_a > y^i_a, \qquad i \neq i', i' \in \overline{\mathcal{I}}_a.$$

If $((p^*,q^*),(x^*,y^*))$ is a competitive equilibrium, and $(x,y) \in \mathcal{N}_{x^*,y^*}$, then $(x,y) \in \mathcal{D}(p,q)$ if and only if there is $(p,q,\pi,\rho) \in \mathcal{N}$ such that $\widehat{d^i}(p,q,c^i(\pi,\rho)) = (x^i,y^i)$, $i \in \mathcal{I}$, and $\widehat{z}(p,q,\pi,\rho) = (0,0)$. The function $\widehat{z}$ is Lipschitz continuous because of the differentiability of the functions $\widehat{d^i}$ and the Lipschitz continuity of the functions c^i. It is differentiable at each $(p,q,\pi,\rho) \in \mathcal{N}$ where all components of π and ρ are non-zero. For each sign vector r without zero components,

$$\mathcal{N}^r = \left\{(p,q,\pi,\rho) \in \mathcal{N} : \pi_{l,s} r_{l,s} > 0, (l,s) \in \check{\mathcal{L}} \times \mathcal{S},\; \rho_a r_a > 0, a \in \check{\mathcal{A}}\right\}.$$

The function $\widehat{z}$ is differentiable on $\mathcal{N}^r$. The limit of its Jacobian, $\lim_{n\to\infty} \partial\widehat{z}$ $(p^n,q^n,\pi^n,\rho^n) = \partial\widehat{z}^r(p^*,q^*,0,0)$, along a sequence $((p^n,q^n,\pi^n,\rho^n) \in \mathcal{N}^r)$ that converges to $(p^*,q^*,0,0)$, exists;

$$\partial_{\check{p},\check{q}}\widehat{z}^r_{l,s}(p^*,q^*,0,0) = \sum_{i\in\mathcal{I}}\partial_{\check{p},\check{q}}\widehat{d}^i_{l,s}(p^*,q^*,0,0) = \partial_{\check{p},\check{q}}z_{l,s}(p^*,q^*),$$

$$\partial_{\check{p},\check{q}}\widehat{z}^r_a(p^*,q^*,0,0) \ = \sum_{i\in\mathcal{I}}\partial_{\check{p},\check{q}}\widehat{d}^i_a(p^*,q^*,0,0) \ = \partial_{\check{p},\check{q}}z_a(p^*,q^*),$$

where $z(p,q)$ denotes the unconstrained total excess demand function for commodities and assets other than the numeraires at prices (p,q). It follows that the Jacobian with respect to $(\check{p},\check{q})$ is independent of r at a competitive equilibrium.

Lemma 1. *If $((p^*,q^*),(x^*,y^*))$ is a competitive equilibrium, such that $\partial z\ (p^*,q^*)$ is of full rank, then, for each sign vector r without zero components, the tangent cone at (p^*,q^*) to the set of price systems having a local fix-price equilibrium with state of the markets r is*

$$\big\{(p,q)\in\mathcal{P}\times\mathcal{Q} : (\check{p},\check{q}) = (\partial z(p^*,q^*))^{-1}\partial_{\pi,\rho}\widehat{z}^r(p^*,q^*,0,0)(\pi,\rho),$$

$$\pi_{l,s}r_{l,s} > 0, (l,s)\in\check{\mathcal{L}}\times\mathcal{S},\ \rho_a r_a > 0, a\in\check{\mathcal{A}}\big\}.$$

Proof.. The restriction of $\widehat{z}$ to $\mathcal{N}^r$ extends to a differentiable function $\widetilde{z} : \mathcal{N} \to \mathbb{R}^{LS+A}$ as follows. For $i\in\mathcal{I}$, the function $\widetilde{c}^i$ is defined by $\widetilde{c}^i_{l,s}(\pi,\rho) = \pi_{l,s}$ if $i\in\underline{\mathcal{I}}_{l,s}$, $r_{l,s} = -1$, or $i\in\overline{\mathcal{I}}_{l,s}$, $r_{l,s} = +1$, $\widetilde{c}^i_{l,s}(\pi,\rho) = 0$ otherwise, and $\widetilde{c}^i_a(\pi,\rho) = \rho_a$ if $i\in\underline{\mathcal{I}}_a$, $r_a = -1$, or $i\in\overline{\mathcal{I}}_a$, $r_a = +1$, and $\widetilde{c}^i_a(\pi,\rho) = 0$ otherwise. The function $\widetilde{z}$ is defined as $\widehat{z}$ with c replaced by $\widetilde{c}$. Since $\partial z(p^*,q^*)$ is of full rank, it follows by the implicit function theorem that the solution to $\widetilde{z}(p,q,\pi,\rho) = (0,0)$ determines p and q as a function of π and ρ in a neighborhood of $(0,0)$. The derivative of this function at $(0,0)$ with respect to π and ρ is given by $(\partial z(p^*,q^*))^{-1}\partial_{\pi,\rho}\widetilde{z}(p^*,q^*,0,0)$. The expression in the proposition follows immediately if one takes into account that only π's and ρ's satisfying $\pi_{l,s}r_{l,s} > 0$, $(l,s)\in\check{\mathcal{L}}\times\mathcal{S}$, and $\rho_a r_a > 0$, $a\in\check{\mathcal{A}}$, should be considered. □

Proposition 2 in Geanakoplos and Polemarchakis (1986) shows that the assumption that $\partial z(p^*,q^*)$ has full rank at every competitive equilibrium holds generically in endowments. Lemma 3.2 characterizes the tangent cones to the regions in the price space having a fix-price equilibrium with state of the markets r in the neighborhood of a competitive equilibrium. It guarantees neither that the closures of these tangent cones cover the price space nor that the tangent cones are full-dimensional nor that the tangent cones do not intersect. If this were the case, local uniqueness would result. In general, an increase in a price causes a different individual to be rationed as a decrease in a price. Since $\partial_{\pi,\rho}\widehat{z}^r$, and therefore the tangent cone, depend on $\partial_{\pi,\rho}\widehat{d}^i$ for the individual i that is rationed, the fact that the tangent cones need not fit nicely together does not come as a surprise. In abstract terms, the fact that different individuals get rationed at different prices in the neighborhood of a competitive equilibrium, creates non-differentiabilities in the function $\widehat{z}$ at competitive prices. At a point of non-differentiability, the implicit function theorem need not apply, and local uniqueness may fail. The generalized Jacobian of a Lipschitz continuous function f at a point x is the convex hull of all matrices that are the limits of the sequence $(\partial f(x^n))$, where (x^n) is a convergent sequence with $\lim_{n\to\infty} x^n = x$

and f is differentiable at x^n. A restriction of the fundamentals of the economy, the utility functions of individuals and the matrix of asset payoffs is required to guarantee that, generically, competitive equilibrium allocations are differentiably locally unique as fix-price equilibrium allocations. If a function f is Lipschitz continuous, $f(\widehat{x},\widehat{y}) = 0$, and every matrix M in $\partial_x f(\widehat{x},\widehat{y})$ has full rank, then there exist a neighborhood $\mathcal{N}_{\widehat{x},\widehat{y}}$, a neighborhood $\mathcal{N}_{\widehat{y}}$, and a Lipschitz continuous function g on $\mathcal{N}_{\widehat{y}}$ such that $(x,y) \in \mathcal{N}_{\widehat{x},\widehat{y}}$ and $f(x,y) = 0$ if and only if $y \in \mathcal{N}_{\widehat{y}}$ and $x = g(y)$.

Assumption 2. *For endowments in Ω^*, an open set of full Lebesgue measure, if $((p^*,q^*),\ (x^*,y^*))$ is a competitive equilibrium, then the determinants of the matrices $\partial_{\pi,\rho}\widehat{z}^r(p^*,q^*,0,0)$, with r sign vectors without zero components, are either all equal to -1 or all equal to $+1$.*

By an argument similar to the one in the proof of Theorem 1, Laroque (1981), the competitive equilibrium allocation is differentiably locally unique as a fix-price equilibrium allocation.

Remark. An example of an economy that satisfies differentiable local uniqu- eness for all endowments and, à fortiori, satisfies Assumption 2. There are three states of the world, two commodities and two assets. The utility functions have an additively separable representation $u^i = \sum_{s\in\mathcal{S}} \pi_s u^i_s$ with

$$u^i_s(x_s) = \alpha^i \ln x_{1,s} + (1-\alpha^i)x_{2,s}, \quad 0 < \alpha^i < 1,$$

and a uniform probability measure π over the states of the world. The payoffs of the assets are $R_{\cdot 1} = (1,0,0)'$, and $R_{\cdot 2} = (0,1,0)'$. Endowments are chosen such that $\mid \underline{\mathcal{I}}_{l,s} \mid = \mid \overline{\mathcal{I}}_{l,s} \mid = 1$, $(l,s) \in \check{\mathcal{L}} \times \mathcal{S}$, and $\mid \underline{\mathcal{I}}_a \mid = \mid \overline{\mathcal{I}}_a \mid = 1$, $a \in \check{\mathcal{A}}$, so they belong to a set of full Lebesgue measure by Lemma 4.4. Competitive equilibrium prices are (p^*,q^*). All partial derivatives are evaluated at $(p^*,q^*,0,0)$. It holds that $\partial_{\pi_{1,s}}\widehat{z}^r = \partial_{\pi_{1,s}}\widehat{d}^{i(1,s)}$, where $\{i(1,s)\} = \underline{I}_{1,s}$ if $r_{1,s} = -1$, and $\{i(1,s)\} = \overline{I}_{1,s}$ if $r_{1,s} = +1$. An increase in $\pi_{1,s}$ corresponds to the introduction of demand rationing or the disappearance of supply rationing on commodity $(1,s)$, which decreases the demand for commodity $(1,s)$, so $\partial_{\pi_{1,s}}\widehat{z}^r_{1,s}$ is negative. The change in income spent on commodity $(1,s)$ equals $p^*_{1,s}\partial_{\pi_{1,s}}\widehat{z}^r_{1,s}$. The individual $i(1)$ is the one affected by rationing in the asset market, so $\{i(1)\} = \underline{I}_1$ if $r_1 = -1$, and $\{i(1)\} = \overline{I}_1$ if $r_1 = +1$. From the properties of the Cobb-Douglas utility function, it follows that

$$\partial_{\pi_{1,1}} \widehat{d}^{i(1,1)}_{1,2} = \frac{-\alpha_1^{i(1,1)} p^*_{1,1} q^*_1 \partial_{\pi_{1,1}} \widehat{z}^r_{1,1}}{p^*_{1,2} q^*_2 \left(2-\alpha_1^{i(1,1)}\right)},$$

$$\partial_{\pi_{1,1}} \widehat{d}^{i(1,1)}_{1,3} = 0, \quad \partial_{\pi_{1,1}} \widehat{d}^{i(1,1)}_{1} = \frac{p^*_{1,1} \partial_{\pi_{1,1}} \widehat{z}^r_{1,1}}{\left(2-\alpha_1^{i(1,1)}\right)},$$

$$\partial_{\pi_{1,2}} \widehat{d}^{i(1,2)}_{1,1} = \frac{-\alpha_1^{i(1,2)} p^*_{1,2} q^*_1 \partial_{\pi_{1,2}} \widehat{z}^r_{1,2}}{p^*_{1,1} q^*_2 \left(2-\alpha_1^{i(1,2)}\right)},$$

$$\partial_{\pi_{1,2}} \widehat{d}^{i(1,2)}_{1,3} = 0, \quad \partial_{\pi_{1,2}} \widehat{d}^{i(1,2)}_{1} = \frac{-p^*_{1,2} q^*_2 \partial_{\pi_{1,2}} \widehat{z}^r_{1,2}}{q^*_1 \left(2-\alpha_1^{i(1,2)}\right)},$$

$$\partial_{\pi_{1,3}} \widehat{d}^{i(1,3)}_{1,1} = 0, \quad \partial_{\pi_{1,3}} \widehat{d}^{i(1,3)}_{1,2} = 0, \quad \partial_{\pi_{1,3}} \widehat{d}^{i(1,3)}_{1} = 0,$$

$$\partial_{\rho_1} \widehat{d}^{i(1)}_{1,1} = \frac{\alpha_1^{i(1)} \partial_{\rho_1} \widehat{z}^r_1}{p^*_{1,1}}, \quad \partial_{\rho_1} \widehat{d}^{i(1)}_{1,2} = \frac{-\alpha_1^{i(1)} q^*_1 \partial_{\rho_1} \widehat{z}^r_1}{p^*_{1,2} q^*_2}, \quad \partial_{\rho_1} \widehat{d}^{i(1)}_{1,3} = 0.$$

The sign of the determinant of $\partial_{\pi,\rho}\widehat{z}^r$ does not change by premultiplying it by the strictly positive row vector $(p^*_{1,1}q^*_1, p^*_{1,2}q^*_2, 1, q^*_1)$ and postmultiplying it by the strictly positive column vector $\left(\left(2-\alpha_1^{i(1,1)}\right) / -p^*_{1,1}q^*_1\partial_{\pi_{1,1}}\widehat{z}^r_{1,1}, \left(2-\alpha_1^{i(1,2)}\right) / -p^*_{1,2}q^*_2\partial_{\pi_{1,2}}\widehat{z}^r_{1,2}, 1/-\widehat{z}^r_{1,3}, 1/-q^*_1\partial_{\rho_1\widehat{z}^r_1}\right)'$. The resulting matrix is given by

$$\begin{bmatrix} \alpha_1^{i(1,1)}-2 & \alpha_1^{i(1,2)} & 0 & -\alpha_1^{i(1)} \\ \alpha_1^{i(1,1)} & \alpha_2^{i(1,2)}-2 & 0 & \alpha_1^{i(1)} \\ 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & -1 \end{bmatrix}$$

and its determinant equals

$$\left(4-2\alpha_1^{i(1,1)}-2\alpha_1^{i(1,2)}\right)\left(1-\alpha_1^{i(1)}\right) > 0.$$

The determinant of $\partial_{\pi,\rho}\widehat{z}^r$ is positive, irrespective of the sign vector r. It follows that the competitive equilibrium allocation is differentiably locally unique as a fix-price equilibrium allocation. □

As in Laroque (1981), whenever there are two sign vectors without zero components r^1 and r^2 such that the determinants of $\partial_{\pi,\rho}\widehat{z}^{r^1}(p^*, q^*, 0, 0)$ and $\partial_{\pi,\rho}\widehat{z}^{r^2}$ $(p^*, q^*, 0, 0)$ have opposite signs and $\partial z(p^*, q^*)$ has full rank, then for every neighborhood $\mathcal{N}_{x^*,y^*}$ there exists for every neighborhood $\mathcal{N}_{p^*,q^*}$ a price system $(p, q) \in \mathcal{N}_{p^*,q^*}$ with at least two fix-price equilibrium allocations in $\mathcal{N}_{x^*,y^*}$. Assumption 2 is "almost necessary" for the differentiable local uniqueness of competitive equilibrium allocations. Local uniqueness of fix-price equilibrium allocations at competitive equilibria is not too strong a requirement. It is less demanding than the requirement of uniqueness of fix-price equilibrium allocations at prices in a neighborhood of competitive prices. The latter requirement guarantees a certain degree of generality of our results. The function $(\widehat{x}, \widehat{y}) : \mathcal{N}_{p^*,q^*} \rightarrow \mathbb{R}^{I(L+1)S+I(A+1)}$ associates the unique fix-price equilibrium allocation in $\mathcal{N}_{x^*,y^*}$ to $(p, q) \in \mathcal{N}_{p^*,q^*}$. The indirect utility function of an individual at a locally unique fix-price equilibrium is defined by

$$v^i(p,q) = u^i\left(\widehat{x}^i(p,q)\right), \quad (p,q) \in \mathcal{N}_{p^*,q^*}.$$

Lemma 2. *If* $((p^*, q^*), (x^*, y^*))$ *is a competitive equilibrium, then the indirect utility function* $v^i : \mathcal{N}_{p^*,q^*} \rightarrow \mathbb{R}$ *is differentiable and*

$$\partial_{p_{l,s}} v^i(p^*, q^*) = -\partial_{x_{L+1,s}} u^i\left(x^{i*}\right)\left(x^{i*}_{l,s} - e^i_{l,s}\right), \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S}.$$

Proof.. For every sign vector r, the restriction of v^i to $\mathcal{N}_{p^*,q^*} \cap \mathcal{PQ}(r)$, denoted v^{i^r}, is differentiable. From the differentiation of the budget constraints

$$q\widehat{y}^{i^r}(p,q) = 0 \text{ and } p_s\left(\widehat{x}^{i^r}_s(p,q) - e^i_s\right) = R_s.\widehat{y}^{i^r}(p,q), \quad s \in \mathcal{S},$$

with respect to $p_{\bar{l},\bar{s}}$, and the first order conditions for individual optimization at a competitive equilibrium,

$$\partial_{x^i_{l,s}} u^i\left(x^{i*}\right) = \partial_{x^i_{L+1,s}} u^i\left(x^{i*}\right) p^*_{l,s}, \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S},$$

and

$$\sum_{s \in \mathcal{S}} \partial_{x^i_{L+1,s}} u^i\left(x^{i*}\right) R_{s.} = \mu^i q^*, \quad \text{for some } \mu^i > 0,$$

it follows that

$$\partial_{p_{\bar{l},\bar{s}}} v^{i^r}(p^*, q^*) = -\partial_{x^i_{L+1,\bar{s}}} u^i\left(x^{i*}\right)\left(x^{i*}_{\bar{l},\bar{s}} - e^i_{\bar{l},\bar{s}}\right).$$

Since the derivative is independent of the sign vector r, the result follows. □

Appendix 3: Pareto improving price regulation

Price regulation can Pareto improve on a competitive equilibrium $((p^*, q^*), (x^*, y^*))$ if there exist prices of commodities p such that a fix-price equilibrium of commodities at prices of commodities and assets (p, q^*) Pareto dominates the allocation x^*. The ambiguity introduced by the possibility of multiple fix-price equilibrium allocations at prices (p, q^*) is circumvented by considering local variations at competitive equilibrium allocations that are differentiably locally unique as fix-price equilibria. Pareto improvement by price regulation is possible only if the asset market is incomplete. Another necessary requirement is that the economy allows for heterogeneous individuals. **Assumption 3.** $A + 1 < S$ *and* $I > 1$. The function φ is defined by

$$\varphi(x,\widetilde{\lambda},\widetilde{p},e)=\begin{pmatrix}\partial u^i\left(x^i\right)-\widetilde{\lambda}^i\widetilde{p}, \quad i\in\mathcal{I}\\ \sum_{s\in\mathcal{S}}\widetilde{p}_s\left(x^i_s-e^i_s\right), \quad i\in\mathcal{I}\\ \sum_{i\in\mathcal{I}}\left(x^i_{l,s}-e^i_{l,s}\right), \quad (l,s)\in\mathcal{L}\times\mathcal{S}\setminus\{(L+1,S)\}\\ \sum_{s\in\mathcal{S}}n_s\widetilde{p}_s\left(x^i_s-e^i_s\right), \quad i\in\mathcal{I}\setminus\{1\}\end{pmatrix},$$

where the Lagrangian multiplier $\widetilde{\lambda}^i\in\mathbb{R}$ does not vary with the state of the world, the prices of commodities $\widetilde{p}\in\mathbb{R}^{(L+1)S-1}_{++}\times\{1\}$ are discounted prices, with only the price of commodity $(L+1,S)$ normalized to 1, and $n\neq 0$ is a fixed vector such that $nR=0$. In the standard reformulation of the incomplete markets model in discounted prices, it is assumed that one individual is unconstrained, so that his marginal utility at an optimal choice is proportional to the price system. Pareto optimality implies that the marginal utility vectors of all agents should be proportional to the price system. The function φ is completed by specifying budget constraints and market clearing conditions, and one condition for every individual but one that recognizes the incompleteness of markets: $\sum_{s\in\mathcal{S}}n_s\widetilde{p}_s(x^i_s-e^i_s)=0$. The existence of $n\neq 0$ such that $nR=0$ follows from market incompleteness. It follows that the function φ vanishes at a Pareto optimal competitive equilibrium. For a function f that depends on a vector of variables α and on endowments e, $f_e(\alpha)$ is the function that results from fixing e; for instance, $\varphi_e(x,\widetilde{\lambda},\widetilde{p})=\varphi(x,\widetilde{\lambda},\widetilde{p},e)$.

Lemma 3. *Generically, competitive equilibrium allocations are not Pareto optimal.*

Proof.. A necessary condition for x to be a Pareto optimal competitive equilibrium allocation for an economy e is that $\varphi_e(x,\widetilde{\lambda},\widetilde{p})=0$. Since the dimension of the domain of φ_e is lower than the dimension of the range, whenever φ_e is transverse to 0, a solution to $\varphi_e(x,\widetilde{\lambda},\widetilde{p})=0$ does not exist. By a standard argument, φ is transverse to 0. By the transversal density theorem, the set of economies for which φ_e is transverse to 0 has full Lebesgue measure. By a standard argument, this set can be chosen to be open. □

The function $\psi:\Xi\times\Omega^*\to\mathbb{R}^N$ is defined by

$$\psi(\xi,e)=\begin{pmatrix}\partial_{x^i_s}u^i(x^i)-\lambda^i_s p_s, \quad i\in\mathcal{I},s\in\mathcal{S}\\ p_s(x^i_s-e^i_s)-R_{s\cdot}y^i, \quad i\in\mathcal{I},s\in\mathcal{S}\\ \lambda^i R-\mu^i q, \quad i\in\mathcal{I}\\ \sum_{i\in\mathcal{I}}(x^i_{l,s}-e^i_{l,s}), \quad (l,s)\in\check{\mathcal{L}}\times\mathcal{S}\\ \sum_{i\in\mathcal{I}}y^i_a, \quad a\in\check{\mathcal{A}}\\ qy^i, \quad i\in\mathcal{I}\end{pmatrix},$$

$\xi = (x, \lambda, y, \mu, \check{p}, \check{q})$ and $\Xi = \mathbb{R}^{I(L+1)S}_{++} \times \mathbb{R}^{IS}_{++} \times \mathbb{R}^{I(A+1)} \times \mathbb{R}^{I} \times \check{P} \times \check{Q}$. The dimension of Ξ is N. When ξ^* is consistent with a competitive equilibrium, it is necessarily the case that $\psi_e(\xi^*) = 0$. The function $h : \Xi \times \mathbb{R}^I \times \Omega^* \to \mathbb{R}^{LS+1}$ is defined by

$$h(\xi, \alpha, e) = \begin{pmatrix} \sum_{i \in \mathcal{I}} \alpha^i \lambda^i_s (x^i_{l,s} - e^i_{l,s}), \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S} \\ \\ \sum_{i \in \mathcal{I}} (\alpha^i)^2 - 1 \end{pmatrix}.$$

A competitive equilibrium can be Pareto improved by price regulation if the matrix of partial derivatives of the indirect utility functions with respect to prices has full rank[6]. By Proposition 4.8, this matrix is guaranteed to have full rank if there is no solution to $\psi_e(\xi) = 0$ in combination with $h_e(\xi, \alpha) = 0$. The function $\widetilde{\psi} : \Xi \times \mathbb{R}^I \times \Omega^* \to \mathbb{R}^{N+LS+1}$ is defined by

$$\widetilde{\psi}(\xi, \alpha, e) = \begin{pmatrix} \psi(\xi, e) \\ h(\xi, \alpha, e) \end{pmatrix}.$$

If $\widetilde{\psi}$ is transverse to 0, then it follows from the transversal density theorem that for a subset of endowments of full Lebesgue measure, $\widetilde{\psi}_e$ is transverse to 0. If $LS \geq I$, then the dimension of the range of $\widetilde{\psi}_e$ exceeds that of the domain. Transversality of $\widetilde{\psi}_e$ implies that there are no solutions to the associated system of equations. It is possible to Pareto improve all competitive equilibria by price regulation.

Proposition 1. *If $LS \geq I$, then, generically, all competitive equilibria of $\mathcal{E}$ can be Pareto improved by price regulation.*

Proof.. One fixes $(\bar{l}, \bar{s}) \in \check{\mathcal{L}} \times \mathcal{S}$ and Ω^{**}, an open subset of endowments in Ω^* of full Lebesgue measure, such that no competitive equilibrium of the associated economy $\mathcal{E}$ is Pareto optimal. The function $\widehat{\psi} : \Xi \times \Omega^{**} \to \mathbb{R}^{N+1}$ is defined by

$$\widehat{\psi}(\xi, e) = \begin{pmatrix} \psi(\xi, e) \\ \\ \sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \sum_{i \in \mathcal{I}} \frac{\lambda^i_s}{\lambda^i_{\bar{s}}} (x^i_{\bar{l},s} - e^i_{\bar{l},s}) \end{pmatrix}.$$

If $\widehat{\psi}(\xi, e) = 0$, then the matrix $\widehat{M}$ of partial derivatives of $\widehat{\psi}$ evaluated at (ξ, e) has full row rank: if $v'\widehat{M} = 0$, then $v = 0$. The components of v are $v_{1,i,l,s}$, $i \in \mathcal{I}$, $(l, s) \in \mathcal{L} \times \mathcal{S}$, $v_{2,i,s}$, $i \in \mathcal{I}$, $s \in \mathcal{S}$, $v_{3,i,a}$, $i \in \mathcal{I}$, $a \in \mathcal{A}$, $v_{4,l,s}$, $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, $v_{5,a}$, $a \in \check{\mathcal{A}}$, $v_{6,i}$, $i \in \mathcal{I}$, and v_9, according to the labelling of the equations defining $\widehat{\psi}$. If v is such that $v'\widehat{M} = 0$, then $0 = v'\partial_{e^i_{L+1,s}} \widehat{\psi}(\xi, e) = -v_{2,i,s}$, $i \in \mathcal{I}$, $s \in \mathcal{S}$. It follows that, for $i \in \mathcal{I}$,

[6] If the matrix of partial derivatives has full rank, it is possible to generate any desired marginal change in utilities by means of price regulation.

$$0 = v'\partial_{e^i_{l,s}}\widehat{\psi}(\xi, e) = -v_{4,l,s}, \quad (l,s) \in (\check{\mathcal{L}} \setminus \{\bar{l}\}) \times \mathcal{S},$$

$$0 = v'\partial_{e^i_{\bar{l},s}}\widehat{\psi}(\xi, e) = -v_{4,\bar{l},s} - v_9\tfrac{\lambda^i_s}{\lambda^i_{\bar{s}}} = 0, \quad s \in \mathcal{S} \setminus \{\bar{s}\},$$

$$0 = v'\partial_{e^i_{\bar{l},\bar{s}}}\widehat{\psi}(\xi, e) = -v_{4,\bar{l},\bar{s}}.$$

Consequently, if $v_{4,\bar{l},\widehat{s}} = 0$ for some $\widehat{s} \in \mathcal{S} \setminus \{\bar{s}\}$, then $v_9 = 0$ and $v_{4,\bar{l},s} = 0$, for all $s \in \mathcal{S} \setminus \{\bar{s}\}$. If, on the contrary, $v_{4,\bar{l},s} \neq 0$, for all $s \in \mathcal{S} \setminus \{\widehat{s}\}$, then

$$\frac{\lambda^i_s}{\lambda^i_{\bar{s}}} = -\frac{v_{4,\bar{l},s}}{v_9} = \frac{\lambda^{i'}_s}{\lambda^{i'}_{\bar{s}}}, \quad i, i' \in \mathcal{I},\ s \in \mathcal{S} \setminus \{\bar{s}\}.$$

Hence, for $i, i' \in \mathcal{I}$, for s^1, $s^2 \in \mathcal{S}$, $\lambda^i_{s^1}/\lambda^i_{s^2} = (\lambda^i_{s^1}/\lambda^i_{\bar{s}})(\lambda^i_{\bar{s}}/\lambda^i_{s^2}) = (\lambda^{i'}_{s^1}/\lambda^{i'}_{\bar{s}})(\lambda^{i'}_{\bar{s}}/\lambda^{i'}_{s^2}) = \lambda^{i'}_{s^1}/\lambda^{i'}_{s^2}$. The economy e has then a Pareto optimal competitive equilibrium induced by ξ, contradicting $e \in \Omega^{**}$. Consequently, $v_{4,\bar{l},s} = 0$, $s \in \mathcal{S} \setminus \{\bar{s}\}$, and $v_9 = 0$. For $i \in \mathcal{I}$, and $(l,s) \in \mathcal{L} \times \mathcal{S}$,

$$0 = v'\partial_{x^i_{l,s}}\widehat{\psi}(\xi, e) = v'_{1,i,\cdot,\cdot}\partial_{x^i_{l,s}}\partial u^i(x^i).$$

It is possible to represent a utility function satisfying A2 by one with $\partial^2 u^i(x^i)$ negative definite on a bounded subset of the consumption set. Then it follows that $v_{1,i,\cdot,\cdot} = 0$. For $i \in \mathcal{I}$, $0 = v'\partial_{y^i_{A+1}}\widehat{\psi}(\xi, e) = v_{8,i}$. Also, for $a \in \check{\mathcal{A}}$, $0 = v'\partial_{y^i_a}\widehat{\psi}(\xi, e) = v_{5,a}$. Finally, $0 = v'\partial_{\lambda^i_s}\widehat{\psi}(\xi, e) = v'_{3,i,\cdot}R'_{s\cdot}$, $i \in \mathcal{I}$, $s \in \mathcal{S}$. Since R has full column rank it follows that $v_{3,i,a} = 0$, $i \in \mathcal{I}, a \in \mathcal{A}$. Therefore, $v = 0$, $\widehat{M}$ has full row rank $N + 1$, and $\widehat{\psi}$ is transverse to 0. The set of endowments such that $\widehat{\psi}_e$ is transverse to zero is denoted $\widehat{\Omega}_{\bar{l},\bar{s}}$. By the transversal density proposition, $\Omega^{**} \setminus \widehat{\Omega}_{\bar{l},\bar{s}}$ has Lebesgue measure zero. For $e \in \widehat{\Omega}_{\bar{l},\bar{s}}$, the dimension of the range of $\widehat{\psi}_e$ exceeds that of the domain, so $\left(\widehat{\psi}_e\right)^{-1}(\{0\}) = \emptyset$. The set $\widehat{\Omega} = \cap_{(l,s)\in\check{\mathcal{L}}\times\mathcal{S}}\widehat{\Omega}_{l,s}$ is of full Lebesgue measure and, by a standard argument, open. Redefine the function $\widetilde{\psi}$ such that endowments belong to $\Omega^* \cap \widehat{\Omega}$. For (ξ, α, e) such that $\widetilde{\psi}(\xi, \alpha, e) = 0$, $\widetilde{M}$ is the matrix of partial derivatives of $\widetilde{\psi}$ evaluated at (ξ, α, e). If v is such that $v'\widetilde{M} = 0$, and the components of v are denoted by $v_{1,i,l,s}$, $v_{2,i,s}$, $v_{3,i,a}$, $v_{4,l,s}$, $v_{5,a}$, $v_{6,i}$, $v_{7,l,s}$, and v_8, then, $0 = v'\partial_{e^i_{L+1,s}}\widetilde{\psi}(\xi, \alpha, e) = -v_{2,i,s}$, $i \in \mathcal{I}$, $s \in \mathcal{S}$. Hence,

$$0 = v'\partial_{e^i_{l,s}}\widetilde{\psi}(\xi, \alpha, e) = -v_{4,l,s} - \alpha^i\lambda^i_s v_{7,l,s}, \quad i \in \mathcal{I},\ (l,s) \in \check{\mathcal{L}} \times \mathcal{S}.$$

Since $\sum_{i\in\mathcal{I}}(\alpha^i)^2 = 1$, there is i' such that $\alpha^{i'} \neq 0$. If there is $\bar{s} \in \mathcal{S}$ such that, for $i \in \mathcal{I} \setminus \{i'\}$, $\alpha^{i'}\lambda^{i'}_{\bar{s}} - \alpha^i\lambda^i_{\bar{s}} = 0$, then, for any $l \in \check{\mathcal{L}}$,

$$0 = \textstyle\sum_{s\in\mathcal{S}\setminus\{\bar{s}\}}\sum_{i\in\mathcal{I}}\alpha^i\lambda^i_s(x^i_{l,s} - e^i_{l,s}) = \sum_{s\in\mathcal{S}\setminus\{\bar{s}\}}\sum_{i\in\mathcal{I}}\frac{\alpha^{i'}\lambda^{i'}_{\bar{s}}}{\lambda^i_{\bar{s}}}\lambda^i_s(x^i_{l,s} - e^i_{l,s})$$

$$= \alpha^{i'}\lambda^{i'}_{\bar{s}}\textstyle\sum_{s\in\mathcal{S}\setminus\{\bar{s}\}}\sum_{i\in\mathcal{I}}\frac{\lambda^i_s}{\lambda^i_{\bar{s}}}(x^i_{l,s} - e^i_{l,s}).$$

Since $\alpha^{i'} \neq 0$, $\sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \sum_{i \in \mathcal{I}} (\lambda^i_s / \lambda^i_{\bar{s}})(x^i_{l,s} - e^i_{l,s}) = 0$, a contradiction since $e \in \widehat{\Omega}$. Consequently, for every $s \in \mathcal{S}$, there is $i \in \mathcal{I} \setminus \{i'\}$ such that $\alpha^{i'} \lambda^{i'}_s - \alpha^i \lambda^i_s \neq 0$. For $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, $(\alpha^{i'} \lambda^{i'}_s - \alpha^i \lambda^i_s) v_{7,l,s} = 0$, so $v_{7,l,s} = 0$, and, thus $v_{4,l,s} = 0$. Also, $0 = v' \partial_{\alpha^{i'}} \widetilde{\psi}(\xi, \alpha, e) = 2\alpha^{i'} v_8$, so, since $\alpha^{i'} \neq 0$, $v_8 = 0$. It follows as in the first part of the proof that $v_{1,i,l,s} = 0$, $i \in \mathcal{I}$, $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, that $v_{6,i} = 0$, $i \in \mathcal{I}$, that $v_{5,a} = 0$, $a \in \check{\mathcal{A}}$, and that $v_{3,i,a} = 0$, $i \in \mathcal{I}$, $a \in \mathcal{A}$. Therefore, $\widetilde{M}$ has rank $N + LS + 1$ and $\widetilde{\psi}$ intersects 0 transversally. If $\widetilde{\Omega}$ is the set of economies such that $\widetilde{\psi}_e$ is transverse to 0, then $\Omega^* \setminus \widetilde{\Omega}$ has Lebesgue measure zero by the transversal density theorem. Openness follows by a standard argument. □

Generically, it is possible to make every individual better off by choosing appropriate price regulations on the spot markets when asset markets are incomplete. One needs at least as many instruments, LS, as individuals, I. Proposition 1 makes clear that this is all one needs. This is not the case in the constrained suboptimality result of Geanakoplos and Polemarchakis (1986), which applies when $2L \leq I \leq L(S-1)+1$. A competitive equilibrium can be Pareto improved by uniform price regulation if the matrix of partial derivatives of the indirect utility functions with respect to uniform price regulation has full rank. The function $k : \Xi \times \mathbb{R}^I \times \Omega^* \to \mathbb{R}^{L+1}$ is defined by

$$k(\xi, \alpha, e) = \begin{pmatrix} \sum_{s \in \mathcal{S}} h_{l,s}(x, \lambda, \alpha, e), \quad l \in \check{\mathcal{L}} \\ \\ \sum_{i \in \mathcal{I}} (\alpha^i)^2 - 1 \end{pmatrix}.$$

The matrix of partial derivatives of the indirect utility functions with respect to uniform price regulation is guaranteed to have full rank if there is no solution to $\psi_e(\xi) = 0$ in combination with $k_e(\xi, \alpha) = 0$.

Corollary 1. *If $L \geq I$, then, generically, all competitive equilibria of $\mathcal{E}$ can be Pareto improved by uniform price regulation.*

Proof.. The argument follows that in the proof of Proposition 5.3. The equations related to h that characterize Pareto improving price regulation are replaced by the equations related to k that characterize Pareto improvements by uniform price regulation. This defines a function $\overline{\psi}$. The matrix $\overline{M}$ gives the partial derivatives of $\overline{\psi}$ evaluated at some (ξ, α, e) with $\overline{\psi}(\xi, \alpha, e) = 0$. If $v'\overline{M} = 0$, by considering the partial derivatives with respect to $e^i_{l,s}$, it follows that $v_{2,i,s} = 0$, $i \in \mathcal{I}, s \in \mathcal{S}$, and $v_{4,l,s} + \alpha^i \lambda^i_s v_{7,l} = 0$, $i \in \mathcal{I}$, $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$. If i' is such that $\alpha^{i'} \neq 0$, and if $\bar{s} \in \mathcal{S}$ such that, for $i \in \mathcal{I} \setminus \{i'\}$, $\alpha^{i'} \lambda^{i'}_{\bar{s}} - \alpha^i \lambda^i_{\bar{s}} = 0$, then

$$0 = \sum_{i \in \mathcal{I}} \alpha^i \sum_{s \in \mathcal{S}} \lambda^i_s (x^i_{l,s} - e^i_{l,s}) = \alpha^{i'} \lambda^{i'}_{\bar{s}} \sum_{i \in \mathcal{I}} \sum_{s \in \mathcal{S}} \frac{\lambda^i_s}{\lambda^i_{\bar{s}}} (x^i_{l,s} - e^i_{l,s})$$

$$= \sum_{i \in \mathcal{I}} \sum_{s \in \mathcal{S}} \frac{\lambda^i_s}{\lambda^i_{\bar{s}}} (x^i_{l,s} - e^i_{l,s}) = \sum_{i \in \mathcal{I}} \sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \frac{\lambda^i_s}{\lambda^i_{\bar{s}}} (x^i_{l,s} - e^i_{l,s}), \quad l \in \check{\mathcal{L}},$$

which contradicts $e \in \widehat{\Omega}$. It follows that $v_{4,l,s} = 0$, $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, and $v_{7,l} = 0$, $l \in \check{\mathcal{L}}$. The remainder of the proof follows the argument in the proof of Proposition 6. □

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13

On Behavioral Heterogeneity*

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Summary. An index of "behavioral heterogeneity" for every finite population of households is defined. It is shown that the higher the index of behavioral heterogeneity the less sensitive depends the aggregate consumption expenditure ratio upon prices. As a consequence, a high index implies a tendency for the Jacobian of aggregate demand to have a dominant negative diagonal.

Key words: Aggregation, Behavioral heterogeneity, Mean demand.

JEL Classification Numbers: D 11, E 10.

13.1 Introduction

It is well-known that the hypothesis of utility maximization on the individual level does not imply any identifiable properties of mean demand for a sufficiently large population H of households, in particular, the Jacobian matrix $\partial F(p) = \left(\partial_{p_j} F_i(p)\right)_{i,j=1,\ldots,l}$ of mean demand

$$F(p) = \frac{1}{\#H} \sum_{h \in H} f^h\left(p, x^h\right) \tag{13.1}$$

has no identifiable structure, such as negative definiteness or diagonal dominance (Sonnenschein, 1973; Andreu, 1983; Chiappori and Ekeland, 1996). As usual, in relation (13.1) $f^h(\cdot, \cdot)$ and x^h denote the demand function and total budget (income) of household $h \in H$.

Consequently, in order to obtain some useful properties of mean demand, one has to make additional assumptions either on the individual utility functions or on the joint distribution of demand functions and income, that is to say, on the composition

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of the population of households. For the first approach see, for example, the literature on additively separable utility functions or Mitjuschin and Polterovich (1978). The distributional approach is treated in Hildenbrand (1983), Härdle, Hildenbrand and Jerison (1991), Grandmont (1992), Quah (1997, 2001), Kneip (1999), Jerison (1999), Giraud and Maret (2002) and K. Hildenbrand and John (2003). In this paper we pursue the distributional approach.

Without denying the possible interdependence among various markets, one often assumes in applied economic analysis, in particular in partial equilibrium analysis, that this interdependence can be restricted. For example, one assumes the following properties of the dependence of mean demand upon prices:

(a) *The demand for commodity i is more effected by a change in its own price p_i than by all other price changes p_j, $i \neq j$.*
(b) *A price change in commodity i has a larger effect on demand of commodity i than on all other commodities j different from i.*

How can assertions (a) and (b) be expressed by the Jacobian matrix $\partial F(p)$ of mean demand $F(p)$?

Clearly, (a) does not mean that $|\partial_{p_i} F_i(p)| > |\partial_{p_j} F_i(p)|$ for all $j \neq i$, since a price change from p_k to $p_k + \Delta$ has a different implication if the price p_k is small or large. One has to compare the effect of a percentage change of prices, i.e.

$$\partial_\lambda [F_i(p_1, \ldots, \lambda p_k, \ldots, p_l)]_{\lambda=1} = p_k \partial_{p_k} F_i(p)$$

Hence, assertion (a) can be interpreted as $|p_i \partial_{p_i} F_i(p)| > |p_j \partial_{p_j} F_i(p)|$ for every $j \neq i$ or more strongly as

$$p_i |\partial_{p_i} F_i(p)| > \sum_{j \neq i} p_j |\partial_{p_j} F_i(p)| \tag{13.2}$$

Furthermore, (b) does not mean that $|\partial_{p_i} F_i(p)| > |\partial_{p_i} F_j(p)|$ for all $j \neq i$, since one cannot sum up a change in demand of different commodities. One has to compare the expenditure of demand changes, i.e. $p_k \partial_{p_i} F_k(p)$. Thus, assertion (b) can be interpreted as as $|p_i \partial_{p_i} F_i(p)| > |p_j \partial_{p_i} F_j(p)|$ for every $j \neq i$ or more strongly as

$$p_i |\partial_{p_i} F_i(p)| > \sum_{j \neq i} p_j |\partial_{p_i} F_j(p)| \tag{13.3}$$

Consider the matrix $D_p \partial F(p) D_p = (p_i p_j \partial_{p_j} F_i(p))$, where D_p denotes the diagonal matrix with $p_1, \ldots, p_l$ on the diagonal. Then, properties (13.2) and (13.3) are equivalent to diagonal dominance[3] of the matrix $D_p \partial F(p) D_p$.

Note that, if the diagonal elements of $D_p \partial F(p) D_p$ are negative, than a dominant diagonal implies that $D_p \partial F(p) D_p$, and hence also $\partial F(p)$ are negative definite. Thus the mean demand $F(p)$ satisfies the *Law of Demand*, i.e.

[3] A matrix $A = (a_{ij})$ is said to be diagonal dominant if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ and $|a_{ii}| > \sum_{j \neq i} |a_{ji}|$ for every i.

$$(p-q)\cdot(F(p)-F(q))<0$$

However, $D_p\partial F(p)D_p$ to have a dominant negative diagonal is a stronger property than the Law of Demand, since a negative definite matrix does not need to have a dominant negative diagonal.

Let us analyze the matrix $D_p\partial F(p)D_p$ more closely. The mean income of the population H is given by $X := \operatorname{mean}_{h\in H} x^h$, where $\operatorname{mean}_{h\in H}$ is used as an abbreviation of $\frac{1}{\#H}\sum_{h\in H}$. The aggregate consumption expenditure ratio for commodity i is denotes by $W_i(p)$, i.e. $W_i(p) = p_i \cdot F_i(p)/X$. One easily verifies that the definition of mean demand $F(p)$ by (13.1) implies that

$$D_p\partial F(p)\,D_p = X\left(-D_{W(p)} + S(p)\right) \tag{13.4}$$

with

$$S_{ij}(p) = \operatorname{mean}_{h\in H}\frac{x^h}{X}s^h_{ij}(p,x^h), \tag{13.5}$$

where $s^h_{ij}(p,x^h)$ is the rate of change in the household's budget share $w^h_i(p,x^h) = \frac{1}{x^h}p_i f^h_i(p,x^h)$ for commodity i with respect to a percentage change in the price p_j, i.e.

$$s^h_{ij}(p,x^h) = p_j\partial_{p_j}w^h_i(p,x^h)$$

Consequently, by (13.4), a tendency towards a dominant negative diagonal of $D_p\partial F(p)\,D_p$ prevails if all elements $|S_{ij}(p)|$ tend to be small as compared to $W_i(p)$ and $W_j(p)$.

Apart from diagonal dominance, small elements $|S_{ij}(p)|$ also imply a strong restriction on the magnitude of price-elasticities of mean demand. Indeed, let $E_{ij}(p)$ denote the price-elasticity of mean demand for commodity i with respect to the price p_j. Then one obtains

$$\begin{aligned} E_{ii}(p) &= -1 + S_{ii}(p)/W_i(p) \\ E_{ij}(p) &= S_{ij}(p)/W_i(p) \qquad \text{if } i\neq j. \end{aligned}$$

Consequently, a low bound for $|S_{ij}(p)|$ implies aggregate own price-elasticities close to -1, while aggregate cross-price elasticities are around zero.

Which feature of a population of households implies a tendency for $|S_{ij}(p)|$ to adopt small values? In this paper we want to discuss this question.

If we exclude the trivial case, where $|s^h_{ij}(p,x^h)|$ is already small for every p and every household in the population, then there remain two (not mutually exclusive) possibilities for $|S_{ij}(p)|$ to become small, namely

- there is a balancing sign effect across the population; some $s^h_{ij}(p,x^h)$ are positive and others are negative
- for every price system p there is a large subpopulation $H(p)$ of H for which $|s^h_{ij}(p,x^h)|$ is small. Since the subpopulation $H(p)$ depends on p, for no household $h\in H$ $\sup_p|s^h_{ij}(p,x^h)|$ needs to be small.

Let us first discuss the balancing sign effect. The budget identity and homogeneity of the budget share function $w(p,x)$ imply that

$$\sum_{i=1}^{l}\sum_{j=1}^{l} s_{ij}^{h}\left(p,x^{h}\right)=0.$$

Thus, if $s_{ij}^{h}\left(p,x^{h}\right)$ is not zero for all i,j then, for a given household h, some $s_{ij}^{h}\left(p,x^{h}\right)$ are positive and others are negative. One might argue that "behavioral heterogeneity" of a population should imply that, for given i,j, the sign of $s_{ij}^{h}\left(p,x^{h}\right)$ alternates across the population. The sign of $s_{ij}^{h}\left(p,x^{h}\right)$ changes across the population if, for $i=j$, the own price-elasticity of demand for commodity i is spread around minus one and if, for $i\neq j$, the cross price elasticity is spread around zero. Why should this prevail for every commodity?

For example, let us assume that all households have an additive utility function $u^{h}(z)=u_{1}^{h}(z_{1})+\cdots+u_{l}^{h}(z_{l})$, $z\in\mathbb{R}_{+}^{l}$. By a well-known result (see e.g. Varian, 1985) it follows that all $|s_{ii}^{h}(p,x^{h})|$ are less than $w_{i}^{h}(p,x^{h})$ and for every i, all $s_{ij}^{h}(p,x^{h})$, $j\neq i$, have the same sign. Furthermore, $s_{ij}^{h}(p,x^{h})$ is positive (negative) if $\rho_{i}^{h}\left(f_{i}^{h}(p,x^{h})\right)$ is smaller (greater) than one, where $\rho_{i}^{h}(\zeta)=-\frac{\left(u_{i}^{h}(\zeta)\right)''}{\left(u_{i}^{h}(\zeta)\right)'}$. Thus, the sign of $s_{ij}^{h}(p,x^{h})$ alternates across the population H if $\left\{\rho_{i}^{h}\left(f_{i}^{h}(p,x^{h})\right)\right\}_{h\in H}$ is spread around 1.

We believe that to a certain extent a balancing sign effect always prevails, yet this balancing effect is not sufficiently general and strong in order to base on it alone the desired conclusion of a sufficiently small $|S_{ij}(p)|$.

In this paper we want to find the smallest possible upper bound for $\sup_{p}|S_{ij}(p)|$ without relying on the potentiality of a balancing sign effect.

If one just knows for every household the income x^{h} and $d_{ij}^{h}:=\sup_{p}|s_{ij}^{h}\left(p,x^{h}\right)|$ then the best possible upper bound is given by

$$\sup_{p}|S_{ij}(p)|\leq\frac{1}{\#H}\sum_{h\in H}\frac{x^{h}}{X}d_{ij}^{h}=:D_{ij}. \tag{13.6}$$

We remark that d_{ij}^{h} does not depend on the income level x^{h} if the budget share function w^{h} is homogeneous of degree zero in (p,x).

It is our goal to improve upon inequality (13.6). To achieve this, one needs some information about the characteristics of the population. The question now is: which feature of the population H leads to an improvement of inequality (13.6)?

In Section 2 we shall associate to every population H of households an index $\gamma(H)$ with $0\leq\gamma(H)\leq 1-\frac{1}{\#H}$, which will be interpreted as a *degree of behavioral heterogeneity* of the population H. By definition, $\gamma(H)=0$ for a population where all households have the same demand function and the same income. The index $\gamma(H)$ is also zero for a population where all households have a Cobb-Douglas demand function even if they are different across the population. This population is heterogeneous, yet not behaviorally heterogeneous since all households have the same price-elasticities. The degree $\gamma(H)$ of behavioral heterogeneity is positive yet less than one if the households $h \in H$ react differently to price changes.

In Section 3 we show that the index $\gamma(H)$ can be used to improve upon inequality (13.6). For example, consider the special case of a population $\{w^h, x^h\}_{h\in H}$ with $x^h = x$ and $\sup_p p_j |\partial_{p_j} w_i^h(p, x^h)| = d_{ij}$ for all $h \in H$. Then we obtain (Proposition 1)

$$\sup_p |S_{ij}(p)| \leq (1 - \gamma(H))\, d_{ij}.$$

For a general population the relation between the degree of behavioral heterogeneity $\gamma(H)$ and a bound for $\sup_p |S_{ij}(p)|$ is more complicated and is given in Proposition 2.

Finally, in Section 4 we discuss the potentiality of $\gamma(H)$ to become essentially greater than zero.

13.2 An index of behavioral heterogeneity

Notation. A population H of households $h \in H$ is defined by $\{f^h, x^h\}_{h\in H}$, where $(p, x) \longmapsto f^h(p, x) \in \mathbb{R}^l_+$ denotes the demand function and $x^h > 0$ the income of household h. The price vector $p \in \mathbb{P}^l = (0, \infty)^l$.

Mean demand $F(p)$ of the population H is defined by

$$F(p) = \text{mean}_{h\in H} f^h(p, x^h) := \frac{1}{\#H} \sum_{h\in H} f^h(p, x^h).$$

The *consumption expenditure ratio of household h* for commodity i (also called the budget share) and the *aggregate consumption expenditure ratio* are defined by

$$w_i^h(p, x) := \frac{1}{x} p_i \cdot f_i^h(p, x) \qquad i = 1, \dots, l$$

and

$$W_i(p) := \frac{1}{X} p_i F_i(p),$$

where X denotes mean income $X = \text{mean}_{h\in H} x^h$ of the population.

Thus,

$$W_i(p) = \text{mean}_{h\in H} \frac{x^h}{X} w_i^h(p, x^h).$$

Assumption 1. $w(p, x)$ is continuously differentiable in $p \gg 0$ and $x > 0$ and

$$d_{ij}(w, x) = \sup_p |p_j \partial_{p_j} w_i(p, x)| < \infty.$$

Note that $\sup_p |\partial_{p_j} w_i(p, x)|$ is typically not finite. For this reason we consider the rate of change of $w_i(p, x)$ for a percentage change of the j-th price.

Let s_{ij}^{h} and S_{ij} denote the rate of change of w_i^h and W_i, respectively, with respect to a percentage change of the price p_j, i.e.,

$$\begin{aligned} s_{ij}^{h}\left(p,x\right) &:= \partial_\lambda w_i^h\left(p_1,\ldots,\lambda p_j,\ldots,p_l\right)|_{\lambda=1} = p_j \partial_{p_j} w_i^h\left(p,x\right) \\ S_{ij}\left(p\right) &:= \partial_\lambda W_i\left(p_1,\ldots,\lambda p_j,\ldots,p_l\right)|_{\lambda=1} = p_j \partial_{p_j} W_i\left(p\right). \end{aligned}$$

As explained in the introduction we want to find an upper bound for $\sup_p \left|S_{ij}\left(p\right)\right|$, improving upon inequality (13.6), which is related to behavioral heterogeneity. To achieve this we now define an index $\gamma(H)$ for every population H which will be interpreted as a *degree of behavioral heterogeneity*.

Let

$$A_{ij}^{\varepsilon}\left(w,x\right) := \left\{p \in \mathbb{P}^l \middle|\ p_j\left|\partial_{p_j} w_i\left(p,x\right)\right| \geq \varepsilon \cdot d_{ij}\left(w,x\right)\right\} \tag{13.7}$$

where $d_{ij}\left(w,x\right) = \sup_p \left\{p_j\left|\partial_{p_j} w_i\left(p,x\right)\right|\right\}$. That is to say, the set $A_{ij}^{\varepsilon}\left(w,x\right)$ is the domain of price vectors p for which the absolute value of the rate of change of the budget share w_i for commodity i with respect to a percentage change of the price p_j is larger than or equal to $\varepsilon \cdot d_{ij}\left(w,x\right)$. Note that $A_{ij}^{0}\left(w,x\right) = \mathbb{P}^l$, $A_{ij}^{\varepsilon}\left(w,x\right) \neq \emptyset$ and is decreasing in ε for $0 < \varepsilon < 1$. Furthermore, $A_{ij}^{\varepsilon}\left(w,x\right) = \emptyset$ for $\varepsilon > 1$.

The sets $A_{ij}^{\varepsilon}\left(w,x\right)$ will play a crucial role in defining our notion of "behavioral heterogeneity". Therefore it is important to emphasize that the set $A_{ij}^{\varepsilon}\left(w,x\right)$ depends in an essential way on the budget share function w and the income level x.

The intuitive notion of "behavioral heterogeneity" of a population H of households – however it is made precise – undoubtedly implies that households' characteristics $\left(w^h, x^h\right)$ must be different across the population. This alone, however, is not sufficient, and the above arguments open a way to quantify the structurally important differences. "Behavioral heterogeneity" should imply that the sets $A_{ij}^{\varepsilon}\left(w^h, x^h\right)$ are different across the population in the sense that they are "located in different regions" in $\mathbb{R}^l$, which can be made precise by requiring that these sets possess only few intersections. In terms of the sets $A_{ij}^{\varepsilon}\left(w,x\right)$ we now define the *intersection frequency:* for every $\varepsilon \geq 0$ we define

$$I_{ij}^{\varepsilon}\left(p\right) := \frac{1}{\#H}\#\left\{h \in H \middle|\ p \in A_{ij}^{\varepsilon}\left(w^h, x^h\right)\right\}.$$

Obviously, $I_{ij}^{0}\left(p\right) = 1$ and $I_{ij}^{\varepsilon}\left(p\right)$ is a decreasing step-function in ε.

A behaviorally heterogeneous population will now be characterized by the property that for every price vector p the intersection frequency $I_{ij}^{\varepsilon}\left(p\right)$ is small. Since we do not want to emphasize a particular value of ε we shall consider the shaded area $\gamma_{ij}\left(p\right)$ in Figure 13.1, i.e.,

$$\gamma_{ij}\left(p\right) := 1 - \int_0^1 I_{ij}^{\varepsilon}\left(p\right) d\varepsilon$$

and define the *degree of behavioral heterogeneity* with respect to i, j by

$$\gamma_{ij} := \inf_p \gamma_{ij}\left(p\right).$$

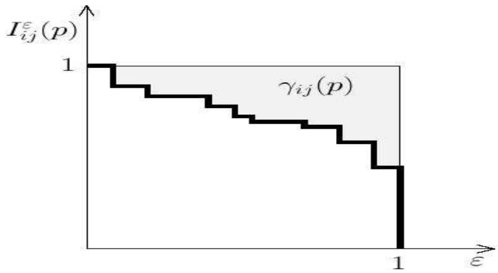

Fig. 13.1. Intersection frequencies for different ε

Definition. The *degree of behavioral heterogeneity of the population* H is defined by

$$\inf_{p,i,j} \gamma_{ij}(p) =: \gamma(H).$$

The minimal degree of behavioral heterogeneity, i.e., $\gamma(H) = 0$, is attained for a population H if

- all households have the same demand function and the same income
- all households have the same demand function which is linear in income
- all households have a Cobb-Douglas demand function which can be different across the population.

Obviously, $\gamma(H) \leq 1 - \frac{1}{\#H} < 1$ for every population H.

13.3 Upper bounds for $\sup_p |S_{ij}(p)|$

We now rely upon the index $\gamma(H)$ in order to improve upon inequality (13.6).

Proposition 1. *For every population* H *with* $d^h_{ij} = d_{ij}$ *and* $x^h = x$ *one obtains*

$$\sup_p |S_{ij}(p)| \leq (1 - \gamma(H)) \, d_{ij}.$$

Proof. Define $G^{\varepsilon}_{ij}(p) := 1 - I^{\varepsilon}_{ij}(p)$. Then

$$G^{\varepsilon}_{ij}(p) = \frac{1}{\#H} \# \left\{ h \in H | \ |s^h_{ij}(p,x)| < \varepsilon d_{ij} \right\}.$$

Thus, $G^{\varepsilon}_{ij}(p)$ is the cumulative distribution function of $|s^h_{ij}(p,x)|/d_{ij}$. Consequently,

$$|S_{ij}(p)| \leq \frac{1}{\#H} \sum_{h \in H} \left| s^h_{ij}(p,x) \right| = d_{ij} \cdot \frac{1}{\#H} \sum_{h \in H} \left| s^h_{ij}(p,x) \right| / d_{ij}$$
$$= d_{ij} \cdot \int \varepsilon \, dG^{\varepsilon}_{ij}(p).$$

Since $\int \varepsilon \, dG_{ij}^{\varepsilon}(p) = 1 - \gamma_{ij}(p)$ we obtain

$$|S_{ij}(p)| \leq (1 - \gamma_{ij}(p))\, d_{ij}. \qquad \square$$

We remark that one obtains

$$\sup_p |S_{ij}(p)| = (1 - \gamma(H))\, d_{ij}$$

if for all households $h \in H$ and all price vectors p either $s_{ij}^h(p, x^h) \geq 0$ or $s_{ij}^h(p, x^h) \leq 0$.

Next we consider a general population H. It follows from the definition of $S_{ij}(p)$ that

$$\begin{aligned} |S_{ij}(p)| &\leq \frac{1}{\#H} \sum_{h \in H} \left| s_{ij}^h(p, x^h) \right| x^h / X \\ &= \frac{1}{\#H} \sum_{h \in H} \frac{\left| s_{ij}^h(p, x^h) \right|}{d_{ij}^h} \cdot \frac{d_{ij}^h x^h}{X}. \end{aligned}$$

Since $\frac{1}{\#H} \sum_{h \in H} \frac{\left| s_{ij}^h(p, x^h) \right|}{d_{ij}^h} = 1 - \gamma_{ij}(p)$, we obtain with $\frac{1}{\#H} \sum_{h \in H} \frac{d_{ij}^h x^h}{X} =: D_{ij}$

$$|S_{ij}(p)| \leq (1 - \gamma_{ij}(p))\, D_{ij} + cov_H\left(\frac{\left| s_{ij}^h(p, x^h) \right|}{d_{ij}^h}, \frac{d_{ij}^h x^h}{X} \right). \tag{13.8}$$

Hence, if the covariance were negative, we would obtain a result that is analogous to Proposition 1, namely

$$\sup_p |S_{ij}(p)| \leq (1 - \gamma(H))\, D_{ij}.$$

Of course, the above covariance need not to be negative. Yet one can show that it becomes small for sufficiently large $\gamma(H)$.

Proposition 2. *For every population H of households with $\gamma(H) \geq \frac{1}{2}$ it follows that*

$$\sup_p |S_{ij}(p)| \leq (1 - \gamma(H))\, D_{ij} + \left[\gamma(H)(1 - \gamma(H))\right]^{\frac{1}{2}} \cdot \frac{1}{X} \cdot \left[var_H d_{ij}^h x^h \right]^{\frac{1}{2}}.$$

Proof. By Cauchy-Schwarz inequality we have

$$\left(cov_H\left(\frac{\left| s_{ij}^h(p, x^h) \right|}{d_{ij}^h}, \frac{d_{ij}^h x^h}{X} \right) \right)^2 \leq var_H\left(\frac{\left| s_{ij}^h(p, x^h) \right|}{d_{ij}^h} \right) \cdot var_H\left(\frac{d_{ij}^h \cdot x^h}{X} \right). \tag{13.9}$$

Let $z_h := \frac{\left| s_{ij}^h(p, x^h) \right|}{d_{ij}^h}$. Then $0 \leq z_h \leq 1$ for $h \in H$ and $mean_H(z_h) = 1 - \gamma_{ij}(p)$. Since $var_H(z_h) = mean_H(z_h^2) - (mean_H(z_h))^2$, we obtain

$$var_H(z_h) \leq (1-\gamma_{ij}(p)) - (1-\gamma_{ij}(p))^2 = \gamma_{ij}(p)(1-\gamma_{ij}(p)).$$

The function $\xi(1-\xi)$ is decreasing on the interval $\left[\frac{1}{2},1\right]$. Consequently we obtain for $\gamma(H) = \inf_{i,j,p} \gamma_{ij}(p) \geq \frac{1}{2}$ that $var_H(z_h) \leq \gamma(H)(1-\gamma(H))$. Hence, from (13.8) and (13.9) we obtain the claimed inequality of Proposition 2. □

The following example illustrates Proposition 2 and shows that fairly high degrees of heterogeneity can already be achieved for small populations with unequal values d_{ij}.

Example 1. *A heterogeneous population*

For simplicity, we consider the case of CES demand functions for two commodities. Recall that the budget share functions are independent of income and are defined by

$$w_1(p_1,p_2) = \frac{a^\sigma p_1^{1-\sigma}}{a^\sigma p_1^{1-\sigma} + (1-a)^\sigma p_2^{1-\sigma}} \tag{13.10}$$

$$w_2(p_1,p_2) = \frac{(1-a)^\sigma p_2^{1-\sigma}}{a^\sigma p_1^{1-\sigma} + (1-a)^\sigma p_2^{1-\sigma}} \tag{13.11}$$

where (a,σ) are parameters with $0 \leq a \leq 1$ and $\sigma > 0$.

The underlying population consists of $\#H = 14$ households with equal income x and parameters (a,σ) =$(0.001, 0.65)$, $(0.001, 0.8)$, $(0.001, 0.9)$, $(0.001, 3.5)$, $(0.002, 4.5)$, $(0.005, 5.0)$, $(0.015, 5.0)$, $(0.04, 5.0)$, $(0.1, 5.5)$, $(0.5, 0.01)$, $(0.93, 5.5)$, $(0.98, 4.5)$, $(0.999, 0.8)$, $(0.999, 0.9)$.

Some straightforward numerical calculations then yield:

a) The individual values of $d_{ij}^h := \sup_p |s_{ij}^h(p,x^h)|$ vary between 0.025 and 1.124. The mean is $D_{ij} = 0.531$, while $\sqrt{var_H d_{ij}^h} = 0.505$. Individual price elasticities $e_{ij}^h(p)$ are bounded by $\max_h \max_i \sup_p |e_{ii}(p)^h| = 5.5$ and $\max_h \max_{i \neq j} \sup_p |e_{ij}(p)^h| = 4.5$
b) $\gamma(H) = 0.903$, $\max_{i,j} \sup_p |S_{ij}(p)| = 0.077$, and $-W_i(p) + S_{ii}(p) \leq -0.221$ for all p and $i = 1,2$. Own-price elasticities $E_{ii}(p)$ of mean demand vary between -0.944 and -1.241 for all p, while aggregate cross-price elasticities are bounded by $\max_{i \neq j} \sup_p |E_{ij}(p)| = 0.241$

One recognizes that for this population the degree of behavioral heterogeneity $\gamma(H) = 0.903$ is close to the upper bound $1 - \frac{1}{\#H} = \frac{13}{14} = 0.929$. Moreover, $\sup_p |S_{ij}|(p) = 0.077$ is much smaller than $D_{ij} = 0.531$.

Since $-W_i(p) + S_{ii}(p) < 0$, and since $\min_i \inf_p |-W_i(p) + S_{ii}(p)| = 0.221 > \max_{i,j} \sup_p |S_{ij}(p)| = 0.077$, the matrix $-D_{W(p)} + S(p)$ in (13.4) possesses a dominant negative diagonal for every p. One can conclude that the Jacobian matrix $\partial F(p)$ of mean demand of the population is negative definite for all p.

It is also of interest to consider the structure of aggregate elasticities. Results a) and b) show that, different from individual elasticities, own price and cross price elasticities of mean demand are not very far from -1 and 0. As outlined in the introduction, this property may be seen as a natural consequence of a high degree of heterogeneity.

In this example $(1-\gamma_{ij}(p))\,D_{ij}=0.051$ is smaller than $\sup_p S_{ij}(p)=0.077$, but Proposition 2 only provides the fairly crude upper bound $(1-\gamma(H))\,D_{ij}+[\gamma(H)(1-\gamma(H))]^{\frac{1}{2}}\cdot\left[var_H d_{ij}^h\right]^{\frac{1}{2}}=0.200$.

13.4 Sparseness

By definition, a high index $\gamma(H)$ requires a low intersection frequency of the sets $A_{ij}^{\varepsilon}\left(w^h,x^h\right)$ across the population. This in turn necessitates that these sets are not arbitrarily large and are located in different regions. To study this question we consider instead of $A_{ij}^{\varepsilon}(w,x)$ the set

$$B_{ij}^{\varepsilon}(w,x):=\left\{q\in\mathbb{R}^l \mid \ \left|\partial_{q_j}w_i(\exp(q),x)\right|\geq\varepsilon\cdot d_{ij}(w,x)\right\}. \qquad (13.12)$$

One easily verifies that $p\in A_{ij}^{\varepsilon}(w,x)$ if and only if $\log p\in B_{ij}^{\varepsilon}(w,x)$.

Proposition 3. *If the budget share function w (or, equivalently, the demand function f) is homogeneous of degree zero in (p,x), then*

$$\begin{aligned}B_{ij}^{\varepsilon}(w,x)&=B_{ij}^{\varepsilon}(w,1)+\log x\cdot\mathbf{1}\\&=\left\{(q_1+\log x,\ldots,q_l+\log x)\,|q\in B_{ij}^{\varepsilon}(w,1)\right\}.\end{aligned}$$

If, in addition, $w(p,x)$ does not depend on the income level x (or equivalently, the demand function $f(p,x)$ is linear in x), then the set $B_{ij}^{\varepsilon}(w,x)$ does not depend on x and $q\in B_{ij}^{\varepsilon}(w)$ implies $q+\lambda\mathbf{1}\in B_{ij}^{\varepsilon}(w)$ for all $\lambda\in\mathbb{R}$.

Proof. First we remark that homogeneity of w in (p,x) implies that $\sup_p\left|p_j\partial_{p_j}w_i(p,x)\right|=d_{ij}(w,x)$ does not depend on the income level x. Homogeneity of w in (p,x) implies $w_i(p,x)=w_i\left(\frac{1}{x}p,1\right)$, and hence

$$\partial_\lambda w_i(p_1,\ldots,\lambda p_j,\ldots,p_l,x)\,|_{\lambda=1}=\partial_\lambda w_i(p_1/x,\ldots,\lambda p_j/x,\ldots,p_L/x,1)\,|_{\lambda=1}.$$

Since $d_{ij}(w,x)$ does not depend on x we obtain that $p\in A_{ij}^{\varepsilon}(w,x)$ if and only if $\frac{1}{x}p\in A_{ij}^{\varepsilon}(w,1)$, and hence $q\in B_{ij}^{\varepsilon}(w,x)$ if and only if $\log q-\log x\cdot 1\in B_{ij}^{\varepsilon}(w,1)$, which proves the first part of Proposition 1.

If $w(p,x)$ does not depend on x then, by definition of the set $B_{ij}^{\varepsilon}(w,x)$, this set does not depend on x and we write $B_{ij}^{\varepsilon}(w)$. Hence, if $q\in B_{ij}^{\varepsilon}(w)$, then the first part of Proposition 1 implies $q-\log x\cdot 1\in B_{ij}^{\varepsilon}(w)$ for every $x>0$. □

The dependence of the set $B_{ij}^{\varepsilon}(w,x)$ on the budget share function w is complex. We just give two examples.

Example 2. Consider two budget share functions w and w^{α} which are linked by

$$w^{\alpha}(p,x)=w(\alpha * p,x) \qquad (\alpha)$$

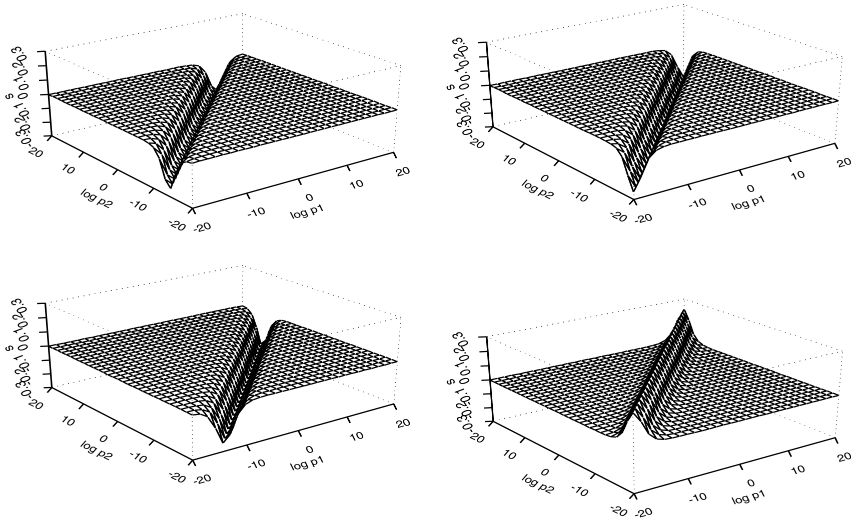

Fig. 13.2. Values of $s_{11}(\cdot)$ for different CES demand functions

for some $\alpha \in \mathbb{P}^l$ where $\alpha * p = (\alpha_1 \cdot p_1, \ldots, \alpha_l \cdot p_l)$. The corresponding demand function f and f^α then satisfy

$$f^\alpha(p, x) = \alpha * f(\alpha * p, x).$$

This transformation of demand functions has been considered by Mas-Colell and Neuefeind (1977), E. Dierker, H. Dierker and Trockel (1984) and Grandmont (1992).

One easily shows that (α) implies

$$B_{ij}^\varepsilon(w^\alpha, x) = B_{ij}^\varepsilon(w, x) - \log \alpha \ ,$$

that is to say, $B_{ij}^\varepsilon(\omega^\alpha, x)$ is just a translation of $B_{ij}^\varepsilon(\omega, x)$ by the vector $(\log \alpha_1, \ldots, \log \alpha_l)$. Indeed, we first observe that $d_{ij}(w, x) = d_{ij}(w^\alpha, x)$. Let $q \in B_{ij}^\varepsilon(w^\alpha, x)$, i.e., $|\partial_{q_j} w_i^\alpha(\exp q, x)| \geq \varepsilon d_{ij}(w^\alpha, x)$. Since $\partial_{q_j} w_i^\alpha(\exp q, x) = \partial_{q_j} w_i(\alpha * \exp q, x) = \partial_{q_j} w_i(\exp(\log \alpha + q), x)$ it follows that $\log \alpha + q \in B^\varepsilon(w, x)$. Thus we showed that $B_{ij}^\varepsilon(w^\alpha, x) \subset B^\varepsilon(w, x) - \log \alpha$. The opposite inclusion is shown analogously.

Example 3. *CES demand functions*

We only consider the case of two commodities and CES budget share functions defined by (13.10) and (13.11).

The functions $s_{ij}(p_1, p_2) = p_j \partial_{p_j} \omega_i(p_1, p_2)$ are either everywhere positive or negative, depending on the parameter values (a, σ).

Figure 13.2 shows the graph of $s_{11}(\cdot)$ as a function of $(\log p_1, \log p_2)$ for the parameter values $(a, \sigma) = (0.95, 2), (0.5, 2), (0.05, 2)$ and $(0.5, 0.1)$.

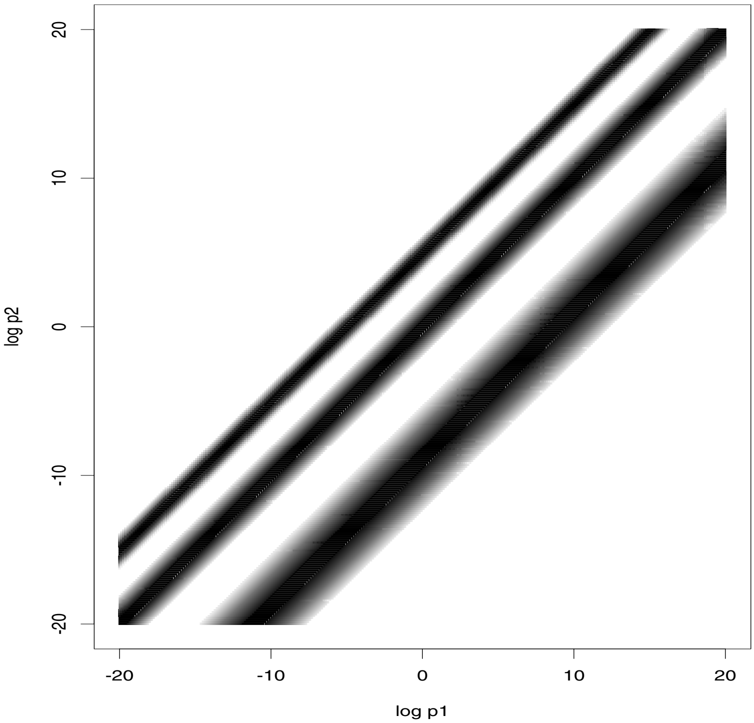

Fig. 13.3. Sets $B_{11}^{0.5}(a,\sigma)\cap Q$ for 3 different CES functions

It follows from Proposition 3 that the set $B_{11}^{\varepsilon}(a,\sigma)$ is a strip parallel to the diagonal in $\mathbb{R}^2$. Thus,

$$B_{11}^{\varepsilon}(a,\sigma)=\left\{(u,v)\in\mathbb{R}^2|\ v-u\in[z-b_l,z+b_r]\right\}.$$

One can compute the interval $[z-b_l,z+b_r]$ and show that the length of this interval only depends on the parameter σ and ε, while the location additionally depends on the parameter a.

In Figure 13.3 are plotted the sets $B_{11}^{\varepsilon}(a,\sigma)\cap Q$ for $\varepsilon=0.5$ and the parameter values $(a,\sigma)=(0.95,2.5)\,,(0.5,2)$ and $(0.05,1.5)\,,$ and the cube $Q=[-20,20]^2$. By a closer look at Figure 3 one recognizes that for all three functions the corresponding set of prices $\log p\in B_{11}^{\varepsilon}(a,\sigma)$ only covers a small fraction of the whole cube $[-20,20]^2$.

The property of $B_{ij}^{\varepsilon}(w,x)$ illustrated in Figure 13.3 is not specific for CES functions, but holds for all budget share functions w satisfying Assumption 2 below. Indeed, we shall show that for $\varepsilon>0$ the set $B_{ij}^{\varepsilon}(w,x)$ is sparse in $\mathbb{R}^l$, that is to say, for any cube $Q=[a,b]^l$ in $\mathbb{R}^l$

$$\frac{\lambda^l\left(B_{ij}^{\varepsilon}(w,x)\cap Q\right)}{\lambda^l(Q)}\underset{(b-a)\to\infty}{\longrightarrow}0$$

where λ^l denotes the Lebesgue measure on $\mathbb{R}^l$.

Assumption 2. The budget share function $w(p,x)$ satisfies $0 \leq w_i(p,x) \leq 1$. Furthermore, there is an integer m such that for all budget share functions that we shall consider and for every $i,j \in \{1,\ldots,l\}$ and every $\overline{p} \in \mathbb{P}^l$ the derivative of the function

$$p_j \longmapsto w_i(\overline{p}_1,\ldots,p_j,\ldots,\overline{p}_l,x)$$

changes its sign at most m times.[4]

Proposition 4. *If* $d_{ij}(w,x) > 0$ *and* $\varepsilon > 0$ *then for every* $\overline{q} \in \mathbb{R}^l$

$$\lambda^1\left\{q_j \in \mathbb{R} |\ (\overline{q}_1,\ldots,q_j,\ldots,\overline{q}_l) \in B_{ij}^{\varepsilon}(w,x)\right\} \leq \frac{m+1}{\varepsilon \cdot d_{ij}(w,x)}$$

Corollary. *The set* $B_{ij}^{\varepsilon}(w,x)$ *is sparse for* $\varepsilon > 0$, more specifically,

$$\frac{\lambda^l\left(B_{ij}^{\varepsilon}(w,x) \cap Q\right)}{\lambda^l(Q)} \leq \frac{m+1}{\varepsilon \cdot d_{ij}(w,x)(b-a)}.$$

Proof of Proposition 4. Let $v(\xi) := w_i(\exp \overline{q}_1,\ldots, \exp \overline{q}_{j-1}, \exp \xi, \exp \overline{q}_{j+1},\ldots,x)$. By definition of the set $B_{ij}^{\varepsilon}(w,x)$ we obtain

$$\begin{aligned}&\{q_j \in \mathbb{R} |\ (\overline{q}_1,\ldots,q_j,\ldots,\overline{q}_l) \in B_{ij}^{\varepsilon}(w,x)\}\\ &= \{\xi \in \mathbb{R} |\ |v'(\xi)| \geq \varepsilon d_{ij}(w,x)\} =: C.\end{aligned}$$

Assumption 3 on budget share functions implies that there are $m+1$ intervals, $I_1 = (-\infty, z_1),\ldots, I_n = (z_{n-1}, z_n),\ldots, I_{m+1} = (z_m, \infty)$ such that the function v is monotone on every interval. Hence $\int_{I_n} |v'(\xi)|\, d\xi = \left|\int_{I_n} v'(\xi)\, d\xi\right|$. Since $0 \leq v(\xi) \leq 1$ one obtains

$$\left|\int_{I_n} v'(\xi)\, d\xi\right| = |v(z_n) - v(z_{n-1})| \leq 1.$$

Note that $v(z_0) = \lim_{\xi \longrightarrow -\infty} v(\xi)$ and $v(z_{m+1}) = \lim_{\xi \longrightarrow \infty} v(\xi)$ exist. Consequently,

$$1 \geq \int_{I_n} |v'(\xi)|\, d\xi \geq \int_{C \cap I_n} |v'(\xi)|\, d\xi \geq \varepsilon d_{ij}(w,x)\, \lambda^1(C \cap I_n),$$

which implies Proposition 4. □

Our discussion of properties of the sets $B_{ij}^{\varepsilon}(w^h, x^h)$ allows to draw the following conclusion: The sets $B_{ij}^{\varepsilon}(w^h, x^h)$ are sparse, and their exact location in $\mathbb{R}^l$ depends crucially on the households' characteristics (w^h, x^h). If (w^1, x^1) is close

[4] Without loss of economic content one might even assume that $m = 0$, that is to say, the function $p_j \longmapsto s_{ij}^h(\overline{p}_1,\ldots,p_j,\ldots,\overline{p}_l,x^h)$ is either non-negative or non-positive.

to $\left(w^2, x^2\right)$, then $B_{ij}^{\varepsilon}\left(w^1, x^1\right)$ and $B_{ij}^{\varepsilon}\left(w^2, x^2\right)$ are to be found in similar regions. On the other hand, if there are substantial structural differences between the budget share functions across a population, then the corresponding sets $B_{ij}^{\varepsilon}\left(w^h, x^h\right)$ do not intersect, and one will obtain low intersection frequencies and, consequently, a high index $\gamma(H)$. This has already been illustrated for CES demand functions (see Fig. 3) and has been used to construct a population with $\gamma(H) = 0.903$ in Example 1.

The crucial importance of sparseness of the sets $B_{ij}^{\varepsilon}\left(w^h, x^h\right)$ is illustrated by the following example.

Example 4. (Grandmont, 1992, revisited)

Grandmont (1992) considers demand functions which are parameterized as in Example 2. Recall that in this situation $B_{ij}^{\varepsilon}(w^\alpha, x) = B_{ij}^{\varepsilon}\left(w^1, x\right) - \log\alpha$, $\alpha \in \mathbb{R}_+^l$. Following Grandmont, the population is described by the budget-share function w^1, the common income x and the distribution ν of $\log\alpha$ on $\mathbb{R}^l$. The intersection frequency is then given by

$$\begin{aligned} I_{ij}^{\varepsilon}(p) &= \nu\left\{\log\alpha \in \mathbb{R}^l \mid\ \log p \in B_{ij}^{\varepsilon}(w^\alpha, x)\right\} \\ &= \nu\left\{\log\alpha \in \mathbb{R}^l \mid \log\alpha \in B_{ij}^{\varepsilon}\left(w^1, x\right) - \log p\right\} = \nu\left\{B_{ij}^{\varepsilon}\left(w^1, x\right) - \log p\right\}. \end{aligned}$$

To simplify computation, let us assume that ν is a uniform distribution on the cube $Q = [a, b]^l$. Then,

$$\begin{aligned} I_{ij}^{\varepsilon}(p) &= \frac{\lambda^l\left\{\left(B_{ij}^{\varepsilon}\left(w^1, x\right) - \log p\right) \cap Q\right\}}{\lambda^l(Q)} \\ &= \frac{\lambda^l\left\{B_{ij}^{\varepsilon}\left(w^1, x\right) \cap Q(p)\right\}}{\lambda^l(Q(p))} \qquad \text{where } Q(p) := [a + \log p, b + \log p]^l \\ &\leq \frac{m+1}{\varepsilon d_{ij}(b-a)} \qquad \text{where } d_{ij} := d_{ij}(w^1, x) \end{aligned}$$

The last inequality follows from the corollary of Proposition 4. Consequently,

$$\int_0^1 I_{ij}^{\varepsilon}(p)\, d\varepsilon \leq \frac{m+1}{d_{ij}(b-a)}\left(1 + \log\frac{d_{ij}(b-a)}{m+1}\right)$$

Obviously, the larger $b - a$, the larger $\gamma(H) = 1 - \sup_{i,j}\sup_p \int_0^1 I_{ij}^{\varepsilon}(p)\, d\varepsilon$. For example, if $m = 0$, $d_{ij} = 1$, and $b - a = 50$ then $\gamma(H) \approx 0.9$.

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14

Learning of Steady States in Nonlinear Models when Shocks Follow a Markov Chain*

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Summary. Local convergence results for adaptive learning of stochastic steady states in nonlinear models are extended to the case where the exogenous observable variables follow a finite Markov chain. The stability conditions for the corresponding nonstochastic model and its steady states yield convergence for the stochastic model when shocks are sufficiently small. The results are applied to asset pricing and to an overlapping generations model. Large shocks can destabilize learning even if the steady state is stable with small shocks. Relationship to stationary sunspot equilibria are also discussed.

Key words: Bounded rationality, Recursive algorithms, Steady state, Linearization, Asset pricing, Overlapping generations.

JEL Classification Numbers: C62, C61, D83.

14.1 Introduction

We consider stability under adaptive learning of stochastic steady state equilibria for nonlinear expectations models of the form

$$y_t = H(E_t^* G(y_{t+1}, w_{t+1}), w_t), \tag{14.1}$$

where y_t is a scalar endogenous variable and w_t is an exogenous random shock. Expectations $E_t^*(.)$ may not always be rational and under rational expectations we denote them by $E_t(.)$. Explicit stability results for this model have been obtained by [4] under the restrictive assumption that w_t is an *iid* process. In this paper we extend the stability results to the case where w_t is a time-dependent process taking the form

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of a finite Markov chain. The extension is useful since many applied models make the Markov chain assumption for the shock process, see e.g. [7] and [11].

The finiteness of the Markov chain is a limitation in our results, but it allows the formulation of adaptive learning in terms of a finite number of parameters. More general assumptions for shocks would lead to the description of stochastic states in terms of infinite dimensional parameters. For example, if w_t in (14.1) is a general Markov process the stochastic steady state is likely to be a function $y(w_t)$ that cannot in general be expressed in parametric form. Agents would have to be estimate $y(w_t)$ using advanced nonparametric methods. Such an approach has been studied by [1], but they do not develop explicit stability conditions in terms of the properties of the underlying economic model. Moreover, models of adaptive learning are based on the hypothesis of boundedly rational agents and the assumption that such agents are sufficiently sophisticated to use non-parametric techniques seems to go against the spirit of the hypothesis.[3]

The class of models (14.1) with autocorrelated shocks is also important for a different reason. The applied literature very often studies linearizations or log-linearizations of nonlinear Euler equations. The linearization is usually done around a non-stochastic steady state under the assumption that the support of the exogenous shock process is small. We provide existence results for models with small Markov chain shocks and then relate the general stability conditions to the linearized case and thus provide a bridge between the nonlinear and the linearized settings.[4]

The results are applied to a standard model of asset pricing and a stochastic overlapping generations model. The stochastic steady state in the asset pricing model is locally stable under learning irrespective of the size of the shocks. In contrast, in the overlapping generations model it is possible that the steady state is stable under learning with small shocks but unstable with sufficiently large shocks.

In this setting the stochastic steady state y_t for model (14.1) is formally a finite Markov chain. This invites a comparison of the steady state solution to the concept of a stationary sunspot equilibrium (SSE), which has been widely discussed in the literature; see e.g. the survey [2]. In Section 14.7 we discuss the differences and similarities between the stochastic steady state under Markov shocks and SSEs.

14.2 The Model

The precise assumptions for model (14.1) are as follows. $H(.)$ and $G(.)$ are twice differentiable with bounded first and second derivatives in some open rectangles. $G(y_t, w_t)$ is assumed to be observable. w_t is an observable exogenous variable that follows a finite Markov chain with states $\{\hat{w}_1, ..., \hat{w}_K\}$ and transition probabilities $\pi_{ij} = \Pr\{w_{t+1} = \hat{w}_j \,|w_t = \hat{w}_i\}$. The transition matrix $\Pi = (\pi_{ij})$ is assumed to be recurrent, irreducible and aperiodic.

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[3] Instead the agents might use a simpler but mis-specified parametric model. For brevity, we do not consider this possibility further.

[4] [5] study a linear model with an exogenous variable that follows a finite Markov chain.

A *rational stochastic steady state* for the stochastic model (14.1) is defined as a set of points $\lambda_1^*, ..., \lambda_K^*$ such that

$$\text{if } w_t = \hat{w}_k, \text{ then } \lambda_t = \lambda_k^*, \text{ where}$$
$$\lambda_k^* = \sum_{s=1}^{K} \pi_{ks} G(H(\lambda_s^*, w_s), w_k). \tag{14.2}$$

λ_k^* is interpreted as the value of $E_t G(y_{t+1}, w_{t+1})$ when the current state of the shock is k and the value of the endogenous variable is $y_t = H(\lambda_k^*, \hat{w}_k)$ if the exogenous variable has the value $\hat{w}_k$ in the current period t.

We remark that (14.2) is a natural definition of a stochastic steady state in the current setting. This is because the agents, having seen the current state of the shock in period t, take account of its value and correctly predict the expected value of $G(y_{t+1}, w_{t+1})$ conditionally on the current state with the economy being in the steady state next period, so that y_{t+1} has the alternative values $H(\lambda_s^*, w_s)$, $s = 1, ..., K$.

Model (14.1) with zero shocks is assumed to have a non-stochastic steady state, which can defined as a solution $\bar{\lambda}$ to the equation

$$\bar{\lambda} = G(H(\bar{\lambda}, 0), 0). \tag{14.3}$$

Next, we introduce the notation $\bar{G}_1 = D_1 G(H(\bar{\lambda}, 0), 0)$, $\bar{H}_1 = D_1 H(\bar{\lambda}, 0)$, select a norm $\|w\| = \max_{1 \leq k \leq K} |w_k|$ and make the following regularity assumption:

Regularity: $\bar{G}_1 \bar{H}_1$ is non-zero and $(\bar{G}_1 \bar{H}_1)^{-1}$ is not an eigenvalue of Π.

Under the regularity assumption we have existence of stochastic steady states with small Markov chain shocks:

Proposition 1. *Suppose that model (14.1) has a non-stochastic steady state $\bar{\lambda}$ defined by (14.3), the Regularity assumption holds for $\bar{\lambda}$. Assume that the transition matrix Π is recurrent, irreducible and aperiodic. Then $\exists \hat{\varepsilon}$ such that $\forall \varepsilon < \hat{\varepsilon}$ model (14.1) has a stochastic steady state $\lambda_1^*, ..., \lambda_K^*$ defined by equations (14.2) for Markov chain shocks with transition matrix Π and states $\{\hat{w}_1, ..., \hat{w}_K\}$ that satisfy $\|w\| \leq \varepsilon$.*

Proof.. Consider the set of equations (14.2), which we write in vector form $F(\lambda, w) = 0$, where $\lambda = (\lambda_1, ..., \lambda_K)$ and $w = (w_1, ..., w_K)$. By the implicit function theorem this vector equation defines locally a function $\lambda(w)$ around $w = \mathbf{0} \equiv (0, ..., 0)$ if $\det(D_1 F(\bar{\lambda}\mathbf{1}, \mathbf{0})) \neq 0$. Here the notation $\bar{\lambda}\mathbf{1} = \bar{\lambda}(\mathbf{1}, ..., \mathbf{1})$ is used. It is easily computed that

$$D_1 F(\bar{\lambda}\mathbf{1}, \mathbf{0}) = \begin{pmatrix} 1 - \pi_{11}\bar{G}_1\bar{H}_1 & -\pi_{12}\bar{G}_1\bar{H}_1 & \cdots & -\pi_{1K}\bar{G}_1\bar{H}_1 \\ -\pi_{21}\bar{G}_1\bar{H}_1 & 1 - \pi_{22}\bar{G}_1\bar{H}_1 & \cdots & -\pi_{2K}\bar{G}_1\bar{H}_1 \\ \vdots & \vdots & \ddots & \vdots \\ -\pi_{K1}\bar{G}_1\bar{H}_1 & -\pi_{K2}\bar{G}_1\bar{H}_1 & \cdots & 1 - \pi_{KK}\bar{G}_1\bar{H}_1 \end{pmatrix}$$
$$= \bar{G}_1\bar{H}_1[(\bar{G}_1\bar{H}_1)^{-1} I - \Pi],$$

provided $\bar{G}_1\bar{H}_1 \neq 0$. Thus $\det(D_1F(\bar{\lambda}\mathbf{1}, \mathbf{0})) \neq 0$ if $(\bar{G}_1\bar{H}_1)^{-1}$ is not an eigenvalue of Π.

Proposition 1 shows that, under mild assumptions, there exist steady state solutions to the stochastic model (14.1). We do not consider existence further since more general existence results, with shocks that are not small, are usually derived for concrete economic models rather than general classes of models such as (14.1). The asset pricing model with stochastic dividend growth due to [11], which is discussed below, is an example of this approach towards existence of equilibria.

14.3 Convergence of Learning to Steady State

We now formulate the adaptive learning of a stochastic steady state (14.2) for model (14.1). Suppose that the agents do not know the steady state values $\lambda_1^*, ..., \lambda_K^*$ but try to infer them from past data. In the past the exogenous shock has been in the different states $k = 1, ...K$ and agents have perceptions that the economy is in an unknown steady state, as defined in (14.2). A plausible learning rule for this setting is that agents group the data into K different groups conditionally on occurrence of the different values w_k of the exogenous shock and compute the state-contingent averages of the group values of the relevant variable.

Thus let $\lambda_{j,t}$ be the estimate of the steady state value λ_j^* for period $t+1$ when the exogenous variable is in state $\hat{w}_j$ in period t. The temporary equilibrium given the forecasts and the current state $\hat{w}_j$ is then $y_t = H(\lambda_{j,t}, \hat{w}_j)$. Define also the indicator function $\psi_{j,t} = 1$ if $w_t = \hat{w}_j$ and $= 0$ otherwise. The learning rule can be written as

$$\lambda_{j,t} = \lambda_{j,t-1} + t^{-1}\psi_{j,t-1}q_{j,t-1}^{-1}(\sum_{s=1}^{K} \pi_{js}G(H(\lambda_{s,t-1}, w_s), w_j) \tag{14.4}$$
$$-\lambda_{j,t-1} + u_{t-1})$$
$$q_{j,t} = q_{j,t-1} + t^{-1}(\psi_{j,t-1} - q_{j,t-1}), \tag{14.5}$$

for $j = 1, ..., K$.

The equations of the learning algorithm can be interpreted as follows. $q_{j,t}$ is the fraction of observations through $t-1$ in which the state $\hat{w}_j$ has occurred. (14.5) is the recursive form for computing the fraction. (14.4) is the recursive form for computing state contingent averages, except for a small measurement or observation error u_{t-1}. u_{t-1} is assumed to be iid with mean 0 and bounded support. (The existence of u_{t-1} is needed only for the instability result.) We remark that the estimates used by agents at time t are based on observations only through period $t-1$.[5] Equation (14.4) also specifies that $\lambda_{j,t-1}$ is updated only if $w_{t-1} = \hat{w}_j$. We remark that, since the agents formulate the forecasts for next period values for λ_j conditionally on the current state j, they do not use the values λ_i for other states $i \neq j$. Data conditional on j does not provide useful information for estimating the other conditional expectations.

[5] This assumption is often used in the literature. It avoids a simultaneity problem between y_t and the forecasts $E_t^*G(y_{t+1}, w_{t+1})$.

The learning rule formulated above is an example of stochastic recursive algorithms (SRA). We employ the techniques for such systems to derive the conditions for convergence of adaptive learning to the stochastic steady state $(\lambda_1^*, ..., \lambda_K^*)$. It turns out that (see below for details) the conditions for local convergence of SRAs can be studied using the local asymptotic stability of the equilibrium of an associated ordinary differential equation (ODE). The stability conditions for latter can in turn be related to those of another ODE:

$$\frac{d\lambda_j}{d\tau} = \sum_{s=1}^{K} \pi_{js} G(H(\lambda_s, \hat{w}_s), \hat{w}_j) - \lambda_j, j = 1, ..., K. \tag{14.6}$$

Clearly, the steady state $(\lambda_1^*, ..., \lambda_K^*)$ is an equilibrium point of (14.6). We linearize (14.6) and introduce the notation $G_{js}^{(1)} = D_1 G(H(\lambda_s^*, \hat{w}_s), \hat{w}_j)$ and $H_s^{(1)} = D_1 H(\lambda_s^*, \hat{w}_s)$. Linearizing (14.6), we obtain:

Theorem 1. *Assume that no eigenvalue of the matrix*

$$M = \begin{pmatrix} \pi_{11} G_{11}^{(1)} H_1^{(1)} & \pi_{12} G_{12}^{(1)} H_2^{(1)} & \dots & \pi_{1K} G_{1K}^{(1)} H_K^{(1)} \\ \pi_{21} G_{21}^{(1)} H_1^{(1)} & \pi_{22} G_{22}^{(1)} H_2^{(1)} & \dots & \pi_{2K} G_{2K}^{(1)} H_K^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{K1} G_{K1}^{(1)} H_1^{(1)} & \pi_{K2} G_{K2}^{(1)} H_2^{(1)} & \dots & \pi_{KK} G_{KK}^{(1)} H_K^{(1)} \end{pmatrix} \tag{14.7}$$

has real part equal to 1. *Then*
(i) The rational steady state $(\lambda_1^*, ..., \lambda_K^*)$ *is locally stable under learning if all the eigenvalues of* M *have real parts less than* 1.
(ii) The rational steady state is locally unstable under learning if M *has an eigenvalue with real part greater than* 1.

Proof.. The recursive equations for the parameter estimates can be written in the form

$$\theta_t = \theta_{t-1} + t^{-1} \mathcal{H}(\theta_{t-1}, X_t),$$

where $\theta_t = (\lambda_{1,t}, ..., \lambda_{K,t}; q_{1,t}, ..., q_{K,t})'$ is the vector of parameters, $X_t = (\psi_{1,t-1}, ..., \psi_{K,t-1}, u_{t-1})'$ is the vector of state variables, and

$$\begin{aligned} \mathcal{H}_j(\theta_{t-1}, X_t) &= \psi_{j,t-1} q_{i,t-1}^{-1} \left(\sum_{s=1}^{K} \pi_{js} G(H(\lambda_{s,t-1}, w_s), w_j) - \lambda_{j,t-1} + u_{t-1}\right) \\ &\quad \text{for } j = 1, ..., K, \text{ and} \\ \mathcal{H}_j(\theta_{t-1}, X_t) &= \psi_{j,t-1} - q_{j,t-1} \\ &\quad \text{for } j = K+1, ...2K. \end{aligned}$$

It is then possible to apply theorems on the convergence of SRAs, see e.g. part II of [6]. The assumptions made in Section 14.2 can easily be shown to imply the required convergence conditions.[6] The results state that, under the appropriate conditions, convergence of these algorithms is governed by the stability of the associated ODE

$$\frac{d\theta}{d\tau} = h(\theta) \text{ where } h(\theta) = \lim_{t\to\infty} E\mathcal{H}(\theta, X_t). \tag{14.8}$$

To compute the function $h(\theta)$ we first let $\bar{\pi}_1, ..., \bar{\pi}_K$ denote the invariant probabilities of the different states of the Markov chain $\Pi = (\pi_{ij})$. Then it is easily seen that

$$h_j(\theta) = \bar{\pi}_j q_j^{-1}(\sum_{s=1}^{K} \pi_{js} G(H(\lambda_j, \hat{w}_j), \hat{w}_j) - \lambda_j), j = 1, .., K$$
$$h_j(\theta) = \bar{\pi}_j - q_j, j = K+1, ..., 2K.$$

The latter set of differential equations is independent of the former and is clearly globally stable with $q_j \to \bar{\pi}_j$. It follows that $(\lambda_j^*, \bar{\pi}_j), j = 1, ..., K$, is a locally asymptotically stable equilibrium point of the associated differential equation (14.8), provided λ_j^* is locally asymptotically stable equilibrium point for the "small" ODE (14.6). Result (i) follows from linearizing (14.6).

To prove result (ii) we remark that the conditions for a standard instability result for SRAs are also satisfied.[7]

These results provide simple conditions to evaluate whether the steady state is stable under learning. For brevity we refrain from the details on the sense of convergence; see, Chapters 6 and 7 of [6] for a discussion. The relevant conditions are obtained by studying the "small" ODE (14.6). This relationship establishes that a concept known as expectational stability (or E-stability) is the key condition behind stability under learning. An REE is defined to be E-stable if it is a locally asymptotically stable equilibrium point of an ODE between given forecasts and the resulting temporary equilibrium. In model (14.1) the ODE defining E-stability is (14.6). [6] provide an extensive discussion of this connection between convergence of adaptive learning and E-stability for different kinds of models.

14.4 The Case of Small Shocks

It is important to note that the derivatives in matrix M in Theorem 1 are evaluated at different points $(\lambda_j^*, \hat{w}_j)$. Thus it is first necessary to compute the steady state values $(\lambda_j^*, \hat{w}_j)$. However, if the random shocks are small in the sense that the different values $\hat{w}_j$ are near a constant value, say, zero then an important further result can be obtained. We now take up this issue.

Letting $w = (w_1, ..., w_K)$, suppose that the states of the shock process satisfy $\|w\| \le \varepsilon$ for small $\varepsilon > 0$. As in Section 14.2 we can consider the stochastic steady

[6] The arguments on pp. 68-69 of [5] can be applied here to show that conditions for general algorithms are satisfied for the model under study.

[7] The formal details are analogous to those in [3], Proof of Proposition 5.2.

state $\lambda_j^*, j = 1, ..., K$, to be a function of w. By continuity, the non-stochastic steady state $\bar{\lambda}$ is a limiting value, so that $\lambda_j^*(w) \to \bar{\lambda}$ for all j as $\|w\| \to 0$ and the steady state values $\lambda_j^*(w)$ satisfy $\left|\lambda_j^*(w) - \bar{\lambda}\right| < \delta, j = 1, ..., K$, for some $\delta > 0$. Also denote by $M(w)$ the matrix (14.7) in Theorem 1 when the elements are viewed as functions of w. By continuity of eigenvalues we have $\lim_{\|w\| \to 0} M(w) \to \bar{G}_1 \bar{H}_1 \Pi$.

Under the regularity assumption $\bar{G}_1 \bar{H}_1 \neq 1$ and thus we have:

Proposition 2. *Consider a given transition probability matrix Π, a stochastic steady state $\lambda_j^*(w), j = 1, ..., K$ and the corresponding non-stochastic steady state $\bar{\lambda}$.*
(i) If $\left|\bar{G}_1 \bar{H}_1\right| < 1$ for the non-stochastic steady state, then there exists $\bar{\varepsilon} > 0$ such that the stochastic steady state is E-stable for economies when the different states of exogenous shock satisfy $\|w\| < \bar{\varepsilon}$.
(ii) The stochastic steady state for economies with an Markovian exogenous variable satisfying $\|w\| < \varepsilon$ for ε sufficiently small, is stable under learning only if $\bar{G}_1 \bar{H}_1 \leq 1$ for the limit non-stochastic steady state.[8]

Proof.. (i) Since Π is a Markov matrix, 1 is an eigenvalue of Π and all eigenvalues have modulus ≤ 1. Thus, as $\|w\| \to 0$, the limits of the eigenvalues of $M(w)$ are equal to $\bar{G}_1 \bar{H}_1 \mu$, where μ is an eigenvalue of Π. If $\left|\bar{G}_1 \bar{H}_1\right| < 1$, the eigenvalues of $M(w)$ are sufficiently close to the values $\bar{G}_1 \bar{H}_1 \mu$, which then satisfy $\left|\bar{G}_1 \bar{H}_1 \mu\right| < 1$.

(ii) Suppose to the contrary that $\bar{G}_1 \bar{H}_1 > 1$. By continuity, at least one eigenvalue of $M(w)$ is greater than one for all $\|w\|$ sufficiently small, which implies instability under learning by Theorem 1.

Proposition 2 can be seen as the basis for the common practice whereby one linearizes the model (14.1) at the non-stochastic steady state and studies the resulting linear model. Since a finite Markov chain can be written in an autoregressive form, see p.679 of [8]), the linear model can be written, after centering, as

$$y_t = A E_t^* y_{t+1} + B w_{t+1} + C w_t, \tag{14.9}$$
$$w_{t+1} = P w_t + v_t, \tag{14.10}$$

where $A = \bar{G}_1 \bar{H}_1$, $B = \bar{G}_2 \bar{H}_1$ and $C = \bar{H}_2$. This can be analyzed further in the usual way. For example, a sufficient condition for stability under learning is $|A| < 1$, while $A < 1$ is a necessary condition.[9]

14.5 Application I: Asset Pricing

The standard model of asset pricing, see e.g. Chapter 10 of [10] for an exposition, leads to an Euler equation for the price p_t of an asset paying a dividend d_t in each period:

[8] We remark that $\bar{G}_1 \bar{H}_1 < 1$ is the E-stability condition for the non-stochastic model, see [4].

[9] [5] study stability conditions for learning in the special case $B = 0, C = 1$. These special assumptions are inconsequential for the stability and instability results.

$$p_t u'(d_t) = \beta E_t^* p_{t+1} u'(d_{t+1}) + \beta E_t^* d_{t+1} u'(d_{t+1}). \tag{14.11}$$

Here $u(.)$ is the utility function of the representative consumer and β is the discount factor. We do not assume that expectations are necessarily always rational, as indicated by * in the expectations.

Case 1. (Asset pricing with stationary dividends) Following [10], Example 2, we first consider the case where dividends assume a finite set of values $\{\hat{d}_1, ..., \hat{d}_K\}$ and they evolve according to a finite Markov chain with transition probabilities $\pi_{ij} = \Pr\left\{d_{t+1} = \hat{d}_j \middle| d_t = \hat{d}_i\right\}$. For simplicity, this Markov chain is assumed to be known to the representative agent. At time t the agent needs to predict the ex dividend asset price p_{t+1} for the period.

Equivalently, we can assume that the agent make prediction of the quantity $E_t^* p_{t+1} u'(d_{t+1})$. Letting $y_t = p_t u'(d_t)$ and $w_{t+1} = d_{t+1} u'(d_{t+1})$ the model (14.11) becomes

$$y_t = \beta E_t^* (y_{t+1} + w_{t+1}),$$

which is a special case of (14.1) when $H(y, w) = \beta y$ and $G(y, w) = y + w$. We can directly apply the general results and conclude that the REE is E-stable and hence locally stable under adaptive learning.

Case 2. (Asset pricing with dividend growth) [11] formulate a finite state model of dividend growth by assuming that the growth of dividends follows a finite Markov chain. In other words, we assume that

$$d_{t+1} = \mu_t d_t,$$

where μ_t is a finite (K-state) Markov chain with a transition matrix Π. Assume also that the utility function of the representative consumer exhibits constant relative risk aversion, so that

$$u(d) = \frac{d^{1-\gamma}}{1-\gamma}, \gamma > 0.$$

The price dividend ratio p_t/d_t can be shown, see p.240 of [10], to satisfy the equation

$$\frac{p_t}{d_t} = E_t^* \left[\beta(\mu_{t+1})^{1-\gamma}\left(\frac{p_{t+1}}{d_{t+1}} + 1\right)\right], \tag{14.12}$$

where again we have allowed for the possibility that expectations may not always be rational.

Letting $y_t = p_t/d_t$ and $w_t = \mu_t$, it is easily seen that (14.12) is a special case of model (14.1) with $H(y, w) = y$ and $G(y, w) = \beta(w)^{1-\gamma}(y + 1)$. The E-stability of the REE is governed by the matrix

$$\begin{pmatrix} \pi_{11}\beta(w_1)^{1-\gamma} & \pi_{12}\beta(w_2)^{1-\gamma} & \cdots & \pi_{1K}\beta(w_K)^{1-\gamma} \\ \pi_{21}\beta(w_1)^{1-\gamma} & \pi_{22}\beta(w_2)^{1-\gamma} & \cdots & \pi_{2K}\beta(w_K)^{1-\gamma} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{K1}\beta(w_1)^{1-\gamma} & \pi_{K2}\beta(w_2)^{1-\gamma} & \cdots & \pi_{KK}\beta(w_K)^{1-\gamma} \end{pmatrix} \tag{14.13}$$

and E-stability requires all of the eigenvalues of matrix (14.13) must have real parts less than one. The model is no longer linear and in principle the stability of the REE under learning could be affected by the nonlinearity. However, instability under learning can be ruled out using a further argument exploited by [11] to ensure uniqueness of the REE. They impose the further condition that all eigenvalues of the matrix (14.13) should be inside the unit circle. This determinacy condition clearly implies that the unique REE is also E-stable.

We summarize the analysis in the following proposition:

Proposition 3. *The stochastic steady state in the standard asset pricing model is stable under learning when (i) the dividends follows a finite Markov chain and also (ii) when the growth rate of dividends follows a finite Markov chain and the eigenvalues of (14.13) have real parts less than one.*

We emphasize that stability of the REE under learning could in principle be affected by the size of the Markov chain shocks in Case 2 of the asset pricing model. (Case 1 is a linear model in which the magnitude of the shocks does not matter for stability.) The assumption used by [11] to ensure existence and uniqueness of REE is, however, sufficient also to rule out instability of the REE under learning.[10]

14.6 Application II: Overlapping Generations

As a second application of our results we consider the standard overlapping generations model (the so-called Samuelson model) with productivity shocks. This model was suggested by [4], Section 6 and we generalize their analysis to the case where the shock are a Markov chain rather than iid. The analysis will illustrate that the size of the shocks is important for the stability of the steady state under learning.

In the Samuelson model it is assumed that each generation lives for two periods. They work when young and consume only when old. Thus the utility function of the representative consumer born in period t takes the form

$$U(c_{t+1}) - V(n_t),$$

where c_{t+1} denotes his consumption in period $t+1$ (when the consumer is old) and n_t denotes labor supply in period t (when he is young). The utility function for consumption $U(c)$ is assumed to be strictly concave while the disutility of labor supply $V(n)$ is taken to be strictly convex. Both are assumed to be twice continuously differentiable. Output is assumed to be perishable and the production function of the consumer is

$$q_t = n_t + \mu_t,$$

[10] We remark that uniqueness (or determinacy) of REE does not in general imply stability under learning for all models, see Parts III and IV of [6] for a detailed discussion.

where the additive productivity shock μ_t is taken to be nonnegative random variable. It is observable when time t decisions are made. We assume that μ_t follows a finite Markov chain that is recurrent, irreducible and aperiodic.

The budget constraints of the consumer are

$$p_t q_t = M_t,$$
$$p_{t+1} c_{t+1} = M_t,$$

where p_t is the price of output and M_t denotes his savings in the form of money. It is assumed that there is a constant stock of nominal money in the economy. As is well-known, utility maximization and market clearing yield the equation

$$(n_t + \mu_t)V'(n_t) = E_t^*[(n_{t+1} + \mu_{t+1})U'(n_{t+1} + \mu_{t+1})], \tag{14.14}$$

which characterizes the (interior) temporary equilibria when the consumers have the expectations $E_t^*(.)$.

The model (14.14) can be cast in the general framework (14.1) as follows. First, we define $y_t = n_t$, $w_t = \mu_t - E(\mu)$, where $E(\mu)$ is the mean of the invariant distribution of μ_t and set

$$G(y, w) = (y + w + E(\mu))U'(y + w + E(\mu)).$$

Second, note that the left hand (14.14) is strictly increasing in n_t, so that (14.14) can be solved for n_t. Then $y = H(x, w)$ is implicitly defined by the equation

$$x = (y + w + E(\mu))V'(y).$$

With these definitions the basic results, Theorem 1 and Proposition 2 can be applied.

We are interested in studying whether the size of the shocks can affect the stability of the stochastic steady state of the model. To show this we assume that utility functions are isoelastic

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma}, V(n) = \frac{n^{1+\varepsilon}}{1+\varepsilon},$$

in which case (14.14) takes the form

$$(n_t + \mu_t)n_t^{\varepsilon} = E_t^*[(n_{t+1} + \mu_{t+1})^{1-\sigma}].$$

We also let $\mu_t = E(\mu) + v_t$, where $v_t \in \{v_1, v_2\}$ with transition probability matrix Π.

We select the following numerical values for the parameters: $\sigma = 4$, $\varepsilon = 1$, $E(\mu) = 0.6$ leading to a non-stochastic steady state at 0.645 and this steady state is stable under steady state learning, see p.201 of [4]. Note that the non-stochastic steady state is also a limiting case, with $v_1, v_2 \to 0$ for any transition matrix Π, of the stochastic steady state when the Markov chain shocks are present. We now specify the values $\pi_{11} = \pi_{22} = 0.1$ for the transition matrix and $v_1 = 0.05$. We then vary the value of v_2 and consider E-stability of the stochastic steady state when the value of v_2 is increased. Table 1 reports the signs of the trace and determinant of the matrix $M - I$ for different values of v_2.[11]

Table 1. E-stability of the stochastic steady state

v_2	0.1	0.15	0.2	0.25	0.3	0.35
Tr	$-$	$-$	$-$	$-$	$-$	$-$
det	$+$	$+$	$+$	$+$	$-$	$-$

Table 1 illustrates how the steady state becomes unstable when the size of the shock becomes larger. As might have been anticipated on the basis of Theorem 1 and Proposition 2, there are models in which the steady state REE is stable under learning when shocks are small but is not stable with large enough shocks. We summarize the finding in

Remark 1. In stochastic nonlinear models stability under learning of a steady state can be affected by the magnitude of the random shocks. A steady state that is stable under learning when shocks are small can become unstable if the shocks are sufficiently large.

14.7 Comparison to Sunspot Equilibrium

In the stochastic steady state (14.2) the endogenous variable y_t becomes a finite Markov chain with transition matrix Π. We remark that a set of data on y_t that is a finite Markov chain with the same transition probabilities as some exogenous variable w_t can in principle be rationalized as a stochastic steady state with w_t as the exogenous shock as in model (14.1) or, alternatively, as an SSE with w_t playing the role of the extrinsic sunspot. Thus there is a potential problem of observational equivalence as in [12]. Distinguishing between the two solution concepts would require testing whether the influence of w_t on y_t is structural or only expectational (i.e. w_t is only a conditioning variable for expectations).

Some further comments can be made about the two equilibrium concepts. First, stochastic steady states (14.2) exist under very mild regularity conditions, as exemplified by Proposition 1. In contrast, the existence of an SSE near a single steady state requires the fulfillment of the indeterminacy conditions for the steady state. Thus the this kind of steady state modeling may have broader applicability than modeling based on an SSE.

Second, the conditions for stability under learning of a stochastic steady state and of an SSE for frameworks such as (14.1) are formally similar; compare our Theorem 1 and Proposition 2 in [5] or pp. 307-308 of [6] (with noise set identically zero). Moreover, the stability conditions become identical in the limiting case where values of the shock approach zero. From this viewpoint, the two notions of equilibrium have similar characteristics.

[11] In this example M is a 2×2 matrix and we can consider stability by computing the trace and determinant of $M - I$.

14.8 Concluding Remarks

In this paper we have extended the earlier results of stability under learning of steady states in stochastic nonlinear models to an important case, where the exogenous shocks are no longer iid and are instead correlated over time. Explicit local stability and instability results in terms of the underlying economic framework were obtained when the shocks follow a finite state Markov chain. We also discussed the significance of small vs. large shocks, the linearization of the model and compared the stochastic steady state to SSEs.

The assumption that the Markov chain is finite is restrictive, though this case has been fairly often used in applications. The assumption enables the definition of steady states and formulation of adaptive learning, so that agents realistically estimate the values of a finite number of parameters.

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15

The Evolution of Conventions under Incomplete Information*

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Summary. We formulate an evolutionary learning process with trembles for static games of incomplete information. For many games, if the amount of trembling is small, play will be in accordance with the games' (strict) Bayesian equilibria most of the time supporting the notion of Bayesian equilibrium. Often the process will select a specific equilibrium. For two specific games of economic interest we characterize this selection. The first is an extension to incomplete information of the prototype strategic conflict known as "Chicken". The second is an incomplete information bilateral monopoly, which is also an extension to incomplete information of Nash's demand game, or a simple version of the so-called sealed bid double auction. The examples reveal that equilibrium selection by evolutionary learning may well be in favor of Bayesian equilibria where some types of players fail to coordinate, so that the equilibrium outcomes are inefficient.

Key words: Static games of incomplete information, Bayesian games, Evolution, Conventions, Chicken, Bilateral monopoly, Double auction.

JEL Classification Numbers: C72.

15.1 Introduction

This paper suggests an evolutionary learning process in the spirit of Young ([12]) for static games of incomplete information and demonstrates in general how such a process may give justification for the notion of Bayesian equilibrium and how it may give selection among multiple Bayesian equilibria. For two specific examples of games of economic interest the selection is characterized. The first is an extension to incomplete information of the prototype strategic conflict known as the "Chicken" game. The second is an incomplete information bilateral monopoly, which can also

* This paper is an extended version of our article with the same title in *Economic Theory* 25(1), 171 - 185. In addition to the results from the journal article, it contains (in Section 5 and Appendix 1) analysis of a bilateral monopoly games.

be viewed as an extension to incomplete information of Nash's demand game, or as a so-called (sealed bid) double auction. Although the applications are special, a general insight emerges: equilibrium selection by evolutionary learning cannot in general be expected to be in favor of the more efficient equilibria where the different types of the players coordinate well.

It is well known that equilibrium selection by evolutionary processes may under complete information be in favor of inefficient equilbria, e.g., in favor of a risk dominant, but payoff dominated equilibrum in a coordination game. However, the inefficiency obtained here is of a different type linked to miscoordination between types and hence to incomplete information.

There are mainly two motivations for extending the models of evolutionary learning to incomplete information. First, the foundations of Bayesian equilibrium are at least as shaky as those of Nash equilibrium. Any doubt one may have concerning the feature that players use best replies against each other is as relevant for Bayesian equilibrium as it is for Nash equilibrium. For Bayesian equilibrium one may further doubt if the idea that players plan for types they are actually not, is an adequate formalization of how players cope with uncertainty in games. It is therefore of interest if one can give a justification of Bayesian equilibrium for games of incomplete information, like the one given of Nash equilibrium for games of complete information by, e.g., [12]. Second, going from complete to incomplete information in games often adds a dimension of equilibrium multiplicity - in particular it often implies the existence of inefficient equilibria with miscoordination between types - such that equilibrium selection may be even more relevant in games of incomplete information. Since the evolutionary models with small trembles suggested by, e.g., [6] and [12], have proved to be strong devices for equilibrium selection, it is a natural idea to generalize such processes to games of incomplete information.

We generalize Young's process of evolutionary learning to static games of incomplete information. To do this, one has to give a physical meaning to the types and priors that are part of the description of an incomplete information game. We assume that there are two large pools of players containing players to take the "row" and the "column" position in the game respectively, and that each pool is subdivided in types. The existing types are as in the underlying game. Play goes on in subsequent rounds. In each round, one player is picked at random from each pool and these two play the game once, and their actions are observed by everybody, just as assumed in [12]. We assume that all players know the true probability distribution by which opponents' types are picked, and that after each round of play the types of the picked players become known to everybody.

For each type of each player there is a record of the actions a player of that type took in a certain number of earlier rounds where this type was picked for play. After a round of play the records on the two types who played are updated; the oldest observation is deleted and the new one is inserted. The records on other types are unchanged. When a player is about to play, he intends to play a best reply to an expectation on the opponent created from samples from the current records on the opponent. This defines the basic learning process. The perturbed learning process is defined from the basic one by adding a tremble: with a small probability ε a picked player does not (necessarily) play a best response, but takes an arbitrary action. The interest is in the perturbed process for ε small.

A considered game and the process may well be such that when the probability of trembling is small play will be in accordance with the games's strict Bayesian equilibria most of the time. For such games one has thus obtained the kind of support for the notion of Bayesian equilibrium looked for. Further, when a game has several strict Bayesian equilibria, it may well be that play will be in accordance with a specific one most of the time. A selection among the Bayesian equilibria is thus obtained. We study such selection in the Chicken and bilateral monopoly games of incomplete information.

Both games are extensions to incomplete information of the kind of coordination games typically studied in the earlier contributions on evolutionary learning in games of complete information, and both are illustrations of how incomplete information may add an economically important dimension of equilibrium multiplicity. Each game has under complete information several strict Nash equilibria, but these have similar efficiency properties. In Chicken both pure Nash equilibria involve one player taking the tough action and the other taking the cautious one; equilibrium selection is only a matter of who takes which action, i.e., who gets the (main part of the) surplus. In Nash's demand game there is a strict Nash equilibrium for each possible strict division, but all equilibria are efficient since all surplus is exploited. Equilibrium selection is only a matter of how large a share each player gets. Under incomplete information the games may well have equilibria with qualitatively different efficiency properties. In Chicken there are still the two equilibria where (independently of types), one side takes the tough, and the other side takes the cautious, action, but in addition there can be an equilibrium where sometimes (for some types), both players take the tough action, and a waste occurs. Our extension of Nash's demand game to incomplete information is one where the game is interpreted as a bilateral monopoly and the buyer and the seller are uncertain about each others' reservation prices. Such a game may well have both strict equilibria, where the bids of the different types are spread out in a way that ensures that any two types who can trade to mutual benefit do so, and strict equilibria where the bids are clustered in a way so that there are couples of types who could trade to mutual benefit, but do not do so, and the potential gains from trade are not exhausted. The first type of equilibrium is efficient, the latter is inefficient. For both games the type of inefficiency possible in equilibrium is due to incomplete information.

The present paper is closely related to contributions such as [1], [6], and [12], that introduced the approach of evolution, or evolutionary learning, with trembles for static games of complete information. Some papers have studied equilibrium selection by evolutionary models for dynamic games with both incomplete information and sequential moves, most notably for signalling games, [2], [5], and [10]. The present paper is (as far as we know) the first to study an evolutionary process in the same spirit for *static* games of incomplete information.[4]

The assumption of observability of types is fundamental and may be controversial, while the assumption of knowledge of priors seems less controversial since if types can be observed then close to correct priors can be derived from long records. The assumption of observability of types is reasonable if the player pools are large, since then each individual only plays rarely, and thus has no incentive to hide his type after having played. Furthermore, to preserve from Young's work the essential structural equivalence between equilibria and particular states (so-called conventions), records have to list actions type-wise, since a Bayesian equilibrium reports an action for each type of each player. Records of past play can only be type-wise if types can be observed A straight extension of Young's work to incomplete information has to imply observability of types.[5]

In Section 2 we give the definitions of games of incomplete information and Bayesian equilibrium. Section 3 defines the evolutionary learning process in two steps, the basic and the perturbed process, respectively. Furthermore we state some general results about convergence and stochastically stable states. In Sections 4 and 5 we analyze the incomplete information versions of the Chicken and the bilateral monopoly game, respectively, and characterize the long run behaviors supported by evolutionary learning in those games. Section 6 concludes. Proofs relating to the Chicken game are given in Appendix. The calculations in the proofs for the results relating to the bilateral monopoly game are very long in their complete versions and therefore we just provide the basic intuition in Section 5. The complete proofs are available upon request.

15.2 Games of Incomplete Information and Bayesian Equilibrium

We describe a finite static two player game of incomplete information as follows. The Row player, Player 1, has finite action set R, and the Column player, Player 2, has finite action set C. Player 1 is of one type α out of the finitely many in A, while Player 2 is of one type β out of the finitely many in B. Each player knows his own type, but not the opponent's. Player 1's belief concerning Player 2's type is given by a probability measure b over B. Likewise, Player 2's belief concerning Player 1's type is given by a probability measure a over A. If players 1 and 2 of types α and β choose actions $r \in R$ and $c \in C$ respectively, they obtain von Neumann-Morgenstern payoffs $u(r, c, \alpha, \beta)$ and $v(r, c, \alpha, \beta)$.

A probabilistic expectation, or conjecture, of Player 1, concerning Player 2's choice, is a collection $q = (q_\beta)$ of probability measures over C, one for each of

[4] In [9], Myatt and Wallace consider evolution in specific Bayesian games, obtained by adding Harsanyian trembles to the agents' payoffs, and use this framework to obtain equilibrium selection in static games of complete information without introducing trembles.

[5] Of course, an alternative process where only actions can be observed is also of interest. In [5] we have studied such a process for the specific case of signaling games.

Player 2's possible types. Likewise, an expectation of player 2 is $p = (p_\alpha)$, where each p_α is a probability measure over R. The expected payoff of a Player 1 of type α, who holds a conjecture q, from choosing the specific action r, is,

$$U_\alpha(r, q) = \sum_{\beta \in B} b(\beta) \sum_{c \in C} u(r, c, \alpha, \beta) q_\beta(c),$$

and r is a best reply if it maximizes $U_\alpha(r', q)$ over $r' \in R$. Let the set of (pure) best replies be $BR_\alpha(q)$. Define the expected payoff of Player 2, $V_\beta(c, p)$, similarly and let the set of best replies for a Player 2 of type β holding a conjecture p be $BR_\beta(p)$. The sets $BR_\alpha(q)$ and $BR_\beta(p)$ are always non-empty.

A Bayesian equilibrium is a pair $r(\cdot), c(\cdot)$, where $r(\alpha) \in R$ for all $\alpha \in A$, and $c(\beta) \in C$ for all $\beta \in B$, such that if one for each α defines the probability measure p_α by $p_\alpha(r(\alpha)) = 1$, and for each β defines q_β by $q_\beta(c(\beta)) = 1$, then $r(\alpha) \in BR_\alpha(q)$ for all $\alpha \in A$, and $c(\beta) \in BR_\beta(p)$ for all $\beta \in B$. A Bayesian equilibrium is strict if for all the α's and β's, $BR_\alpha(q)$ and $BR_\beta(p)$ are singletons. We only consider games with at least one strict Bayesian equilibrium.

15.3 Evolutionary Learning in Games of Incomplete Information

A Bayesian equilibrium is a Nash equilibrium of the extended game where the players' pure strategies are mappings $r : A \longrightarrow R$, and $c : B \longrightarrow C$, and payoffs associated to such strategies are the expected values of payoffs from the original incomplete information game, where expectations are taken with respect to types. However, the evolutionary learning process we define in this section is not just an application of the process of Young ([12]) to this extended game. Applying Young's process directly to the extended game would imply an assumption that in each round the full strategy of the opponent, and not just the action taken by the relevant type of opponent, is observed, and this is not meaningful.

15.3.1 The Basic Learning Process

Envisage that there are two large (disjoint) pools of player 1s and 2s, both pools partitioned according to types. Players from the two pools play a game in subsequent rounds. In each round one player 1 is picked at random from pool 1, such that type α has probability $a(\alpha)$, and one player 2 is picked randomly from pool 2 with probability $b(\beta)$ of type β. The picked player 1 chooses an action r from R, while the picked player 2 chooses an action c from C. They receive payoffs according to their choices r and c, and their types α and β, as given by the underlying game, that is, $u(r, c, \alpha, \beta)$ to player 1, and $v(r, c, \alpha, \beta)$ to player 2.

We assume that players know the true probability measure, a or b, by which their opponent is picked, and that after a round of play the actions r and c chosen in the round, as well as the true types α and β of the players who took them, become known to everybody.

The individuals keep records of past play. For each type α or β, a record h_α or h_β reports which actions were taken the last m_α or m_β times a player of that type played, $h_\alpha \in R^{m_\alpha}$ and $h_\beta \in C^{m_\beta}$. A state h is a complete description of the records, $h = ((h_\alpha), (h_\beta)) \in H \equiv \prod_{\alpha \in A} R^{m_\alpha} \times \prod_{\beta \in B} C^{m_\beta}$. The state space H is thus finite. After a round of play where a type α of player 1 chose r, against a type β of player 2 who chose c, only h_α and h_β are updated, and this is done by deleting in each of them the oldest observation and inserting as the newest observation r and c respectively. Given a state h, a state h' is a successor to h, if it is possible to go from h to h' in one step according to this procedure by picking α and β, and r and c appropriately.

When a player 1 has been picked, he first samples from the records on player 2; for each β he takes a sample Q_β from h_β, where the sample size $k_\beta = \#Q_\beta$ fulfils $k_\beta \leq m_\beta$. The sampling goes on according to a random procedure, which is such that all observations in h_β have positive probability of ending in Q_β. Let Q be the collection of samples, $Q = (Q_\beta)$. A picked player 2 samples P_α from h_α, where again the sample size is $k_\alpha \leq m_\alpha$, and the set of samples is $P = (P_\alpha)$. Samples P_α and Q_β are converted into probability measures p_α and q_β over R and C respectively in the obvious way: $p_\alpha(r)$ is the number of times r appears in the sample P_α divided by k_α, etc. It should cause no confusion to identify the samples P and Q with the so derived collections of probability measures p and q.

According to the basic learning process, a player 1 picked for play will take an action in $BR_\alpha(Q)$, if he is of type α. If $BR_\alpha(Q)$ has several elements, player 1 will pick one at random according to a full support probability measure on $BR_\alpha(Q)$. Similarly a picked player 2 of type β will take an action in $BR_\beta(P)$.

Given the random procedures by which players are picked, sampling goes on, and ties are broken, there will for each pair of states h and h' be a specific probability of going from h to h' in one step. Call this transition probability $\pi^0(h, h')$. If h' is not a successor of h, then $\pi^0(h, h') = 0$. The matrix of all transition probabilities is Π^0, which defines a homogeneous Markov chain on H.

A convention is defined as a state h with two properties: First, it consists entirely of constant records, that is, for each α, the record h_α is a list of m_α identical actions $r(\alpha)$, and for each β, h_β is a list of m_β identical actions $c(\beta)$. Second, each recorded action is the unique best reply to the only samples that are possible from h: If one for each α lets P_α be the list of k_α times $r(\alpha)$, and for each β lets Q_β be the list of k_β times $c(\beta)$, then $BR_\alpha(Q) = \{r(\alpha)\}$ for each α, and $BR_\beta(P) = \{c(\beta)\}$ for each β.

A convention is an absorbing state for Π^0, that is, a state that one stays in for sure when transitions are governed by Π^0. Since we have assumed that in case of several best replies all have positive probability, it is also true that every absorbing state is a convention. Further, for any convention there is a strict Bayesian equilibrium defined in the obvious way, and every strict Bayesian equilibrium defines exactly one convention. Since we have assumed that there is a strict Bayesian equilibrium, there is at least one convention.

Assume that from any state, there is, according to Π^0, positive probability of reaching a convention in a finite number of steps. From a standard argument there is then probability one of finally reaching a convention irrespectively of initial state,[6] and we say that the basic learning process converges to a convention. One can formulate assumptions on the considered game and the details of the learning process that ensure convergence. The following is a straightforward extension of Proposition 1 in [12]. First define the best reply graph: Each node s is a combination $(r(\alpha), c(\beta))$ of actions, one for each type of each player. There is a directed edge from s to s' if and only if s' is different from s for exactly one type of one player, and for this type the action in s' is the unique best reply to the action combination of the opponent in s. The game is weakly acyclic if, from any node, there is a directed path (through directed edges) to some node out of which there is no edge (a sink). Every sink is a strict Bayesian equilibrium. For each node s, let $L(s)$ be the length of the shortest path to a sink. Let L be the maximum of $L(s)$ over all nodes. Finally, let k be the largest sample size, $k = \max_{\alpha,\beta}(k_\alpha, k_\beta)$, and let m be the shortest record length, $m = \min_{\alpha,\beta}(m_\alpha, m_b)$.

Proposition 0. *If the game is weakly acyclic and for each type* γ: $k_\gamma \leq \frac{m_\gamma}{L+2}$, *then the basic learning process converges to a convention.*

The proof is similar to the proof of Proposition 1 in [12], where it is demonstrated that in the one type model it is sufficient for convergence that the game is weakly acyclic and that for each player $k \leq \frac{m}{L+2}$. Young's proof is constructive, providing a finite sequence of samples that has positive probability and that implies that a convention is reached. In our model only one type of each player plays in each round and a similar sequence of types and samples can be constructed. Since sampling and updating is done for each type of each player separately, it is sufficient that the condition on k_γ is fulfilled typewise.

Convergence implies that there are no other absorbing sets than the singleton sets of conventions. It then follows from a standard result on Markov chains that a stationary distribution for Π^0, a probability measure μ over H, such that $\mu\Pi^0 = \mu$, can only have $\mu(h) > 0$ for conventions h. The reverse, that any distribution with $\mu(h) > 0$ only for conventions h is stationary, is obvious.

15.3.2 The Perturbed Learning Process

The basic process is modified by assuming that there is in each round a small probability ε (independent across rounds and players) that a picked player will not play a best reply to his samples, but take a random action according to a specific full support distribution (independent of ε) over his full strategy set. The remaining probability, $1 - \varepsilon$, is assigned to the best replies as before. This means that in each round, any

[6] If there is, starting from any state, at least probability π of reaching a convention in s steps, then there is at most probability $1 - \pi$ of *not* reaching a convention in s steps, and then there is at most probability $(1-\pi)^t$ of not reaching a convention in st steps. Here $(1-\pi)^t$ goes to zero as t goes to infinity. So, independently of initial state it has probability zero to not eventually reach a convention, or probability one to reach one.

"suboptimal" action has a probability proportional to ε of being played. The described trembling can be interpreted as mistakes or experiments or "mutations". The transition probabilities of the modified process are called $\pi^\varepsilon(h, h')$, and the matrix of transition probabilities, the perturbed Markov chain, is called Π^ε.

The process Π^ε is irreducible: for any pair of states h and h', there is according to Π^ε positive probability of going from h to h' in a finite number of steps; this is just a matter of picking types and actions (which now all have positive probability) appropriately. Further, Π^ε is aperiodic: this follows since Π^ε is irreducible and there is a state (namely a convention) h with $\pi^\varepsilon(h, h) > 0$. Finally, Π^ε is a regular perturbation of Π^0 in the sense of [12], i.e., Π^ε is irreducible and aperiodic, $\Pi^\varepsilon \longrightarrow \Pi^0$ as $\varepsilon \to 0$, and for any transition hh' for which $\pi^0(h, h') = 0$, there is a well defined order of the speed by which $\pi^\varepsilon(h, h')$ goes to zero with ε.

The resistance in a transition hh' is defined as this order: If h' is not a successor of h, then even with trembles it is impossible to go from h to h', and the resistance in hh' is infinite. If $\pi^0(h, h') = 0$, but one can go from h to h' if and only if one type of each player makes a tremble, so two trembles are necessary, then $\pi^\varepsilon(h, h')$ is some constant times ε^2, which goes to zero with a speed of order two, and the resistance in hh' is two. If $\pi^0(h, h') = 0$, but it requires just one tremble (by one type of one player) to go from h to h', then the resistance is one. Finally, if $\pi^0(h, h') > 0$, the resistance is zero. To find the resistance in a transition from one state to another is just a matter of counting the trembles necessary to go from the first to the second.

It is standard from the theory of Markov chains that for each $\varepsilon > 0$, there is a unique stationary distribution μ^ε, $\mu^\varepsilon \Pi^\varepsilon = \mu^\varepsilon$, and if one lets the process run for a long time according to the probabilities of Π^ε, then the relative frequencies by which the states are visited converge to the probabilities of μ^ε with probability one. Our interest is in Π^ε for small ε, and therefore in $\mu^0 \equiv \lim_{\varepsilon \longrightarrow 0} \mu^\varepsilon$. It follows by [12], Lemma 1, that this limit distribution exists and is a stationary distribution for Π^0. The states in the support of μ^0 are called stochastically stable, and these are the states that will be observed frequently for small ε. Since μ^0 is stationary for Π^0, *if the basic learning process converges, then only the conventions of Π^0 can be stochastically stable.*

This has two important implications. In the long run observed play will most of the time be in accordance with the game's strict Bayesian equilibria. This supports the notion of Bayesian equilibrium, in so far as the game and the details of the process are such that the basic learning process is convergent. Further, it will often be the case that only one convention is stochastically stable, so only play according to a specific strict Bayesian equilibrium will be observed frequently. This provides a selection among the Bayesian equilibria.

To actually find the selected equilibria a characterization of the stochastically stable states is useful. The following is a result (Corollary to Proposition 2) from [12], specialized to the situation where all absorbing sets are singletons (conventions). Assume that Π^0 has several conventions $h_1, ..., h_T$. For two conventions h and h', define the resistance in the (indirect) transition hh', as the minimal sum of resistances over all collections of direct transitions leading from h to h'. An h-tree is a collection of transitions $h_l h_n$ between conventions such that each convention other than h stands first in exactly one transition, and for any convention $h' \neq h$, there is a unique way to go from h' to h through the transitions of the collection. For each h-tree one can define the sum of resistances embodied in the transitions of the tree. The stochastic potential of h is the minimal of such total resistances over all h-trees.

Theorem 0. *Assume that the basic learning process converges to a convention. Then the stochastically stable states are exactly the conventions with minimal stochastic potential.*

15.4 Conventional Behavior in a Game of Chicken with Incomplete Information

Below to the left the complete information game "Chicken" is displayed, where $R = C = \{D, H\}$, and $\alpha > 0$ and $\beta > 0$.

	D	H
D	$0, 0$	$0, \beta$
H	$\alpha, 0$	$-1, -1$

	D	H
D	$0, 0$	$1, \beta$
H	$\alpha, 1$	$0, 0$

This is a prototype strategic situation where each player has a cautious action, here D (Dove), and a tough one, here H (Hawk), and it is good to take the tough action against an opponent playing cautiously - the parameters α and β indicate how good - but bad to take it against an opponent who also plays toughly, while D against D is "neutral".[7] The game has exactly two strict Nash equilibria: One is (H, D), where Player 1 plays H, and Player 2 plays D; the other is (D, H).

The game is strategically equivalent to the game at the right, and it follows that if $\alpha > \beta$, then (H, D) "risk dominates" (D, H), and vice versa, cf. [4]. Both [6] and [12] show that the risk dominance selection rule is supported by evolutionary learning. The process of Young ([12]) is as defined in Section 3 with only one type of each player. Assume that the record sizes for both positions in the game are m, and the sample sizes are both k. It should cause no confusion to let a vector of m times H be denoted also by H, etc. The two conventions are (H, D) and (D, H), corresponding to the two strict equilibria. Young shows that if $k < m/3$, then the basic learning process converges, and if further k is sufficiently large, then the convention corresponding to the risk dominating equilibrium is the only stochastically stable state.

We now consider a game of Chicken with incomplete information. The action sets are still $R = C = \{D, H\}$, and uncertainty concerns how good it is for a player to play H against D. Player 1 is of one of two types which, with a slight abuse of notation, are called $1/\alpha$ and α, where α is a number greater than one. Each type has

[7] The assumption that a player using D receives the same payoff when the opponent plays D as when he plays H simplifies some formulas below, but is not essential for the basic result. The Chicken game formalizes, e.g., duopoly situations where the strategic conflict is really a battle over the roles as leader and follower.

probability $1/2$. Similarly, player 2 is of type $1/\beta$ with probability 1/2, and of type β with probability $1/2$, where $\beta > 1$. For the action combinations (D, D) and (H, H) payoffs are independent of types and as given in the complete information game, $(0, 0)$ and $(-1, -1)$ respectively. If the action combination is (H, D), then player 1's payoff is independent of the type of player 2, and it is $1/\alpha$ if player 1 is of type $1/\alpha$, and α if he is of type α, while player 2 gets 0 independently of types. Similarly, if the combination is (D, H), then player 2 gets $1/\beta$ if of type $1/\beta$, and β if of type β, independently of the type of player 1, while player 1 gets zero irrespective of types. For what follows it is convenient that the incomplete information game is (also) given by just two parameters α and β, and that these measure the degree of the players' uncertainty about payoffs.

The incomplete information game has three strict Bayesian equilibria. The first two are: $r(1/\alpha) = r(\alpha) = H, c(1/\beta) = c(\beta) = D$, and $r(1/\alpha) = r(\alpha) = D, c(1/\beta) = c(\beta) = H$. These are counterparts of the strict equilibria of the complete information game; on side plays H, and the other plays D irrespective of types. The players coordinate and clearly the equilibria are efficient ex ante and ex post. The third equilibrium is $r(1/\alpha) = c(1/\beta) = D, r(\alpha) = c(\beta) = H$, where the "low" types play D, and the "high" types play H. This is an equilibrium since against the strategy of player 2, player 1 of type $1/\alpha$ obtains zero in expected payoff from D and $\frac{1}{2}(\frac{1}{\alpha} - 1)$ from H, so D is best, while player 1 of type α obtains zero from D and $\frac{1}{2}(\alpha - 1)$ from H, so H is better, etc. There is a lack of coordination in it, since when both players are of low type or both are of high type, losses occur. The equilibrium is not inefficient ex ante, but it is ex post, and even ex ante it is dominated by a half-half convex combination of the other two equilibria.

Consider the learning process defined in Section 3 for this particular game. Because of the symmetry of the game it is natural to assume $m_{1/\alpha} = m_a = m_{1/\beta} = m_\beta = m$, and $k_{1/\alpha} = k_\alpha = k_{1/\beta} = k_\beta = k$. The three conventions are, in obvious notation, (HH, DD) and (DD, HH) corresponding to the two efficient equilibria with coordination, and (DH, DH) corresponding to the equilibrium with lack of coordination. It is easy to check that the game is weakly acyclic and $L = 2$. Therefore we get directly from Proposition 0,

Proposition 1. *If $k < m/4$, then the basic learning process converges to a convention.*

So, when $k < m/4$ and the trembling probability is small, only behavior in accordance with the three conventions, or strict Bayesian equilibria, can be observed frequently. Theorem 1, which is proved in the Appendix, tells which will be observed,

Theorem 1. *Assume $k < m/4$. If k is sufficiently large, then if $(\alpha - 1)(\beta - 1) < 4$, the conventions (HH, DD) and (DD, HH) are the only stochastically stable states, while if $(\alpha - 1)(\beta - 1) > 4$, the convention (DH, DH) is the only stochastically stable state.*

Thus, if there is substantial difference between the payoffs of the different types (i.e, if α and β are so high, that $(\alpha - 1)(\beta - 1) > 4$), then the process selects the equilibrium (DH, DH), where players sometimes miscoordinate. This happens since high values of α and β imply that in the equilibria (HH, DD) and (DD, HH) it only takes a few trembles by the player using action H for the best response of the high type of the opponent (type α or β) to change, and similarly it only takes a few trembles of the player using action D for the best response of the low type of the opponent (type $1/\alpha$ or $1/\beta$) to change. This is in constrast to what happens in the equlibrium (DH, DH), where each type of each player "goes for his most preferred equilibrium", and it therefore takes more trembles to upset this action being a best response.[8]

15.5 Conventional Behavior in a Bilateral Monopoly with Incomplete Information

Bargaining in a bilateral monopoly with uncertainty about reservation prices can be modelled as a game of incomplete information, as suggested by [3]. There are two players: player 1 is a buyer characterized by a reservation price $\alpha \in A$, and player 2 is a seller characterized by a reservation price (or cost) $\beta \in B$, where A and B are the sets of possible reservation prices. The buyer offers a price $r \in R$, and the seller asks for a price $c \in C$. It is assumed that $R = C = M$, where M is the set of allowed bids. Trade occurs if and only if $r \geq c$, at the trading price $p = \frac{r+c}{2}$.[9] The payoff for the buyer is zero if no trade occurs, and $\alpha - p$ otherwise, and the payoff for the seller is zero if no trade occurs, and $p - \beta$ otherwise. We refer to α and β as the types of the buyer and the seller respectively. Each player knows his own type, but not the opponent's. The belief concerning the latter is given by some probability distribution over A or B, which is independent of the player's own type.

This game has gained much attention in the literature. It captures some important aspects of bargaining under incomplete information. In fact it is an extension to incomplete information of the bargaining game known as Nash's demand game, which appears as the complete information special case, $A = \{1\}$, $B = \{0\}$. Furthermore, it is a possible formalization of what goes on whenever a buyer meets a seller, and the two are not completely sure about the other's eagerness to sell or buy. Finally, it is a simple case of a "sealed bid double auction", a much studied trade mechanism, see [11].

In the special case of Nash's demand game, there is a strict Nash equilibrium for each bid $x \in M$ strictly between zero and one, where both players bid x. All these equilibria are efficient and only differ with respect to the division of the surplus.

[8] It is just an artifact of the payoff specifications above that when $(\alpha - 1)(\beta - 1) < 4$, both of the conventions corresponding to a coordinated equilibrium are stocahstically stable. It is relatively easy to see that if one increases one of the player 1 payoffs a little bit above $1/\alpha$ or α, then only (HH, DD) is stochastically stable.

[9] The set of allowed trading prices is thus the set of all possible midpoints of allowed bids.

Consider for this game the learning process defined in Section 3, just with only one type for each player, and assume that record sizes on both players are m, and sample sizes are both k, and that $k \leq m/2$. There is a convention for each of the strict Nash equilibria, where each convention consists of two records, one for each player, with only the bid x of the equilibrium in them. It is an implication of the main result in [13], that for k sufficiently large, the convention with $x = 1/2$ is the only stochastically stable state (assuming M symmetric with enough bids in it, and $1/2 \in M$).

Under incomplete information there may be both more efficient (strict) equilibria with good coordination between types and less efficient (strict) equilibria with less good coordination between types, This is shown below (and it is well-known from the literature, e.g., [7] and [9]. Often it seems to be taken for granted that the more efficient equilibria are also the more descriptive ones. We will study equilibrium selection by evolutionary learning in a bilateral monopoly game to see if a general conjecture, that the more efficient, highly coordinated equilibria are also the more relevant ones, gains support.

We study a finite version of the game, where $A = \{\frac{1}{3}, \frac{2}{3}, 1\}$, $B = \{0, \frac{1}{3}, \frac{2}{3}\}$, and for each player independently each type has probability $1/3$. Further, $M = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$, where $n = 10^N$ for some natural number N.[10] This is a simplest possible version of the game that still represents a "true" conflict between more and less efficiency in the sense that the game has strict equilibria of two kinds: 3-step equilibria which are efficient, and 2-step equilibria which are inefficient, see below.

For the buyer it is dominated to bid at or above the reservation price α, but any allowed bid x below α is undominated since it is the unique best reply if the seller bids x irrespective of type. Likewise, for a seller of type β, the undominated actions are all bids in M strictly above β. The game has many Bayesian equilibria in pure strategies using only undominated actions. The strategies $r(\alpha) = 0$ for all α, and $c(\beta) = 1$ for all β, form a no-trade equilibrium, which is highly inefficient. Next, for any allowed bid $x \neq 0, 1$, the strategies $r(\alpha) = x$ if $\alpha > x$, and $r(\alpha) = 0$ otherwise, and $c(\beta) = x$ if $\beta < x$, and $c(\beta) = 1$ otherwise, form a one-step equilibrium, which involves some trade, but is still inefficient. A one-step equilibrium is illustrated in Figure 1(a). Zero- and one-step equilibria are not strict: there will always be either a type α of buyer who bids zero, or a seller β, who bids one, which is best reply, but not the only one, since, e.g., the smallest allowed bid above zero is also a best reply for such a buyer. Therefore none of these equilibria can correspond to conventions.

There are, however, strict two-step equilibria. Let x and z be two allowed bids with $x < \frac{1}{3}$, and $z > \frac{2}{3}$. Consider the strategies $r(\cdot)$ and $c(\cdot)$ for the buyer and seller respectively,

$$r(\frac{1}{3}) = r(\frac{2}{3}) = x, \; r(1) = z, \text{ and } c(0) = x, \; c(\frac{1}{3}) = c(\frac{2}{3}) = z, \tag{15.1}$$

as illustrated in Figure 1(b). It is immediate that for $\alpha \in \{\frac{1}{3}, \frac{2}{3}\}$, the action $r(\alpha) = x$ is the unique best reply to $c(\cdot)$. For $r(1) = z$ to be also the unique best reply, a

[10] The assumption, $n = 10^N$, is technical and not essential for our results.

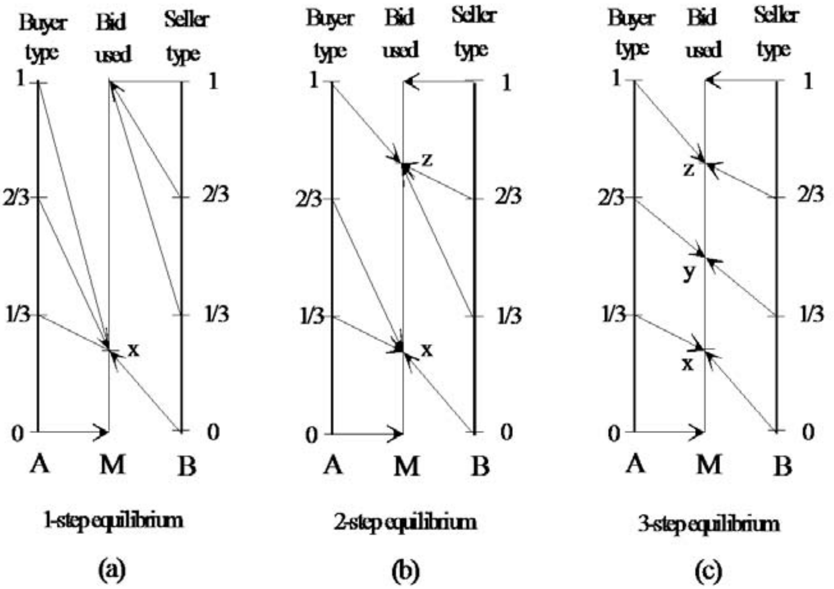
deviation by type $\alpha = 1$ down to x must yield less expected payoff, that is, one must have $\frac{1}{2}(1-z) + \frac{1}{4}(1-\frac{x+z}{2}) > \frac{1}{4}(1-x)$, or $x > 5z - 4$. Likewise, $c(\beta)$ is obviously best reply for $\beta \in \{\frac{1}{3}, \frac{2}{3}\}$, but for $\beta = 0$ to play a unique best reply one must have $\frac{1}{2}x + \frac{1}{4}\frac{x+z}{2} > \frac{1}{4}z$, or $5x > z$. So, under the further restrictions $x > 5z-4$, and $5x > z$, also implying $x > 0$ and $z < 1$, the strategies form a strict Bayesian equilibrium, and such strategies always exist: just set $x = \frac{3}{10}$ and $z = \frac{7}{10}$. The two-step equilibria are not efficient, because if a buyer of type $\alpha = \frac{2}{3}$ happens to meet a seller of type $\beta = \frac{1}{3}$, the two will not trade according to equilibrium behavior, although they could do so to mutual benefit at any price between their reservation prices.[11] There are also equilibria involving trade at two different prices which are not strict. For instance, there exist pairs of allowed bids fulfilling $0 < x < \frac{1}{3} < y < \frac{2}{3}$, and strategies $r(\frac{1}{3}) = x$, $r(\frac{2}{3}) = r(1) = y$, and $c(0) = x$, $c(\frac{1}{3}) = y$, $c(\frac{2}{3}) = 1$, that form an equilibrium, but here the seller of type $\beta = \frac{2}{3}$ does not have a unique best reply.

Finally, there are strict three-step equilibria. Let three allowed bids fulfill $x < \frac{1}{3} < y < \frac{2}{3} < z$, and let the strategies be,

$$r(\frac{1}{3}) = c(0) = x, r(\frac{2}{3}) = c(\frac{1}{3}) = y, r(1) = c(\frac{2}{3}) = z, \tag{15.2}$$

as illustrated in Figure 1(c). Under some restrictions these form a strict equilibrium. Obviously, the buyer type $\alpha = \frac{1}{3}$ plays a unique best reply. For $\alpha = \frac{2}{3}$, it must not pay to deviate from y down to x: $\frac{1}{4}(\frac{2}{3} - y) + \frac{1}{4}(\frac{2}{3} - \frac{x+y}{2}) > \frac{1}{4}(\frac{2}{3} - x)$, or $4 + 3x > 9y$. Similarly one finds that the condition for z being strictly better than y for type $\alpha = 1$, is $y > 2z - 1$. One does not have to check whether a deviation by $\alpha = 1$ all the way down to x could pay, due to an obvious monotonicity property: higher types can only have higher best replies. So, if deviation down to x payed for $\alpha = 1$, so it would have done $\alpha = \frac{2}{3}$. By symmetry (in the above conditions change x to $1 - z$, and z to $1 - x$, and y to $1 - y$), the condition for a seller of type $\beta = 0$ to strictly prefer x to y is $2x > y$, and for $\beta = \frac{1}{3}$ to strictly prefer y to z is $9y > 3z + 2$. Again, due to monotonicity in best reply as a function of type (and since seller types $\beta = \frac{2}{3}$ obviously plays a unique best reply), one does not have to check further. Again the obtained restrictions imply that $x > 0$ and $z < 1$, and strategies obeying all restrictions exist, e.g., $x = \frac{3}{10}, y = \frac{1}{2}, z = \frac{7}{10}$. These strategies form a strict Bayesian equilibrium, which is obviously efficient. We have thus established,

[11] The presence of an unexploited opportunity of trade is not a full proof of inefficiency. One has to demonstrate the existence of better strategies. Change the equilibrium strategies so that $r(\frac{2}{3}) = c(\frac{1}{3}) = y$, where $\frac{1}{3} < y < \frac{2}{3}$, the strategies being otherwise the same. At the new strategies, buyers of types different from $\alpha = \frac{2}{3}$ obviously get higher or the same expected payoff. Type $\alpha = \frac{2}{3}$ gets higher payoff if $\frac{1}{4}(\frac{2}{3} - y) + \frac{1}{4}(\frac{2}{3} - \frac{x+y}{2}) > \frac{1}{4}(\frac{2}{3} - x)$, or $9y < 4 + 3x$. Similarly, only for the seller type $\beta = \frac{1}{3}$ could there be doubt about payoff, and this type gets higher expected payoff if $\frac{1}{4}(\frac{y+z}{2} - \frac{1}{3}) + \frac{1}{4}(y - \frac{1}{3}) > \frac{1}{4}(z - \frac{1}{3})$, or $9y > 2 + 3z$. Under the equilibrium restrictions on x and z, one has $4 + 3x > 2 + 3z$, so one can find a number y, such that $\frac{2+3z}{9} < y < \frac{4+3x}{9}$ (also implying $\frac{4}{9} < y < \frac{5}{9}$), and in fact the set M will always contain an allowed bid with this property. The new strategies are at least as good as the equilibrium strategies for all types and strictly better for some types.



Proposition 2. *There are exactly the following strict Bayesian equilibria of the considered game: two-step equilibria given by the strategies in (15.1) and the restrictions that x and z are both in M, that $0 < x < \frac{1}{3}$, $\frac{2}{3} < z < 1$, and,*

$$5x > z \text{ and } x > 5z - 4, \tag{15.3}$$

and three-step equilibria given by the strategies in (15.2) and the restrictions that x, y, and z are all in M, that $0 < x < \frac{1}{3} < y < \frac{2}{3} < z < 1$, and,

$$4 + 3x > 9y \text{ and } y > 2z - 1 \text{ and } 2x > y \text{ and } 9y > 3z + 2. \tag{15.4}$$

All the two-step equilibria are inefficient, whereas all the three-step equilibria are efficient.

For the evolutionary learning process defined in Section 3 it is natural to specify $m_\alpha = m_\beta = m$, and $k_\alpha = k_\beta = k$ for all α, β, where $k \leq m$, because of the symmetry of the game. A state h thus consists of three records h_α and three records h_β, each being a list of m bids from M. As described in Section 3 each strict Bayesian equilibrium $r(\cdot), c(\cdot)$ defines the convention with "constant" records, $h_\alpha = (r(\alpha), ..., r(\alpha))$ for each α, and $h_\beta = (c(\beta), ..., c(\beta))$ for each β, and there are no other conventions than these .

Proposition 3.[12] *If $k < m/2$, then the basic learning process converges to a convention.*

Our interest is in the perturbed learning process Π^{ε}, given by an $\varepsilon > 0$, for ε small. Since we use the general definition of Π^{ε} to the particular game studied here, a consequence of our formulation is, as discussed in Section 3.2, that in the ε-eventuality of a tremble, all bids, including the dominated ones on the "wrong" side of the reservation price have positive probability. This may be considered problematic, and in the complete proof of Theorem 2 below we therefore show that under the alternative assumption, that trembling can only be to bids on the "right" side of a player's reservation price, the theorem still holds.

Let h^* be the convention that corresponds to the particular two-step equilibrium given by (15.1) with $x = \frac{1}{4}$, and $z = \frac{3}{4}$. From Proposition 1, this is indeed a strict equilibrium as long as $N \geq 2$.

Theorem 2. *Assume $k < m/2$. The convention h^* is stochastically stable. If n and k are sufficiently large, then h^* is the only stochastically stable state.*

In words Theorem 2 says: If conventional behavior evolves gradually as described by the evolutionary learning process with small trembling probability ε, and there is not too little sampling (or too few allowed bids), then the only behavior that will be observed frequently in the long run is behavior in accordance with the inefficient two-step equilibrium given by $x = \frac{1}{4}$ and $z = \frac{3}{4}$.

The formal proof of Theorem 2 is long and tedious. Basically, what is shown is that it is relative more easy (it takes relatively few trembles) to escape from conventions corresponding to three-step equilibria than from conventions corresponding to two-step equilibria. In (conventions corresponding to) three-step equilibria the expected payoffs are relatively high, which contributes to making it take relatively many trembles to make a non-equilibrium bid a best reply. In two-step equilibria each equilibrium bid is made with larger probability, which contributes to making it take relatively many trembles to make a non-equilibrium bid a best reply. The proof demonstrates that this second force (bids made with high frequency) dominates the first force (high expected payoffs), such that a two-step equilibrium is selected.

The considered game is specific, but nevertheless a general lesson is learned. A conjecture that evolutionary learning will generally tend to support the more efficient equilibria with the more spread bids is, by Theorem 2, shown to be wrong. Further, the intuition for Theorem 2 in terms of the two counteracting forces suggests that the possibility that less efficient equilibria with more clustered bids can be the most firm is not specific to the considered example.

Theorem 2 is a statement on the limit behavior that will be observed as $\varepsilon \to 0$, if n and k are sufficiently large. One may rightfully wonder, if there is only a reasonable degree of precision in the prediction pointed to by Theorem 2 for extreme

[12] The proofs of Proposition 3 and Theorem 2 below are very long and have been omitted, but they are available from the authors. The basic intuition for the main result is provided below.

values of ε, n, and k. We therefore performed a simulation with moderate parameter values which rejects this suspicion.[13] The game studied in the simulation is as above, only with $M = \left\{0, \frac{1}{16}, ..., \frac{15}{16}, 1\right\}$. This is the second-smallest symmetric set that includes the bids $\frac{1}{4}$ and $\frac{3}{4}$. For the learning process we used $k = m = 5$, and $\varepsilon = 1\%$. The 100 million iterations were started in the convention corresponding to the three-step equilibrium $x = \frac{1}{4}, y = \frac{1}{2}, z = \frac{3}{4}$. From the proof of Theorem 2, this is, so to say, the closest three-step competitor to h^*. Control simulations revealed, however, that one gets almost the same figures if one starts the iterations in h^* itself. The program counts the number of steps where the state is exactly equal to h^*. However, this underestimates how often the process is really at the two-step equilibrium. Occasionally trembles occur which will in the subsequent rounds pass through the records without making these essentially different from h^*. The program therefore also counts the number of rounds in which the state is such that the best replies of all types are the same as in state h^*. Finally the program counts the number of rounds in which best replies are as in some three-step equilibrium. The results are that in 71% of the time the state is exactly h^*, and in 97% of the time play generated by the state is (best replies are) as in h^*. In 0.37% of the time the state will generate play according to a three-step equilibrium. The conclusion is that also for moderate values of ε, n, and k, play will most of the time correspond to the two-step equilibrium with $x = \frac{1}{4}$ and $z = \frac{3}{4}$.

15.6 Conclusions

The evolutionary learning process we have studied is meant to formalize the view that agents involved in strategic conflicts expect their opponents to act more or less as they usually do in identical or similar conflicts. Therefore they form conjectures on opponents from records of past play. They intend primarily to take actions that are best given such conjectures, but with small probability they take more or less arbitrary actions perhaps to test if they are right in the presumption that alternative actions give poorer results.

We have shown in general how this view may imply support for the notion of Bayesian equilibrium, since the learning process may well generate play in accordance with the game's (strict) Bayesian equilibria most of the time, and selection among multiple Bayesian equilibria, since the learning process may well generate play in accordance with one particular Bayesian equilibrium most of the time.

For two specific incomplete information "surplus division" games we have found the selected equilibria. It turns out that under incomplete information equilibrium selection by evolutionary learning is no longer just a matter of who grasps the surplus. Rather, under incomplete information evolutionary learning may well select in favor of equilibria that have inferior efficiency properties due to bad coordination between the player types.

[13] The GAUSS-programme used for these simulations is available from the authors upon request.

A Proof of Theorem 1

We find the stochastic potentials for all three conventions by systematically computing the resistances in all transitions from one convention to another.

First consider the transition from (HH, DD) to (DH, DH). Obviously, the way to do this transition that requires the fewest trembles (the cheapest way) is either by player 1 type $1/\alpha$ trembling to D thus for possible sampling changing the best reply of player 2 type β to H, or by player 2 type β trembling to H thus changing the best reply of player 1 type $1/\alpha$ to D. Assume Player 1 of type $1/\alpha$ makes l trembles to D, that is, type $1/\alpha$ is picked in l consecutive rounds and plays D in all of them, where $l \leq k$. For a player 2 of type β, who samples all l of the Ds from the record on player 1 type $1/\alpha$, the best reply will have changed to H if $\frac{1}{2}\left[\frac{l}{k}\beta + \frac{k-l}{k}(-1)\right] + \frac{1}{2}(-1) \geq 0$, or if,

$$l \geq \frac{2}{\beta+1}k. \tag{15.5}$$

According to the basic learning process, there is positive probability that in each of the k rounds following the l trembles, player 1 is of type α, so $h_{1/\alpha}$ is unchanged and contains the l times D, and player 2 is of type β and in all k rounds samples all l of the Ds from $h_{1/\alpha}$, and hence plays H in all k rounds. This will gradually insert k times H in h_β. The picked player 1s of type α will with positive probability according to Π^0 have played H in all k rounds (they may not have sampled any Hs from h_β). After this h_β contains k times H. Now it has, according to the basic learning process, positive probability that in each of the next m rounds player 1 is of type $1/\alpha$ and samples k times H from h_β, which gives him best reply D (since $\frac{1}{2}\frac{1}{\alpha} + \frac{1}{2}(-1) < 0$), while player 2 is of type β, and all the time samples at least l times D from $h_{1/\alpha}$, and hence plays H. Note that in these m rounds, when some of the old Ds go out of $h_{1/\alpha}$, new Ds are inserted, and when some of the old Hs go out of h_β, new Hs are inserted, which makes it possible that best replies for possible sampling continues to be D for type $1/\alpha$, and H for type β, during all m rounds. Then the convention (DH, DH) has been reached without further trembling.

Now assume Player 2s of type β make l trembles to H. If player 1 type $1/\alpha$ samples all l of the Hs his best reply will have changed to D if $\frac{1}{2}\frac{1}{\alpha} + \frac{1}{2}\left[\frac{l}{k}(-1) + \frac{k-l}{k}\frac{1}{\alpha}\right] \leq 0$, or if,

$$l \geq \frac{2}{\alpha+1}k. \tag{15.6}$$

After these l trembles it has, as above, positive probability according to the basic process to reach (DH, DH) (first k rounds with types $1/\alpha$ and $1/\beta$, and then m rounds with types $1/\alpha$ and β).

So, the resistance in the transition from (HH, DD) to (DH, DH) is,[14]

[14] Note that here, and in what follows, we are ignoring an integer problem: The resistance is really the smallest integer which is (weakly) above either $\frac{2}{\beta+1}k$ or $\frac{2}{\alpha+1}k$. It is because of the assumption of "k sufficiently large" that this will cause no error (if a resistance as defined here is smaller than another one, then it will also be smaller according to the correct definition if only k is large enough).

$$\rho[(HH, DD) \to (DH, DH)] = \min\{\frac{2}{\beta+1}k, \frac{2}{\alpha+1}k\}.$$

By symmetry it follows that $\rho[(DD, HH) \to (DH, DH)] = \min\{\frac{2}{\alpha+1}k, \frac{2}{\beta+1}k\}$, which is the same.

Next we consider the transition from (DH, DH) to (HH, DD). It is again cheaply brought about either by trembling of type $1/\alpha$ of player 1, or of type β of player 2. Assume player 1s of type $1/\alpha$ tremble l times to H. The best reply of player 2 type β will for positive sampling have changed to D if $\frac{1}{2}\left[\frac{l}{k}(-1) + \frac{k-l}{k}\beta\right] + \frac{1}{2}(-1) \leq 0$, or if,

$$l \geq \frac{\beta-1}{\beta+1}k. \tag{15.7}$$

Again it can be shown that after these l trembles it has, according to the basic process, positive probability to go all the way to (HH, DD) (first k rounds with types α and β, and then m rounds with types $1/\alpha$ and β).

Now assume player 2s of type β tremble l times to D. The best reply of player 1 type $1/\alpha$ will for possible sampling have changed to H if $\frac{1}{2}\frac{1}{\alpha} + \frac{1}{2}\left[\frac{l}{k}\frac{1}{\alpha} + \frac{k-l}{k}(-1)\right] \geq 0$, or if

$$l \geq \frac{\alpha-1}{\alpha+1}k. \tag{15.8}$$

Again, after these l trembles it has, according to the basic process, positive probability to go all the way to (HH, DD). So,

$$\rho[(DH, DH) \to (HH, DD)] = \min\{\frac{\beta-1}{\beta+1}k, \frac{\alpha-1}{\alpha+1}k\}.$$

Due to symmetry, $\rho[(DH, DH) \to (DD, HH)] = \min\{\frac{\alpha-1}{\alpha+1}k, \frac{\beta-1}{\beta+1}k\}$ and the same.

Finally we consider the transition from (HH, DD) to (DD, HH) (and visa versa). It is evident that if this transition is to be initiated by trembles of player 1, then it is obtained most cheaply if it is type α who does it: trembling by type $1/\alpha$ and α are equally effective in making H a possible best reply for player 2, most easily so for type β. However, when this has occurred and h_β has become full of Hs, then the only possible best reply of player 1 type $1/\alpha$ is D, so $h_{1/\alpha}$ is filled up with Ds at no further cost, so it can only work to make the transition cheaper that the Ds, that occur because of trembling, are to be found in h_α.

So, assume that from (HH, DD), player 1s of type α make l trembles to D. From (15.5) the best reply for player 2 type β will have changed to H for possible sampling if $l \geq \frac{2}{\beta+1}k$. After these l trembles it has, according to the basic process, positive probability that in each of the next $2m$ rounds the types are $\frac{1}{\alpha}$ and β, so h_α does not change, and the samples drawn from this record contain in each round all l of the Ds, so each player 2 of type β plays H. After the first m rounds $h_\beta = H$, which makes D the only possible best reply for player 1 type $1/\alpha$, so after the next m rounds also $h_{1/\alpha} = D$.

Now suppose a type $1/\beta$ of player 2 is drawn. With positive probability the sample drawn from h_α will contain all l of the Ds in it. In that case the best reply of type $1/\beta$ will be H if $\frac{1}{2}\frac{1}{\beta} + \frac{1}{2}\left[\frac{l}{k}\frac{1}{\beta} + \frac{k-l}{k}(-1)\right] > 0$. It is already required that $l \geq \frac{2}{\beta+1}k$. By inserting this number of trembles we find that it also suffices for the now considered best reply shift exactly if $-\beta^2 + 2\beta + 3 \geq 0$, that is, if $\beta \leq 3$.

Case $\beta \leq 3$: It has, according to the basic process, positive probability that in each of the next m rounds the types are $1/\alpha$ and $1/\beta$, and that, since already $h_\beta = H$, all the player 1s of type $1/\alpha$ play D, keeping $h_{1/\alpha}$ unchanged, and all the player 2s of type $1/\beta$ play H, so after these rounds $h_{1/\beta} = H$. Finally, with positive probability the types α and β are drawn in the next m rounds, and since now $h_{1/\beta} = h_\beta = H$, with positive probability all the player 1s of type α will play D, which can only work to keep H a best reply for player 2 type β, so all of these play H. The state will then be the convention (DD, HH). Soonly the $\frac{2}{\beta+1}k$ trembles are required for transition.

Case $\beta > 3$: Here further trembling is required. There are two possibilities: (1) type α of player 1 trembles further such that the best reply for type $1/\beta$ of player 2 *does* shift to H (for possible sampling), or (2) type $1/\beta$ trembles such that the best reply for type α changes to D.

(1) Given that already $h_{1/\alpha} = D$, the total number of Ds required in h_α to ensure that H is a possible best reply for type $1/\beta$ is l such that $\frac{1}{2}\frac{1}{\beta} + \frac{1}{2}\left[\frac{l}{k}\frac{1}{\beta} + \frac{k-l}{k}(-1)\right] \geq 0$, or

$$l \geq \frac{\beta - 1}{\beta + 1}k. \tag{15.9}$$

If this were the number of trembles first made by type α, then as above we would with positive probability reach the convention (DD, HH).

(2) The best reply changes to D for type α of player 1, if player 2 type $1/\beta$ makes l trembles to H, where $\frac{1}{2}\left[\frac{l}{k}(-1) + \frac{k-l}{k}\alpha\right] + \frac{1}{2}(-1) \leq 0$, or $l \geq \frac{\alpha-1}{\alpha+1}k$. As above one can verify that after these *further* trembles there is positive probability according to the basic process of reaching the convention (DD, HH). So, the total number of trembles required is,

$$\frac{2}{\beta + 1}k + \frac{\alpha - 1}{\alpha + 1}k. \tag{15.10}$$

We have now shown that if transition from (HH, DD) to (DD, HH) is to be initiated by trembling of player 1, then if $\beta \leq 3$, the number of trembles required is as given by (15.5), whereas if $\beta > 3$, it is the minimum of the expressions in (15.9) and (15.10).

By exactly parallel arguments - substituting player 1 for player 2 and α for β etc. - one can show that if transition from (HH, DD) to (DD, HH) is to be initiated by trembling of player 2, then if $\alpha \leq 3$, the number of trembles required is as given by (15.6), whereas if $\alpha > 3$, it is the minimum $\frac{\alpha-1}{\alpha+1}k$ and $\frac{2}{\alpha+1}k + \frac{\beta-1}{\beta+1}k$.

The resistance in the transition from (HH, DD) to (DD, HH) is the minimum over the numbers of trembles required for transition when trembling is first done by player 1, and when it is first done by player 2. So,

$$\rho[(HH,DD)\to(DD,HH)] = k\cdot\begin{cases}\min\{\frac{2}{\alpha+1},\frac{2}{\beta+1}\} & \text{if } \alpha\le 3,\beta\le 3\\ \min\{\frac{\alpha-1}{\alpha+1},\frac{2}{\alpha+1}+\frac{\beta-1}{\beta+1},\frac{2}{\beta+1}\} & \text{if } \alpha>3,\beta\le 3\\ \min\{\frac{2}{\alpha+1},\frac{2}{\beta+1}+\frac{\alpha-1}{\alpha+1},\frac{\beta-1}{\beta+1}\} & \text{if } \alpha\le 3,\beta>3\\ \min\{\frac{\alpha-1}{\alpha+1},\frac{2}{\alpha+1}+\frac{\beta-1}{\beta+1},\frac{2}{\beta+1}+\frac{\alpha-1}{\alpha+1},\frac{\beta-1}{\beta+1}\} & \text{if } \alpha>3,\beta>3\end{cases}$$

which is equivalent to,

$$\rho[(HH,DD)\to(DD,HH)] = k\cdot\begin{cases}\min\{\frac{2}{\alpha+1},\frac{2}{\beta+1}\} \text{ if } \alpha\le 3,\beta\le 3\\ \min\{\frac{\alpha-1}{\alpha+1},\frac{2}{\beta+1}\} \text{ if } \alpha>3,\beta\le 3\\ \min\{\frac{2}{\alpha+1},\frac{\beta-1}{\beta+1}\} \text{ if } \alpha\le 3,\beta>3\\ \min\{\frac{\alpha-1}{\alpha+1},\frac{\beta-1}{\beta+1}\} \text{ if } \alpha>3,\beta>3\end{cases}$$

From symmetry we get by permuting α and β in the expression above that $\rho[(DD,HH)\to(HH,DD)] = \rho[(HH,DD)\to(DD,HH)]$.

In the following we assume wlog. that $\alpha\ge\beta$ (again by symmetry the result is the same if $\beta\ge\alpha$). The expression above is then reduced to,

$$\rho[(HH,DD)\to(DD,HH)] = \rho[(DD,HH)\to(HH,DD)] = k\cdot\begin{cases}\frac{2}{\alpha+1} & \text{if } \alpha\le 3\\ \min\{\frac{\alpha-1}{\alpha+1},\frac{2}{\beta+1}\} & \text{if } \beta\le 3\le\alpha\\ \frac{\beta-1}{\beta+1} & \text{if } \beta>3\end{cases}.$$

Now note that $\rho[(DD,HH)\to(HH,DD)]\ge\rho[(DD,HH)\to(DH,DH)]$, which together with the symmetry property $\rho[(DH,DH)\to(DD,HH)] = \rho[(DH,DH)\to(HH,DD)]$ imply that their can be no (HH,DD)-tree with less total resistance than the one consisting of the transitions from (DD,HH) to (DH,DH) and from (DH,DH) to (HH,DD). The stochastic potential of (HH,DD) is then,

$$\gamma[(HH,DD)] = \rho[(DD,HH)\to(DH,DH)] + \rho[(DH,DH)\to(HH,DD)].$$

By symmetry, $\gamma[(DD,HH)] = \rho[(HH,DD)\to(DH,DH)] + \rho[(DH,DH)\to(DD,HH) = \gamma[(HH,DD)]$, where the last equality also uses $\rho[(HH,DD)\to(DH,DH)] = \rho[(DD,HH)\to(DH,DH)]$.

Again, since $\rho[(DD,HH)\to(HH,DD)]\ge\rho[(DD,HH)\to(DH,DH)]$ etc., there can be no cheaper way to go to (DH,DH), than to go from each of the other conventions separately, so

$$\gamma[(DH,DH)] = \rho[(HH,DD)\to(DH,DH)] + \rho[(DD,HH)\to(DH,DH)].$$

Then we arrive at the conclusion that both of (HH,DD) and (DD,HH), and only those conventions, are stochastically stable if and only if $\rho[(DH,DH)\to$

$(HH, DD)] < \rho[(HH, DD) \rightarrow (DH, DH)]$, and (DH, DH), and only that, is stochastically stable if $\rho[(DH, DH) \rightarrow (HH, DD)] > \rho[(HH, DD) \rightarrow (DH, DH)]$ (where it is again used that k is sufficiently large).

Finally, from $\alpha \geq \beta$ it follows that $\rho[(DH, DH) \longrightarrow (HH, DD)] = \frac{\beta-1}{\beta+1}k$, and $\rho[(HH, DD) \rightarrow (DH, DH)] = \frac{2}{\alpha+1}k$. Since $\frac{2}{\alpha+1} < \frac{\beta-1}{\beta+1}$, if and only if $(\alpha-1)(\beta-1) > 4$, the conclusion of Theorem 1 follows. $\square$

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16

Group Formation with Heterogeneous Feasible Sets

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Summary. In this paper we consider a model of group formation where group of individuals may have different feasible sets. We focus on two polar cases, *increasing returns*, when the set of feasible alternatives increases if a new member joins the group, and *decreasing returns*, when a new member has an opposite effect and reduces the number of alternatives available for the enlarged group. We consider two notions, *stability* and *strong stability* of group structures, that correspond to Nash and Strong Nash equilibrium of the associated non-cooperative game. We prove existence results for various classes of environments and also investigate the link between dimensionality of feasible sets and the existence of stable structures.

Key words: Feasible sets, Stable partitions, Positive externality, Increasing Returns, Decreasing Returns.

JEL Classification Numbers: C71, C72, D62, D71.

16.1 Introduction

In this paper we consider a model with heterogeneous individuals who have preferences over a given set of alternatives and partnerships. They partition themselves into groups (coalitions), where each group chooses an alternative from its own feasible set. Every individual benefits from the large size of the group she belongs to, however the choice of an alternative by that group could be detrimental for some of its members. In this regard our model related is to the brand of literature on coalition formation that attempts to resolve the conflict between increasing returns to scale in large groups and heterogeneity of individual preferences. Indeed, it might be beneficial to be a member of a large political party due potential benefits and perks of power. However, the structure of a large party could be such that its political platform chosen by its committees or conventions might be very distant from the preferences of some of the party members. In this case the party unity could be under threat of loosing a part of its membership, who may even create their own party. The question that may arise is whether the breakaway group has sufficient resources and a power of conviction to survive on its own.

In general, the problem of group stability is rooted in the comparative power of small versus large groups. In this paper we examine this aspect of stability of group formation by focusing our attention at the set of feasible options available for groups of individuals when they form. The main distinctive feature of our model is that groups of individuals, when they form, have different feasible sets. Most of the existing literature (Guesnerie and Oddou (1879, 1981, 1988), Greenbeg and Weber (1986), Demange (1994), Demange and Guesnerie (1997), Konishi, Le Breton and Weber (1997,a-e), among others)[3] consider the case where the set of available choice for all coalitions is the same. However, one may consider various situations where this assumption does not necessarily hold. For example, a merger of two firms, may generate technological and market opportunities that were not available in the pre-merger environment, and the firm would benefit from increasing returns to scale. The enlargement of the European Union is often supported by the claim that the whole is larger than the sum of its part. Thus, the enlarged union offers its new members prospects and opportunities that are unavailable outside of the union. On the other hand, in the framework of an international conflict, presence of a large number of participants could make it difficult to find a policy acceptable for all parties involved. Even though everybody would prefer a joint action, finding a compromise becomes increasingly difficult when the number of participants is large. In this case one may assume that the set of feasible, or acceptable, alternatives is a *declining* function of the group size. The same situation may occur in business, political or any kind of negotiations where each of participants has a predetermined set of acceptable alternatives. Thus, the question would be whether it is possible to find a suitable compromise that would be unanimously accepted, and the large number of heterogeneous participants can create difficult barriers to overcome.

We also relate our results on stability of group formation to the number of dimensions of the alternative space. It is quite natural to expect that the ability to reach a compromise among various participants is crucially dependent on the dimensionality of the conflict. In general, the severity of the conflict raises when the number of its dimensions increases, in which case the search for a stable group structure that would satisfy all participants becomes more challenging.

We consider two extreme cases, *increasing returns*, when the set of feasible alternatives increases if a new member joins the group, and *decreasing returns*, when a new member has an opposite effect and reduces the number of alternatives available for the enlarged group. We associate our model of group formation with a noncooperative game in strategic form and examine the notions of Nash equilibrium and strong Nash Equilibrium of this game that correspond to *stability* and *strong stability* of group structures. We then identify the classes of environments that admit either stable or strongly stable group structures under increasing or decreasing returns. We also investigate how the stability is linked to the number of dimensions of the alternative set examine the case of *dichotomy* where all groups are divided in two types,

[3] Greenberg and Weber (1993) is an exception.

effective, whose feasible set is the entire alternative set, and ineffective, whose feasible sets are empty. It turns out that the Nakamura number plays an important role in our investigation.

The paper is organized as follows. In the next section we describe the model and introduce the assumptions that will be used to prove our results. In Section 3 we define the noncooperative and cooperative games associated with our model and determine the link between them. Section 4 is devoted to the discussion on the impact of the dimensionality of the set of alternatives on stability of group structures. In Sections 5 and 6 we state our existence results in the case of increasing and decreasing returns, respectively. The proofs of the results are relegated to the Appendix.

16.2 The Model

In our setting, a society (environment) E is defined as a quadruple (N, Ω, Φ, U), where $N = \{1, \ldots, n\}$ is a set of players, Ω is a set of alternatives, Φ is a feasibility correspondence that assigns the set $\phi(S) \subset \Omega$ to every coalition $S \subset N$, and $U = \{u_i\}_{i \in N}$ is a vector of players utility functions that represent players' preferences over pairs of alternatives and coalitions: $u_i : \Omega \times \mathcal{S}^i \to \mathbf{R}_+$, where $\mathcal{S}^i$ denotes the set of coalitions that contain i.

The agents may form different groups (coalitions). Each coalition S has a set of feasible alternatives $\phi(S) \subset \Omega$; thus, if coalition S forms, it can choose an alternative from $\phi(S)$. The notion of feasibility may have different interpretations. If the alternatives under consideration are allocations of goods, feasibility, as usual, means that, due to resource constraints, some alternatives may be out of reach for some individuals or coalitions. But feasibility may also incorporate constraints that are generated by institutions, rules or social norms. These second best constraints limit further the scope of action of coalitions. Such a broad interpretation allows us to examine the feasibility correspondence in a wide range of environments.

We impose two assumptions that hold throughout the rest of the paper. One requires that for every player it is better to be a member of any coalition that chooses a feasible alternative rather than to be a member of a coalition that chooses an alternative outside of its feasible set. Thus, if possible, every individual would prefer to join a coalition choosing a feasible alternative rather that staying in the group that chooses an alternative outside its feasible set.

Desirability of Feasible Alternatives - DFA: For every $i \in N$, two coalitions $S, S' \in \mathcal{S}^i$, and two alternatives $\omega \in \phi(S), \omega' \notin \phi(S')$, we have $u_i(\omega, S) > u_i(\omega', S')$.

The notion of desirability is introduced through the definition of $u_i(\omega, S)$ by assigning very low values to all $u_i(\omega, S)$ for which $\omega \notin \phi(S)$. Similarly, the fact that $\phi(S) = \emptyset$ for some coalition S can be interpreted as meaning that for every player in i in S, each feasible alternative in other coalition would yield i a higher payoff than one she can obtain within S. In short, we want any group of players to rule out any alternative outside of their but in order to formally define a non-cooperative game below, we have to assign a (unacceptably low) value to any alternative outside of the group's feasible set.

We also require that the utility of a player within every coalition would not decline if a new member joins that coalition without altering the chosen alternative.

Positive Externality - PE: For every $i \in N$, every two coalitions $S \subset S'$ with $S \in \mathcal{S}^i$, and every $\omega \in \phi(S) \bigcap \phi(S')$, we have $u_i(\omega, S) \leq u_i(\omega, S')$.

In addition to these two basic assumptions, we introduce some additional requirements that we need for some of our results. In the case of unidimensional set of alternatives, the convexity of preferences corresponds to the classical property of *single-peakedness* that yields the existence of a top alternative for each player within a given coalition, such that her utility would decline the further away she is from her ideal choice. It order to avoid some technical problems, we often turn to a finite set of alternatives.

Single-peakedness - SP: Ω is a finite subset of the real line and for every $i \in N$ and $S \in \mathcal{S}^i$, $u_i(\cdot, S)$ is single-peaked on $\phi(S)$.

The next property, labelled *anonymity*, implies that the utility of each individual depends only on the number but not the identity of other members of the coalition she is in.

Anonymity - AN: For every $i \in N$, every two coalitions $S, S' \in \mathcal{S}^i$ with $|S| = |S'|$, and every $\omega \in \phi(S) \bigcap \phi(S')$, we have$u_i(\omega, S) = u_i(\omega, S')$.

The class of environments that satisfy DFA, PE, SP and AN is denoted by $\mathcal{E}$. In some cases, we consider a stronger assumption:

Separability - SEP: Ω is a finite subset of the real line and for every $i \in N$, the utility function u_i can be represented as[4]

$$u_i(\omega, S) = \begin{cases} v_i(\omega) & if\ \omega \notin \phi(S) \\ v_i(\omega) + h(|S|) & if\ \omega \in \phi(S), \end{cases}$$

where $v_i(\cdot)$ is single-peaked over $\phi(S)$, and $h(\cdot)$ is strictly increasing on the set of positive integers. (Note that the function h does not depend on i).

To guarantee that the desirability of feasible alternatives assumption holds, we will require that $h(1) > v_i(\omega) - v_i(\omega')$ for every $i \in N$, $\omega, \omega' \in \Omega$. The class of environments that satisfy SEP is denoted by $\mathcal{E}'$.

Sometimes we employ a stronger assumption than separability. Namely, we require that every individual has an ideal alternative ω^i, such that her utility function is represented by the Euclidean distance function from alternative ω to ω^i.

[4] Konishi and Fishburn (1996) provide an axiomatic characterization of separable preferences.

Euclidean preferences: Suppose that for every i there exists a unique peak, ω^i, such that the utility function u_i is given by

$$u_i(\omega, S) = \begin{cases} -|\omega - \omega^i| & if\ \omega \notin \phi(S) \\ -|\omega - \omega^i| + h(|S|) & if\ \omega \in \phi(S), \end{cases}$$

where, again, $h(\cdot)$ is strictly increasing on the set of positive integers, and for every $i \in N, \omega, \omega' \in \Omega$, $h(1) > v_i(\omega) - v_i(\omega')$.

The class of environments that satisfy EP is denoted by $\mathcal{E}''$. We have the following inclusion:

$$\mathcal{E}'' \subset \mathcal{E}' \subset \mathcal{E}.$$

Denote by $\mathcal{E}_l$ the classes of environments that belong to $\mathcal{E}$ and satisfy Assumption $l = IR, DR$. Similar notation is used for $\mathcal{E}'$ and $\mathcal{E}''$.

As we mentioned in the introduction, the crucial role in our analysis will be played by the feasibility correspondence. In particular, we distinguish between two cases: one, when every new member of a coalition expands its feasible set, and another, when an entry of a new member shrinks the existing feasible set.[5] Formally,

Increasing Returns - IR: For every two coalitions $S \subset S'$ we have $\phi(S) \subset \phi(S')$.
Decreasing Returns -DR: For every two coalitions $S \subset S'$ we have $\phi(S') \subset \phi(S)$.

Occasionally we will mention the case of *constant returns* which would simply describe the environments for which both Assumptions IR and DR hold. This simply means that all nonempty coalitions have the same feasible set.

In the next section we introduce both non-cooperative and cooperative descriptions of players' interaction and establish an important link between two approaches.

16.3 Non-cooperative and Cooperative Framework

First, we consider a noncooperative game Γ where the strategy set of each player is given by the set Ω. Each player simultaneously and independently select an alternative. These choices determine a partition of the players into several groups to which we will refer hereafter as a *group structure*: two players are in the same group if they have selected the same alternative. A physical outcome consists of a group structure together with an alternative for each group. If every player i chooses a strategy ω_i this gives rise to the strategy profile $x = (\omega_1, \ldots, \omega_n)$. For every $\omega \in \Omega$ denote by $N_\omega(x)$ the set of players who choose ω in x. Those choices generate a partition $P(x)$ of the set N into pairwise disjoint sets $\{S_1, \ldots, S_K\}$, each consisting of the players choosing the same alternative in x. The payoff function of player i is given by $u_i(\omega_i, N_{\omega_i}(x))$, so that the payoff of player i depends both on the alternative selected by that player and the subset of players selecting the same alternative.

[5] Intermediate cases, like those familiar in the theory of clubs where increasing returns can be bounded by congestion effects could be considered - see Konishi, Le Breton and Weber (1997b).

An important feature of this formulation is the independence of the player's payoff with respect to choices different from hers. That is, whenever groups are formed, any change in one of them has no impact on the others as long their choices remain different. This important property has been studied under different names ("orthogonality" in Guesnerie and Oddou (1988), "games without externalities" in Bloch (1996), or "games without spillovers" in Konishi, Le Breton and Weber (1997e)). It obviously rules out some important group formation problems like cartel or association agreements in industrial organization (Belleflamme (2000), Bloch (1995)), custom unions (Yi (1997)) and environmental agreements (Ray and Vohra (2001)), among others. To emphasize this feature,[6] we usually reserve the term group formation for this case in contrast to the terminology coalition formation, which covers any strategic setting where independent players have the opportunity to form coalitions and make bidding agreements.

It is worth to pointing out that an outcome of the game consists not only of a group structure but also of a vector of alternatives, where one alternatives is chosen by one group. These chosen alternatives play an important role in many cases. But there are environments where these alternatives do not play a direct role and simply represent, as mentioned above, coordination devices. These games correspond to the particular case where the utility function $u_i(\cdot, \cdot)$ only depends only on the set of players choosing the same alternative but not on the alternative itself. Those so-called "hedonic games"[7] games (Banerjee, Konishi and Sömnez (2001), Bogomolnaia and Jackson (2002), Milchtaich and Winter (2002)).[8]

We now introduce two notions of equilibrium (or stability) that will be used in this paper. Consider, first, the notion of Nash equilibrium in pure strategies. In our framework, this notion imposes the following natural requirements. First, even if the set of feasible alternatives for individual i is nonempty, she would not be better off by staying alone. Secondly, suppose that player i contemplates joining another coalition S in the existing partition $P(x)$ all members of which choose alternative ω. If ω is not feasible for the coalition $S\bigcup\{i\}$, the move obviously would not take place. But even if the alternative ω is feasible for the coalition $S\bigcup\{i\}$, the move should not be beneficial for player i.

Definition 3.1: A strategy profile $x = (\omega_1, \ldots, \omega_n)$ is a Nash equilibrium if
(i) $u_i(\omega_i), N_{\omega_i}(x)) \geq \max_{\omega\in\phi(\{i\})} u_i(\omega, \{i\})$ for every $i \in N$;
(ii) For every individual i and every alternative $\omega \neq \omega_i$ with $N_\omega(x) \neq \emptyset$, either $\omega \notin \phi(N_\omega(x)\bigcup\{i\})$ or $u_i(\omega_i, N_{\omega_i}(x)) \geq u_i(\omega, N_\omega(x)\bigcup\{i\})$.

[6] It is worthwhile to remind that this assumption is implicit in all cooperative game theory where the use of the concept of characteristic function itself presumes the group's payoff is not affected by actions of its complement.

[7] The term was introduced in Drèze and Greenberg (1980. It captures the fact that every player in this group formation game cares about her partners in the group.

[8] This type of games also emerges in the sequential setting, where groups form first and then according to some mechanism (voting, social planner, market) each group selects an alternative. After solving the game backwards, we eliminate the second stage and end up, in essence, with a hedonic group formation game.

The resulting partition $P(x)$ is called *stable*.

The notion of a strong Nash equilibrium (Aumann (1959)) is more demanding. It requires that the profile of strategies is immune to any deviation by any coalition. There are no restrictions on the deviation except for the fact that it must be profitable to all the members of the deviating coalition[9]. In our context, a strategy profile is a strong Nash equilibrium if there exists no coalition, which is not necessarily an element of the existing partition, that possesses a feasible alternative that is beneficial for all its members. Strong Nash equilibrium is an appealing concept that solves an array of coordination problems. However, given its stringent requirements, there are not many environments that admit the existence of a strong Nash equilibrium. Some classes of environments, for which it does exist, are indicated below. Formally,

Definition 3.2: A strategy profile $x = (\omega_1, \ldots, \omega_n)$ is a strong Nash equilibrium if there is no coalition $S \subset N$ and a profile $x' = (\omega'_1, \ldots, \omega'_n)$ with $\omega'_i = \omega_i$ for all $i \notin S$ such that $u_i(\omega'_i, N_{\omega'_i}(x')) > u_i(\omega_i, N_{\omega_i}(x))$ for all $i \in S$.
The resulting partition $P(x)$ is called *strongly stable*.

Let us now describe a cooperative variant of our model by associating a cooperative game in characteristic form v with the noncooperative game of group formation described above. For every $S \subset N$ the characteristic function of game v yields the value $v(S)$ determined by:

$$v(S) = \{u \in \mathbf{R}^n : \exists \omega \in \phi(S) \text{ s.t. } u_i \leq u_i(\omega, S) \forall i \in S\}.$$

We use the standard definition of the core:

Definition 3.3: A vector of payoffs u is in the *core* of the game v if $u \in V(N)$ and there is coalition $S \subseteq N$ and $u' \in v(S)$ such that $u'_i > u_i$ for all $i \in S$.

Denote by $\mathcal{P}$ the set of all partitions or group structures[10] of N and consider a partition $P = (S_k)_{1 \leq k \leq K} \in \mathcal{P}$. Then the set of feasible payoffs is given by the following intersection:

$$v(P) = \cap_{1 \leq k \leq K} v(S_k).$$

This allows us to define the *superadditive cover* $\widehat{v}$ of the game v:

$$\widehat{v}(S) = \cup_{P_S \in \mathcal{P}(S)} \cap_{T \in P_S} v(T)$$

for all $S \subseteq N$, where $\mathcal{P}(S)$ denotes the set of partitions of S. The game $\widehat{v}$ describes the set of payoffs feasible for that coalition when the group can be partitioned in any

[9] One may examine an alternative concept of coalition-proof Nash equilibrium (Bernheim, Peleg and Whinston (1987),) that aims to introduce robustness with respect to potential coalitional deviations. While, in general, strong Nash equilibria and coalition-proof Nash equilibria yield different sets of equilibria, the two sets coincide for a large class of group formation games (see Konishi, Le Breton and Weber (1997c)).

[10] Cooperative games with a coalition structure were introduced first by Aumann and Drèze (1974). We refer the reader to Greenberg (1994) for an excellent, though somewhat outdated, review of the literature on coalition structures.

arbitrary way. The games v and $\widehat{v}$ coincide if v is superadditive, where, to recall, the game v is *superadditive* if $v(T) \cap v(S \setminus T) \subseteq v(S)$ for every $S \subseteq N$ and every $T \subseteq S$.

Since an element is in the core only if it is associated with an outcome immune against coalitional deviation, it follows that when v is not superadditive, the definition of core has to be modified. Following Guesnerie and Oddou (1979), the natural way to proceed is to consider payoff vectors in the core of $\widehat{v}$. The following proposition states the connection between the noncooperative group formation game Γ and the cooperative game $\widehat{v}$:

Proposition 3.4: Let E be a society satisfying DFA,PE and IR. The strategy profile $x = (\omega_1, \ldots, \omega_n)$ is a strong Nashequilibrium of Γ if and only if the vector $u \in \mathbf{R}^n$, where $u_i = u_i(\omega_i, N_{\omega_i}(x))$ for all $i \in N$, is in the core of $\widehat{v}$.

Proof: See Appendix.

Since the main goal of this paper is to investigate the existence of a Nash equilibrium and Strong Nash equilibrium in pure strategies in the case of heterogeneous feasible sets, the usefulness of Proposition 3.4 lies in the fact that it allows to import results from cooperative game theory to use them in the non-cooperative setting.

16.4 On Dichotomy, Role of Dimensions and the Nakamura Number

In this section we review some existing results on the nature of the group(s) that would form at equilibrium and offer offering some observations on the role of the dimension or cardinality of the set of alternatives in the existence of a strong Nash equilibrium. Formally, we consider two alternative assumptions, *Convexity* and *Finiteness*:

Convexity - CNV: Ω is a subset of $\mathbf{R}^m$. For every group S the feasible set $\phi(S)$ is a convex subset of $\mathbf{R}^m$ and for every $i \in S$, the utility function $u_i(\cdot, S)$ is quasi-concave on $\phi(S)$.

Alternatively, we consider a finite variant of the model:

Finiteness - FIN: The set of alternatives Ω is a finite set.

In terms of applications, one can easily construct environments where one of the variants (finite or continuous) provides a more appropriate description of the problem.

Under the assumption of positive externality and increasing returns, there are good reasons for the grand coalition N to form. However, it is not difficult to construct societies for which this is not true. As already alluded to, this has to do with the severity of the potential conflicts among the individuals. When groups have to reach a consensus on multidimensional issues, it is natural to expect that the larger the dimension m (or the cardinality of the set of alternatives Ω in the finite case), the more difficult it would be to find such a consensus. Thus, these two values could be considered measures of the severity of the conflict.

We start with the notion of an efficient outcome:

Definition 4.1: An *efficient outcome* is a vector $u \in \mathbf{R}^n$ such that
(i) there is group structure $P \in \mathcal{P}$ for which $u \in v(P)$,
(ii) there is no other group structure P' and $u' \in v(P')$ such that $u'_i > u_i$ for all $i \in N$. A group structure P is *efficient* is there is an efficient outcome $u \in v(P)$. A group structure P is *universally efficient* if any efficient outcome $u \in v(P)$.

Guesnerie and Oddou (1988) have introduced the following general condition on the profile of utility functions:

Definition 4.2: A profile of utility functions $U = \{U_i\}_{i \in N}$ satisfies the condition of $(q+1)$ multilateral agreements in N if for any group structure $P = (S_1, \ldots, S_K)$, for any $u \in v(P)$ and any group T with $q+1$ members there exists an alternative $\omega \in \phi(N)$ such that $u_i \leq U_i(\omega, N)$ for all $i \in T$.

Proposition 4.3 (Guesnerie and Oddou (1988): Let E be a society satisfying DFA, PE and CNV. If $n > m$, then N is universally efficient if and only if the profile of utility functions satisfies the condition of $(m+1)$ multilateral agreements.

We have already pointed out that the superiority of large groups over smaller ones relies on ability of members of a large group to reach an unanimously acceptable and beneficial compromise. It may be quite difficult and demanding to check for all possible proposals and the main contribution of Proposition 4.3 is to limit the verification of the conditions to a proper subset of coalitions; the smaller is the number m, the smaller is the subset of coalitions to be examined. In particular, in the unidimensional case the condition of $(m+1)$ multilateral agreements corresponds to a *condition of bilateral agreements* which can be formulated as follows:

For any two individuals $i, j \in N$, for any group S with $i \in S$ and $j \in N \backslash S$, there exists an alternative $\omega \in \phi(N)$ such that

$$\max_{\omega' \in \phi(S)} U_i(\omega', S) \leq U_i(\omega, N) \text{ and } \max_{\omega' \in \phi(N \backslash S)} U_j(\omega', N \backslash S) \leq U_j(\omega, N \backslash S).$$

The condition implies that if the entire population is divided into two groups and if individuals i and j were selected to act as dictators in their respective groups, then these two individuals should be able to find a compromise when considering to merge their groups. It is important to note that the condition of bilateral agreements requires a finite number of inequalities to be checked, which is not the case in the multidimensional framework. Hence, in the unidimensional setting Proposition 4.3 formalizes and simplifies the intuitive notion of a profile of preferences that exhibit a limited degree of conflict.

The above result examines the universal efficiency of the grand coalition and does not provide any information of its stability[11], and moreover, it does not consider

[11] In the specific second best taxation problem considered by Guesnerie and Oddou (1981), it turns out that universal efficiency and strong stability coincides. A proof based on bal-

alternative group structures. To offer some results in these directions, we need the following definition:

Definition 4.4: A feasible correspondence ϕ is *dichotomic* if it satisfies IR and for every $S \subseteq N$ either $\phi(S) = \Omega$ (effective group) or $\phi(S) = \emptyset$ (ineffective).

Dichotomic feasible correspondences represent a specific class among feasible correspondences displaying increasing returns: every group is either completely effective or completely ineffective. While extreme, this discontinuous form of increasing returns to size occurs in many important examples. Let $\mathcal{C}$ be the family of effective coalitions, i.e., $S \in \mathcal{C}$ if and only if $\phi(S) = \Omega$.

Definition 4.5: Let the family of efficient coalitions $\mathcal{C}$ be given. Let $A(\mathcal{C})$ be the set of integers defined by

$$A(\mathcal{C}) \equiv \{K | \exists \text{ groups } S_1, \ldots, S_K \in \mathcal{C} \text{ such that } \cap_{1 \leq k \leq K} S_k = \emptyset\}.$$

The *Nakamura number* of the family $\mathcal{C}$ is the integer $\nu(\mathcal{C})$ defined by

$$\nu(\mathcal{C}) = \begin{cases} \min\{K | K \in A(\mathcal{C})\} & if\ A(\mathcal{C}) \neq \emptyset \\ +\infty & if\ A(\mathcal{C}) = \emptyset. \end{cases}$$

This integer, introduced by Nakamura (1979) to study voting in committees, provides a useful combinatorial information about the effective groups and the nature of the increasing returns to size described by the feasible correspondence ϕ. If returns to size are already exhausted by relatively small coalitions, the Nakamura number is likely to be small and it would become more difficult to obtain a stable group structure. In what follows, we denote by $\nu(\phi)$ the Nakamura number of the family of effective coalitions induced by the feasible correspondence ϕ.

Proposition 4.6: Let E be a society satisfying DFA, PE, CON and DIC. Suppose also that the family $\mathcal{C}$ satisfies *exclusive effectiveness*: $S \in \mathcal{C}$ implies $N \backslash S \notin \mathcal{C}$. Then the grand coalition N is universally efficient. If, moreover, $m \leq \nu(\phi) - 2$, it is also strongly stable.

Proof: See Appendix.

It is possible to show that if $m \geq \nu(\phi) - 1$ and there are too many dimensions in the decision problem (given groups' feasible sets), then there exist societies without a strongly stable group structure. The exclusive effectiveness in Proposition 4.6 rules out a possibility that each of two disjoint groups has the same feasibility range as the grand coalition. In some situations, the power of smaller groups may enhance efficiency as the group structure can contain several groups making their own choices. However, in presence of many small powerful groups, stability is more difficult to obtain.

It is worth to point out that replacing convexity by finiteness in Proposition 4.6, will generate similar results. The condition $m \leq \nu(\phi) - 2$ should simply be replaced

ancedness is provided in Greenberg and Weber (1982). Some results are also contained in Demange and Guesnerie (1997).

by the condition $|\Omega| \;|\leq \nu(\phi) - 1$. Under the finiteness assumption the relation between the existence of stable structures and the cardinality of the feasible set has been studied by Deb, Weber and Winter (1996). They examined the class of *quota games*, where S is effective if and only if its size reaches an integer threshold q and, obviously, if $q \leq \frac{n}{2}$, the exclusive effectiveness is violated. Deb, Weber and Winter have explicitly calculated the integer $\rho(\phi)$ such that if $|\Omega| \leq \rho(\phi) - 1$, then there exists a strongly stable group structure.

We are not aware of a study of the general case in the continuous and convex setting, but we could conjecture that a result similar to Proposition 4.6 can be obtained. However, if $m \geq 2$, then there exist societies satisfying the convexity assumption for which there are no strongly stable group structure. In this context, Le Breton and Weber (1995) considered the case where $q = 2$, $\Omega = \mathbf{R}^2$ and $U_i(\omega, S) = - \parallel \omega - \theta_i \parallel$ for all $i \in N$ and all $S \subseteq N$ where $\theta_i \in \mathbf{R}^2$ is the "ideal" point of individual i. They provide a necessary and sufficient condition on the matrix of pairwise distances between ideal points to guarantee the existence of a strongly stable group structure. The characterization provides information about the nature and the magnitude of the disagreement between individual preferences leading to a strongly stable coalition structure despite the fact that very small coalitions are very powerful. But the question of a general characterization of the class of societies for which a strongly stable coalition structure exists remains open.

Finally, in this dichotomic setting, there are environments where the feasible set of a coalition is not so much related to the number of individuals in the coalition but to the joint characteristics (or types) of the individuals in the coalition. A complete characterization of the families $\mathcal{C}$ for which a strongly stable group structure always exists is provided in Kaneko and Wooders (1982) and Owen, Le Breton and Weber (1992).

One lesson from the results obtained in the dichotomic case is that the existence of a strongly stable coalition structure is intimately related to a comparison between a combinatorial index summarizing the dispersion of power across potential groups and the dimension (or the cardinality) of the set of alternatives. It has been observed that when alternatives are described by more than one dimension, it becomes very difficult to ensure existence: for instance, when $m = 2$ and feasibility only depends on the number of individuals in the group, groups with less than $\frac{2}{3}$ of the entire population must be powerless.

Those insights raise a number of open problems. Is it possible to obtain similar results in a nondichotomic setting? For instance, it would be interesting to examine the case with three types of groups, effective, ineffective, and intermediate, whose feasible set is a given subset of Ω. Furthermore, we have focused here on the existence of strongly stable group structures but one could examine stable structures as well. Some partial answers to these questions are provided in the next two sections.

16.5 Increasing Returns

Now, let us turn to the case of increasing returns. Without loss of generality, let $\phi(N)$ be the entire nonempty set Ω. The case of increasing returns has attracted most of the attention but very few general results are available. Within the class $\mathcal{E}$, Greenberg and Weber (1993) show that a strong Nash equilibrium always exist if for all $i \in N$, $\omega \in \Omega$ and for any two coalitions $S, T \in \mathcal{S}^i$, $u_i(\omega, S) = u_i(\omega, T)$ as long as ω is feasible for both S and T. Within the class $\mathcal{E}'$, Konishi, Le Breton and Weber (1997a) have proved that a Nash equilibrium always exists whenever ϕ exhibits constant returns to scale.

Our first example shows that within the class $\mathcal{E}'_{IR}$, and therefore the class $\mathcal{E}_{IR}$, a Nash equilibrium may fail to exist.

Example 5.1 We construct a quadruple E that belongs to $\mathcal{E}'_I$. Let $\Omega = \{a, b, c\}$ and $N = \{1, 2, 3, 4, 5, 6\}$. The feasible set $\phi(S) = N$ for all coalitions S except that $\phi(\{3\}) = \{a, c\}$. The (single-peaked) preferences of the players are given by:

$$v_1(a) = 8,\ v_1(b) = 3,\ v_1(c) = 1$$
$$v_2(a) = 45,\ v_2(b) = 48,\ v_2(c) = 0$$
$$v_3(a) = 0,\ v_3(b) = 50,\ v_3(c) = 25$$
$$v_4(a) = 50,\ v_4(b) = 0,\ v_4(c) = 0$$
$$v_5(a) = 0,\ v_5(b) = 0,\ v_5(c) = 50$$
$$v_6(a) = 0,\ v_6(b) = 0,\ v_6(c) = 50.$$

The function $h(\cdot)$ is given by

$$h(1) = 100,\ h(2) = 102,\ h(3) = 108,\ h(4) = 116,\ h(5) = 117,\ h(6) = 118.$$

Suppose there is a Nash equilibrium $(\omega_1, \ldots, \omega_6)$. Then $\omega_4 = a, \omega_5 = \omega_6 = c$ and $\omega_2 \neq c$.

Suppose that $\omega_2 = b$. Then $\omega_3 = b$ and the best response of player 1 is a. But (a, b, b, a, c, c) is not a Nash equilibrium since player 2 would rather switch to a.

Suppose that $\omega_2 = a$. Since player 1 would not choose b, player 3 must choose c. In this case player 1 will select c. But (c, a, c, a, c, c) is not a Nash equilibrium since player 2 would rather switch to b. Thus, this environment does not admit a Nash equilibrium.

Even though the set of Nash equilibria is, in general, empty within the class $\mathcal{E}'_{IR}$, there exists an interesting subset of this class for which the existence of a Nash equilibrium can be rescued. As in Greenberg (1979), we require that all coalitions of the same size have identical feasible sets:

Anonymity of the feasible correspondence - AFC: For every two coalitions S, S' with $|S| = |S'|$, we have $\phi(S) = \phi(S')$.

Then we have the following:

Proposition 5.2: Let $E \in \mathcal{E}'_{\mathcal{IR}}$, and suppose that AFC holds. Then there exists a Nash equilibrium.

Proof: See Appendix.

An important example of group formation game with increasing returns is the second best taxation game introduced by Guesnerie and Oddou (1981):

Example 5.3: There is an economy with n agents and two goods, one private and one pure public good. The direct preferences of player i over consumption plans $(x_i, y) \in \mathbf{R}^2_+$, where x_i and y her consumption of the private and public good, respectively, are represented by the utility function $U_i(x_i, y)$. Player i has an initial endowment w_i in private good and the public good is produced from the private good through a constant returns technology normalized to 1. Lump sum transfers cannot be used to finance public good production and the only of financing of production of the public good is through taxes proportional to initial endowments in private good. Two games can be considered depending on how Ω is defined. First is a *taxation game*, which was the focus of the Guesnerie and Oddou study), where players form groups according to the taxation rate $t \in [0, 1]$ they select. The set of alternatives is $\Omega = [0, 1]$, the utility functions are $u_i(\omega, S) = U_i((1 - \omega)w_i, \omega \sum_{j \in S} w_j)$, and $\phi(S) = [0, 1]$ for all $S \subseteq N$.

Alternatively, there is a *production game*, where players form groups according to the level of production y of the public good they select. Then $\Omega = \mathbf{R}_+$ and $u_i(\omega, S) = U_i(w_i - \omega \frac{w_i}{\sum_{j \in S} w_j}, \omega)$ and $\phi(S) = \left\{\omega \in \mathbf{R}_+ : \omega \leq \sum_{j \in S} w_j\right\}$ for each group S.

It is straightforward to show that from the point of view of strong Nash equilibria, the two games are equivalent. However, they are not when Nash equilibria are considered. In both games, PE is satisfied and SP is satisfied, too, whenever U_i is quasi-concave. AN is satisfied only when the initial wealth is distributed uniformly, i.e., $w_i = w_j$ for all $i, j \in N$. Separability follows from separability assumptions on U_i.

Under the assumption of equal wealth, both games are in $\mathcal{E}$ and satisfy AFC. If, moreover, $U_i(x_i, y) = v_i(x_i) + g(y)$ for all $i \in N$, then both games belong to $\mathcal{E}'$, by Proposition 5.2. both admit Nash equilibria. However, as demonstrated in Weber and Zamir (1985) and Konishi, Le Breton and Weber (1998), both games, which are equivalent in that respect, may fail to possess strong Nash equilibria.

If the initial wealth is not uniformly distributed across players, then, as demonstrated in Konishi, Le Breton and Weber (1998), the production game may fail to possess a Nash equilibrium, even if we assume quasi-linearity with respect to the private good i.e. $U_i(x_i, y) = x_i + g_i(y)$ for all $i \in N$.

It is clear that the production game can be extended in many directions. One could consider the production of several public goods and assume their indivisibility. That would alter the nature of the problem and allow us to introduce effective groups, dichotomy of feasible correspondence and even exclusive effectiveness. But all these issues go beyond the scope of this paper and are left for future research.

Even though the existence of strong Nash equilibria is, in general, difficult to obtain, we show that the Euclidean preferences assumption PE yields the desirable existence:

Proposition 5.4: Within the class $\mathcal{E}''_{IR}$, there exists a strong Nash equilibrium.

Proof: See Appendix. As already pointed out in our examination of the Guesnerie and Oddou taxation and production games, the existence of a strong Nash equilibrium cannot be extended to the classes $\mathcal{E}_{IR}$ and $\mathcal{E}'_{IR}$. However, the Greenberg and Weber (1993) result, mentioned in the beginning of this section, yields the existence of a strong Nash equilibrium within the class $\mathcal{E}'_{IR}$, provided that the utility of every individual i derived from a given alternative ω within any coalitions S and S' is the same as long as ω is feasible for both S and S'. Formally,

Proposition 5.5 (Greenberg and Weber (1993): Suppose that the environment $E \in \mathcal{E}'_{IR}$ is such that the utility function of individual i is represented by

$$u_i(\omega, S) = \begin{cases} v_i(\omega) & if\ \omega \notin \phi(S) \\ v_i(\omega) + h & if\ \omega \in \phi(S) \end{cases}$$

where $v_i(\cdot)$ is single-peaked over $\phi(S)$, and the number h is such that for every $\omega, \omega' \in \Omega\ h > v_i(\omega) - v_i(\omega')$. Then E admits a strong Nash equilibrium.

16.6 Decreasing Returns

In this section we consider the case of decreasing returns. The assumption DR of decreasing returns should not be confused with the assumption of *negative externality*, stating that $u_i(\omega, S) \geq u_i(\omega, S')$ for every $i \in N$, every two coalitions with $i \in S \subset S'$ and every $\omega \in \phi(S) \bigcap \phi(S')$. This is the opposite of the positive externality assumption maintained through this paper.[12].

The assumption of decreasing returns means something different as every player benefits from a given alternative to be chosen by a larger group. To provide some intuition, assume that each player i has a predetermined list $\phi(\{i\})$ of conceivable or acceptable alternatives. Then the set of possible compromises $\phi(S)$ for group S is the intersection of all individually acceptable choices of its members $\phi(S) = \cap_{i \in S} \phi(\{i\})$. With that interpretation of feasibility, large coalitions may find difficult to reach a conceivable compromise. Then to look for equilibria simply amounts to search among the partitions of N into conceivable groups, where group is conceivable if its feasible set is nonempty. The conflict here is obvious: everybody wants to be in a large group - it is just difficult to reach a compromise there.

The result in Konishi, Le Breton and Weber (1997a) obtained under constant returns implies that, within the class $\mathcal{E}_{DR}$, a Nash equilibrium may fail to exist. However, we are able to show the existence of a Nash equilibrium for all environments in the class $\mathcal{E}'_{DR}$.

[12] Under negative externality, anonymity and constant returns, there always exists a strong Nash equilibrium in the group formation game (Milchtaich (1996), Konishi, Le Breton and Weber (1997d)

Proposition 6.1: Within the class $\mathcal{E}'_{DR}$, there always exists a Nash equilibrium.

Proof: See Appendix.

As far as a strong Nash equilibrium is concerned, it turns out that even within the class the class $\mathcal{E}''_{DR}$ it may fail to exist. Consider the following example:

Example 6.2: We construct a quadruple E that belongs to $\mathcal{E}''_{DR}$. Let $\Omega = \{a, b, c\}$ and $N = \{1, 2, 3\}$. The alternatives a, b, c, that represent the ideal points of individuals $1, 2, 3$, respectively, are located on the line. Let $a = b - 1 = c - 2$. The feasible set $\phi(\{i\}) = N$ for all singletons, $\phi(\{1, 3\}) = \{a, c\}$, $\phi(\{1, 2\}) = \phi(\{2, 3\}) = \{b\}$ and $\phi(N)$ is empty. The function $h(\cdot)$ is given by $h(1) = 3, \; h(2) = 6, \; h(3) = 7$. Suppose there is a strong Nash equilibrium $(\omega_1, \omega_2, \omega_3)$. There are five possible candidates: (b, b, c) - blocked by coalition $(1, 3)$ via alternative a.
(a, b, b) - blocked by coalition $(1, 3)$ via alternative c.
(a, b, a) - blocked by coalition $(2, 3)$ via alternative c.
(c, b, c) - blocked by coalition $(1, 2)$ via alternative a.
(a, b, c) - blocked by coalition $(1, 2)$ via alternative a.
Thus, there exists no strong Nash equilibrium.

The existence of a strong Nash equilibrium can be rescued if we impose some additional conditions on the feasible correspondence and feasible sets. First, we require that every individually feasible set $\phi(\{i\})$ is nonempty and is *consecutive* in the sense that for every three alternatives $\omega, \omega', \omega''$ with $\omega < \omega' < \omega''$, $\omega, \omega'' \in \phi(\{i\})$ imply $\omega' \in \phi(\{i\})$. The consecutiveness will be also imposed across feasible sets by requiring that an alternative that belongs to the feasible sets for two individuals, should be feasible for all intermediate individuals between these two. Formally,

Consecutiveness of the Feasibility Correspondence - CFC: (i) For every individual i, the set $\phi(\{i\})$ is a consecutive subset of Ω containing i's ideal point ω^i;
(ii) For every three individuals $i < j < k$ and an alternative $\omega \in \phi(\{i\}) \bigcap \phi(\{k\})$, it follows that $\omega \in \phi(\{j\})$;
(iii) The feasible set of every coalition $S \subset N$ is the intersection of the individual feasible sets of all of its members: $\phi(S) = \bigcap_{i \in S} \phi(\{i\})$.

Then

Proposition 6.3: If CFC holds then every environment within the class $\mathcal{E}''_{D}$, admits a strong Nash equilibrium.

Proof: See Appendix.

16.7 Appendix

Proof of Proposition 3.4: It is straightforward to show that the statement in Proposition 3.4 follows from the fact that under the assumptions that E is a society satisfying DFA,PE and IR, a strategy profile $x = (\omega_1, \ldots, \omega_n)$ is a strong Nash equilibrium if and only if there is no coalition $S \subset N$ and an alternative $\omega \in \phi(S)$ such that $u_i(\omega, S) > u_i(\omega_i, N_{\omega_i}(x)$ for all $i \in S$.

First, let $x = (\omega_1, \ldots, \omega_n)$ be a strong Nash equilibrium. We have to show that there is no coalition $S \subset N$ and an alternative $\omega \in \phi(S)$ such that $u_i(\omega, S) > u_i(\omega_i, N_{\omega_i}(x))$ for all $i \in S$. Assume, in negation, that there is a coalition $S \subset N$ and an alternative $\omega \in \phi(S)$ with $u_i(\omega, S) > u_i(\omega_i, N_{\omega_i}(x))$ for all $i \in S$. Let $x' = (\omega'_1, \ldots, \omega'_n)$ be such that $\omega'_i = \omega$ for all $i \in S$ and $\omega'_i = \omega_i$ for all $i \notin S$. By construction and IR, it follows that $N_\omega(x') \supseteq S$. Since, by positive externality, $u_i(\omega, N_\omega(x')) \geq u_i(\omega, S) > u_i(\omega_i, N_{\omega_i}(x))$ for all $i \in S$, we contradict our assumption that $x = (\omega_1, \ldots, \omega_n)$ is a strong Nash equilibrium.

Second, let $x = (\omega_1, \ldots, \omega_n)$ be a strategy profile such that there is no coalition $S \subset N$ and an alternative $\omega \in \phi(S)$ with $u_i(\omega, S) > u_i(\omega_i, N_{\omega_i}(x)))$ for all $i \in S$. We have to prove that x is a strong Nash equilibrium. Assume, on the contrary, that there exists $x' = (\omega'_1, \ldots, \omega'_n)$ such that $\omega'_i = \omega_i$ for all $i \notin S$ and $u_i(\omega'_i, N_{\omega'_i}(x')) > u_i(\omega_i, N_{\omega_i}(x))$ for all $i \in S$. Note that $\omega'_i \in \{\omega_1, \ldots, \omega_n\}$ for all $i \in S$. Otherwise, our original assertion would be violated via the group $N_{\omega'_i}(x')$ choosing the alternative ω'_i. This implies that the size of one of the existing groups, say, the group containing player j, has strictly increased. That is, $N_{\omega_j}(x') \supset N_{\omega_j}(x)$ and, in particular, $S \cap N_{\omega_j}(x') \neq \emptyset$. By construction, $u_i(\omega_j, N_{\omega_j}(x')) > u_i(\omega_i, N_{\omega_i}(x))$ for all $i \in S \cap N_{\omega_j}(x')$ and, by PE, $u_i(\omega_j, N_{\omega_j}(x')) > u_i(\omega_i, N_{\omega_i}(x))$ for all $i \in N_{\omega_j}(x') \backslash S$. Therefore, $u_i(\omega, N_{\omega_j}(x')) > u_i(\omega_i, N_{\omega_i}(x))$ for all $i \in N_{\omega_j}(x')$ with $\omega \equiv \omega_j$,a contradiction.□

Proof of Proposition 4.6: If the family $\mathcal{C}$ satisfies exclusive effectiveness, then either $\phi(S) = \emptyset$ or $\phi(N \backslash S) = \emptyset$. Therefore, the condition of bilateral agreement is equivalent to the following: for every i and $S \in \mathcal{S}'$ and $\phi(S) = \Omega$, there exists $\omega \in \phi(N)$ such that $\max_{\omega' \in \Omega} U_i(\omega', S) \leq U_i(\omega, N)$. But the last inequality is guaranteed by PE.

Now let $m \leq \nu(\phi) - 2$. The continuous version of the Nakamura's theorem[13] implies that the core of the NTU cooperative game $\overline{v}$, defined by[14]:

$$\overline{v}(S) = \{u \in \mathbf{R}^n : \exists \omega \in \phi(S) \text{ s. t. } u_i \leq U_i(\omega, N) \; \forall i \in S\}$$

is nonempty. Let u be an element of the core and $\omega \in \phi(N)$ be such that $u = U_i(\omega, N)$ for all $i \in N$. We claim that the profile $(\omega, \omega, \ldots, \omega)$ is a strong Nash equilibrium.

Suppose it is not. Then there exists a coalition S and $\omega' \in \phi(S)$ such that $U_i(\omega', S) > U_i(\omega, N)$ for all $i \in S$. From the positive externality assumption, we

[13] The first version of the Nakamura's theorem is due to Greenberg (1979) who studied families of effective coalitions $\mathcal{C}$ satisfying AFC. The extension to an arbitrary $\mathcal{C}$ is due to Le Breton (1987), Schofield (1984) and Strnad(1985). It should be pointed out that in this setting existence of the core does not follow in general from a balancedness argument, as the game is not always balanced (Le Breton (1989)).

[14] It should be noted that the game $\overline{v}$ is different from the game v defined earlier.

deduce that $U_i(\omega', N) > U_i(\omega, N)$ for all $i \in S$. Let $u' \in \mathbf{R}^n$ be defined as follows: $u'_i = U_i(\omega', N)$ for all $i \in S$ and $u'_i = u_i$ for all $i \notin S$. By construction, $u' \in \overline{v}(S)$, but since $u'_i > u_i$ for all $i \in S$, it contradicts the assumption that u is in the core of $\overline{v}$.□

Proof of Proposition 5.2: The proof is carried out by using the potential function approached pioneered by Rosenthal (1973). (See also Monderer and Shapley (1996), Konishi, Le Breton and Weber (1997a)).

AFC implies that for every ω there exists a positive integer $n(\omega)$ such that $\omega \in \phi(S)$ for every S with $|S| \geq n(\omega)$ and $\omega \notin \phi(S)$ for every S with $|S| < n(\omega)$.

Consider the function Ψ defined on the set of all strategy profiles:

$$\Psi(x) = \sum_{\omega \in \Omega} [\sum v_i(\omega) + \sum_{k=n(\omega)}^{|N_x(\omega)|} h(k)]$$

(If $n(\omega) > |N_x(\omega)|$ then the corresponding sum is set to zero.)

Take a maximum of the function Ψ, $x = (\omega_1, \ldots .\omega^n)$. We shall show that x is a Nash equilibrium. First, there is at least one i such that $\omega_i \in \phi(N_x(\omega_i))$. Otherwise, DFA implies that all players choosing the same alternative would generate a higher value of function Ψ. Moreover, DFA implies that $\omega_i \in \phi(N_x(\omega_i))$ for each i. Suppose now that there is a player i who wishes to switch from ω_i to ω. Then $\omega \in \phi(N_x(\omega) \cup \{i\})$ and

$$v_i(\omega_i) + h(|N_x(\omega_i)|) < v_i(\omega) + h(|N_x(\omega)| + 1).$$

This would imply that $\Psi(x) < \Psi((x_{-i}, \omega))$, where (x_{-i}, ω) is the profile with i choosing ω and all other players choosing the same strategies as in x. But this is a contradiction to x being a maximum of the function Ψ.□

Proof of Proposition 5.4: Assume that all players are ordered with respect to their ideal points so that $\omega^1 \leq \omega^2 \leq \ldots \leq \omega^n$.

By Greenberg and Weber (1986), there exists a strategy profile x such that every coalition in $P(x)$ is consecutive and there is no consecutive coalition S and $\omega \in \phi(S)$ such that

$$-|\omega^i - \omega| + h(|S|) > -|\omega^i - \omega_i| + h(|N_x(\omega_i)|) \text{ for all } i \in S.$$

We shall say then that S blocks x.

In order to show that x is a strong Nash equilibrium, it remains to demonstrate that there is no nonconsecutive coalition S and $\omega \in \phi(S)$ such that

$$-|\omega^i - \omega| + h(|S|) > -|\omega^i - \omega_i| + h(|N_x(\omega_i)|) \text{ for all } i \in S.$$

Suppose, in negation, that there is a nonconsecutive coalition S that blocks x. Nonconsecutiveness of S implies that there exist three players $i < j < k$ such that i and j belong to S whereas j does not. That is, there is an alternative ω such that

$$-|\omega^i - \omega| + h(|S|) > -|\omega^i - \omega_i| + h(|N_x(\omega_i)|) \tag{16.1}$$
$$-|\omega^k - \omega| + h(|S|) > -|\omega^k - \omega_k| + h(N_x(|\omega_k)|). \tag{16.2}$$

Let us introduce the degree of nonconsecutiveness $n(S)$ of a coalition C. Let $m(C)$ be its lowest member and $M(C)$ the highest. Then $n(C)$ represents the maximal size of a consecutive coalition C' such that $C' \subset \{m(C), \ldots, M(C)\}$ and $C' \bigcap C = \emptyset$. Obviously $n(C) > 0$ if and only C is nonconsecutive.

Assume, without loss of generality, that (i) the degree of nonconsecutiveness of S is the lowest among nonconsecutive coalitions that block (P, A), (ii) $|S| = \min_{C|n(C)=n(S)} |C|$, and (iii) $k - i = n(S) + 1$.

Note then that neither the coalition $N_x(\omega_j) \bigcup \{i\}$ nor the coalition $N_x(\omega_j) \bigcup \{k\}$ can block x as both $n(N_x(\omega_j) \bigcup \{i\})$ and $n(N_x(\omega_j) \bigcup \{k\})$ are smaller than $n(S)$. In particular, they cannot use ω_j in order to block. By DFA and PE, we have

$$-|\omega^i - \omega_j| + h(|N_x(\omega_j| + 1) \leq -|\omega^i - \omega_i| + h(|N_x(\omega_i|) \tag{16.3}$$
$$-|\omega^k - \omega_j| + h(|N_x(\omega_j| + 1) \leq -|\omega^k - \omega_k| + h(|N_x(\omega_k|). \tag{16.4}$$

Combining (16.1) - (16.4) we have

$$-|\omega^i - \omega_j| + h(|N_x(\omega_j| + 1) < -|\omega^i - \omega| + h(|S|) \tag{16.5}$$
$$-|\omega^k - \omega_j| + h(|N_x(\omega_j| + 1) < -|\omega^k - \omega| + h(|S|). \tag{16.6}$$

The inequalities (16.5) and (16.6) imply that

$$-|\omega^j - \omega_j| + h(|N_x(\omega_j| + 1) < -|\omega^j - \omega| + h(|S|) \tag{16.7}$$

Let $T = S \bigcup \{j\}$. Since DFA implies $\omega \in \phi(T)$. and h is increasing, it follows that T blocks x, a contradiction. Indeed, if T is consecutive, by our assumption, it cannot block x. If T is nonconsecutive then $n(T) \leq n(S)$, whereas $|S| < |T|$, contrary to our choice of S.□

Proof of Proposition 6.1: We again apply the potential functions approach. Consider the function Ψ defined on the set of all strategy profiles:

$$\Psi(x) = \sum_{\omega \in \Omega} [\sum v_i(\omega) + \lambda(x, \omega)],$$

where

$$\lambda(x, \omega) = \begin{cases} 0 & if\ \omega \notin \phi(N_x(\omega)) \\ \sum_{k=1)}^{|N_x(\omega)|} h(k) & if\ \omega \in \phi(N_x(\omega)) \end{cases}$$

Let x be a maximum of the function Ψ. If it is not a Nash equilibrium, there is an individual i that wishes to switch from a to b. In this case, the alternative b must be feasible for coalition $N_x(b) \bigcup \{i\}$.

There could not be the case where individual i is assigned to the group $N_x(a)$ choosing alternative a outside its feasible set, while in the same time there an alternative b such that $b \in \phi(N_x(b)) \bigcup \{i\}$. Moreover, a must be feasible for $N_x(a)$, otherwise the reassignment of i from a to b would have increased the value of the function Ψ, a contradiction to x being its maximum. But then

$$\Psi(x_i, b) - \Psi(x) = v_i(b) + h(|N_x(b)| + 1) - v_i(a) - h(|N_x(a)|) > 0,$$

a contradiction to x being a maximum.□

Proof of Proposition 6.3: As in the proof of Proposition 5.4, consider a strategy profile x such that every coalition in $P(x)$ is consecutive and there is no consecutive coalition S that blocks x. We shall show that there is no nonconsecutive coalition that blocks x.

Suppose, in negation, that there is a nonconsecutive coalition S that blocks x. Again, there are three players $i < j < k$ such that i and j belong to S whereas j does not. There is ω such that

$$-|\omega^i - \omega| + h(|S|) > -|\omega^i - \omega_i| + h(|N_x(\omega_i)|) \tag{16.8}$$
$$-|\omega^k - \omega| + h(|S|) > -|\omega^k - \omega_k| + h(|N_x(\omega_k)|). \tag{16.9}$$

Again, assume, without loss of generality, that (i) the degree of nonconsecutiveness of S is the lowest among nonconsecutive coalitions that block (P, A), (ii) $|S| = \min_{\{C|n(C)=n(S)\}} |C|$, and (iii) $k - i = n(S) + 1$.

Note then that neither the coalition $N_x(\omega_j) \bigcup \{i\}$ nor the coalition $N_x(\omega_j) \bigcup \{k\}$ can block x by using ω_j, as both $n(S(j) \bigcup \{i\})$ and $n(S(j) \bigcup \{k\})$ are smaller than $n(S)$. Here we have to deviate from the proof of the Proposition 5.4. The fact that one of the two coalitions may not block x, could be due to the feasibility constraints. In other words, if $\omega_j \in \phi(N_x(\omega_j) \bigcup \{i\})$ and $\omega_j \in \phi(N_x(\omega_j) \bigcup \{k\})$, then we can complete our proof as before.

Suppose therefore that $\omega_j \notin \phi(N_x(\omega_j) \bigcup \{i\})$. Since i and k can co-exist in coalition S, the intersection of $\phi(\{i\})$ and $\phi(\{k\})$ is nonempty. Since both sets are consecutive it follows that $\omega_j \in \phi(\{k\})$. Thus, the fact that $N_x(\omega_j) \bigcup \{k\}$ cannot block x by using ω_j, is not due to feasibility and

$$-|\omega^k - \omega_j| + h(|N_x(\omega_j)| + 1) \leq -|\omega^k - \omega_k| + h(|N_x(\omega_k)|). \tag{16.10}$$

Combining (16.9) and (16.10), we have

$$-|\omega^k - \omega_j| + h(|N_x(\omega_j)| + 1) < -|\omega^k - \omega| + h(|S|). \tag{16.11}$$

However, the alternative ω is feasible for $S \bigcup \{j\}$. But, by our assumptions, this coalition could not block x and we have:

$$-|\omega^j - \omega_j| + h(|N_x(\omega_j)|) \geq -|\omega^j - \omega| + h(|S| + 1) \tag{16.12}$$

Inequalities (11) and (12) imply that

$$|\omega^k - \omega_j| - |\omega^k - \omega| > |\omega^j - \omega_j| - |\omega^j - \omega|. \tag{16.13}$$

Since $\omega^k \geq \omega^j$ it follows that $\omega_j < \omega$. However, since $\omega^i \leq \omega^j$ and $\omega \in \phi(\{i\})$, it follows that $\omega_j \in \phi(\{i\})$ as well. But it would contradict the fact that ω_j is not feasible for the coalition $N_x(\omega_j) \bigcap \{i\}$.□

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17

Monotone Risk Aversion*

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Summary. This paper defines decreasing absolute risk aversion in purely behavioral terms without any assumption of differentiability and shows that a strictly increasing and risk averse utility function with decreasing absolute risk aversion is necessarily differentiable with an absolutely continuous derivative. A risk averse utility function has decreasing absolute risk aversion if and only if it has a decreasing absolute risk aversion density, and if and only if the cumulative absolute risk aversion function is increasing and concave. This leads to a characterization of all such utility functions. Analogues of these results also hold for increasing absolute and for increasing and decreasing relative risk aversion.

Key words: Absolute risk aversion, Relative risk aversion, Decreasing risk aversion, Increasing risk aversion, Cumulative absolute risk aversion, Cumulative relative risk aversion.

JEL Classification Numbers: D81.

17.1 Introduction

Decreasing absolute risk aversion means that the decision maker behaves in a less risk averse fashion the larger his wealth. Pratt [6] defined it by the condition that the equivalent risk premium $\pi(x, z)$ should be a decreasing function of initial wealth x, for every random addition z to wealth. Dybvig and Lippman [5], following Yaari [11], defined decreasing risk aversion by requiring that "gambles accepted at a given level of wealth will be accepted at all higher levels of wealth." These definitions are equivalent, and they can be directly interpreted in terms of preferences or choice behavior.

An important fact about decreasing absolute risk aversion is that it can be expressed in terms of the Arrow-Pratt coefficient of absolute risk aversion. Indeed, a

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utility function exhibits decreasing absolute risk aversion if and only if the coefficient of absolute risk aversion is decreasing. This was shown by Pratt [6], Pratt and by Dybvig and Lippman [5]. Arrow [1] stated this result as well, except that his terminology was inverted relative to Pratt's. Arrow used the decreasing coefficient of absolute risk aversion as a definition of decreasing absolute risk aversion, and he then argued that it is equivalent to more risk averse behavior at higher wealth levels.

The definition of the coefficient of absolute risk aversion assumes that the utility function is twice differentiable. Pratt's proof that one utility function is more risk averse than another if and only if it globally has a larger coefficient of risk aversion assumes that the utility functions are twice continuously differentiable. The same assumption therefore underlies the proof that decreasing absolute risk aversion is equivalent to a decreasing coefficient of absolute risk aversion. Since Dybvig and Lippman use Pratt's results, their proof also relies on this assumption.

However, a utility function with decreasing absolute risk aversion is not necessarily twice continuously differentiable. If it is strictly increasing, then it has to be everywhere differentiable with positive derivative. If, in addition, it is concave and differentiable, then it is automatically continuously differentiable; but the second derivative may not exist everywhere.

This paper defines the concept of an absolute risk aversion density, which is a generalization of the coefficient of absolute risk aversion. The main result, Theorem 1, says that a risk averse utility function has decreasing absolute risk aversion if and only if it has a decreasing absolute risk aversion density. It generalizes the result of Pratt [6], Arrow [1], and Dybvig and Lippman [5] to utility functions that are not assumed to be twice continuously differentiable.

If the utility function is risk averse and exhibits decreasing absolut risk averison, then the marginal utility function is absolutely continuous and has a density function. Since the marginal utility function is absolutely continuous, the same is true of what we call the cumulative absolute risk aversion function, the negative of the logarithm of the marginal utility. This allows us to define the absolute risk aversion density as the density of the cumulative absolute risk aversion function. It equals the negative of the density of the marginal utility divided by the marginal utility itself, and it coincides with the usual coefficient of absolute risk aversion whenever the latter is defined.

An idea similar to the absolute risk aversion density appeared in Vickson [8, 9, 10], where a DARA utility function was defined as one which has, in our terminology, a non-negative, decreasing, and "piece-wise smooth" absolute risk aversion density.

Theorem 1 also says that a risk averse utility function has decreasing absolute risk aversion if and only if its cumulative absolute risk aversion function is increasing and concave. This allows a complete characterization of risk averse utility functions with decreasing absolute risk aversion. It says that such a utility function is uniquely determined by its cumulative absolute risk aversion function, which can be any increasing concave function, and an additive constant. This is so because the utility function can be recovered, up to an additive constant, by integrating the exponential of the cumulative absolute risk aversion function.

Pratt [6] wrote that "..., convenient utility functions for which [the coefficient of absolute risk aversion] is decreasing are not so very easy to find." Our characterization represents a way of finding all such utility functions.

The results also hold for increasing absolute risk aversion, except that the cumulative absolute risk aversion function will be convex and the absolute risk aversion density will be increasing.

Analogues of all the results hold for relative risk aversion, where increasing or decreasing relative risk aversion is defined in behavioral terms without assuming that the utility function is twice differentiable.

The cumulative relative risk aversion function of a utility function defined on the positive half-line is defined as the composition of the exponential function and the cumulative absolute risk aversion function. If it is absolutely continuous, then we call its density the relative risk aversion density.

If the utility function exhibits increasing or decreasing relative risk aversion, then it is necessarily differentiable with positive derivative, so that the cumulative absolute and the cumulative relative risk aversion functions are well defined. If, in addition, the utility function is risk averse, then the marginal utility functions, the cumulative absolute and the cumulative relative risk aversion functions are all absolutely continuous.

Theorem 2 says that a risk averse utility function has increasing (decreasing) relative risk aversion if and only if it has an increasing (decreasing) relative risk aversion density. The theorem also says that a risk averse utility function has increasing (decreasing) relative risk aversion if and only if its cumulative relative risk aversion function is increasing and convex (concave). This implies that a risk averse utility function with increasing (decreasing) absolute risk aversion is uniquely determined by its cumulative absolute risk aversion function, which can be any increasing and convex (concave) function, and an additive constant.

17.2 Monotone absolute risk aversion

Consider a decision model where the set of outcomes is an open interval I on the real line, unbounded above. The outcomes may be interpreted as levels of future wealth or future consumption. Typically, $I = \mathbb{R}$ or $I = (0, \infty)$.

A random variable z will be called a *binary lottery* if it has at most two distinct values.

The utility function u will be assumed to be strictly increasing. Recall that u is said to be *risk averse* if

$$Eu(x + z) \leq u(x)$$

whenever x is in I and z is a binary lottery with $Ez \leq 0$ and such that $x + z \in I$ with probability one. It is well known that u is risk averse if and only if it is concave.

Recall that utility function u is *more risk averse* than a utility function v if

$$Eu(x + z) \leq u(x)$$

whenever x is in I and z is a binary lottery such that $x + z \in I$ with probability one and such that $Ev(x + z) \leq v(x)$.

We say that u exhibits decreasing absolute risk aversion (respectively, increasing absolute risk aversion) if it is less risk averse (respectively, more risk averse) at higher wealth levels. Decreasing or increasing absolute risk aversion are referred to jointly as monotone absolute risk aversion.

To express this formally, for every $d \geq 0$, define a utility function $u_{[d]}$ on I by

$$u_{[d]}(x) = u(x + d)$$

Say that u exhibits

- *decreasing absolute risk aversion* if for all $d \geq 0$, u is more risk averse than $u_{[d]}$ on I
- *increasing absolute risk aversion* if for all $d \geq 0$, $u_{[d]}$ is more risk averse than u on I
- *constant absolute risk aversion* if u exhibits both increasing and decreasing absolute risk aversion.

According to Proposition 1, a strictly increasing utility function with increasing or decreasing absolute risk aversion must be differentiable. The intuition is that kinks are not possible, because somehow the utility function would be infinitely risk averse at the kink. It would then have to be infinitely risk averse at any higher wealth level also, in the case of decreasing absolute risk aversion, or at any lower wealth level, in the case of increasing absolute risk aversion. This is not possible.

In the case of increasing absolute risk aversion, the result in Proposition 1 is known from Lemma 8.4.7 of Dubins and Savage [3, 4]. We provide a proof of Proposition 1 in the appendix.

Proposition 1. *Let u be a strictly increasing utility function on I. Suppose u exhibits decreasing or increasing absolute risk aversion. Then u is differentiable with $u' > 0$.*

Proposition 2 is a generalization of a part of the proof of Pratt [6, Theorem 1]. Pratt assumed the utility functions to be twice continuously differentiable.

Proposition 2. *Let u and v be differentiable utility functions on an interval I with $u' > 0$ and $v' > 0$. Then u is more risk averse than v if and only if u'/v' is continuous and*

$$\ln u'(r) - \ln u'(s) \geq \ln v'(r) - \ln v'(s)$$

whenever $r, s \in I$, $r < s$.

Proof.. Let $k : v(I) \to \mathbb{R}$ be the function such that $u = k \circ v$. It is strictly increasing. Since $k = v^{-1} \circ u$, k is differentiable, and

$$u'(x) = k'(v(x))v'(x)$$

for $x \in I$.

Now u is more risk averse than v if and only if k is concave, which is the case if and only if k' is continuous and decreasing. But k' is continuous and decreasing if and only if the function u'/v' is continuous and decreasing. Take logs and rearrange to see that u'/v' is decreasing if and only if

$$\ln u'(r) - \ln u'(s) \geq \ln v'(r) - \ln v'(s)$$

whenever $r, s \in I$, $r < s$. □

Because of Proposition 2, we call $-\ln u'$ the *cumulative absolute risk aversion function* of u, provided that u is differentiable with positive derivative.

If u the cumulative absolute risk aversion function exists (u is differentiable with positive derivative), then u is risk averse (concave) if and only if the cumulative absolute risk aversion function is increasing.

Recall that a function $G : I \to \mathbb{R}$ is *absolutely continuous* if and only if there exists a measurable function $g : I \to \mathbb{R}$ such that

$$\int_s^r |g(t)|\, dt < \infty$$

and

$$G(r) - G(s) = \int_s^r g(t)\, dt$$

for $r, s \in I$. If so, then we may call g a *density* of G. A density of G is unique almost everywhere.

If u is differentiable with $u' > 0$ and the cumulative absolute risk aversion function $-\ln \circ u'$ is absolutely continuous, then a density $R_A^*(u)$ of $-\ln \circ u'$ will be called an *absolute risk aversion density* for u. If u has an absolute risk aversion density $R_A^*(u)$, then $R_A^*(u)$ is unique almost everywhere.

Corollary 1. *Let u and v be differentiable utility functions on an interval I with $u' > 0$ and $v' > 0$. Assume that $R_A^*(u)$ and $R_A^*(v)$ exist. Then u is more risk averse than v if and only if*

$$R_A^*(u) \geq R_A^*(v)$$

almost everywhere.

Proof.. It follows from Proposition 2 that u is more risk averse than v if and only if

$$\int_r^s R_A^*(u)(t)\, dt \geq \int_r^s R_A^*(u)(t)\, dt$$

whenever $r, s \in I$, $r < s$. But this holds if and only if

$$R_A^*(u) \geq R_A^*(v)$$

almost everywhere. □

If $R_A^*(u)$ exists, then $-\ln u'$ is differentiable at almost every x, with

$$(-\ln u')'(x) = R_A^*(u)(x)$$

and, hence,

$$R_A^*(u)(x) = -\frac{u''(x)}{u'(x)}$$

at almost every x. So, the usual coefficient of absolute risk aversion is almost everywhere well defined and equal to $R_A^*(u)$. Conversely, if u is twice continuously differentiable, then $R_A^*(u)$ exists and is equal to the usual coefficient of absolute risk aversion almost everywhere.

Proposition 3. *If u is differentiable with $u' > 0$, then $-\ln \circ u'$ is absolutely continuous if and only if u' is absolutely continuous; and $R_A^*(u)$ is an absolute risk aversion density of u if and only if $-u' R_A^*(u)$ is a density of u'.*

Proof.. Suppose $R_A^*(u)$ is an absolute risk aversion density of u. Then

$$\ln \circ u'(s) = \ln \circ u'(r) - \int_r^s R_A^*(u)(t)\, dt$$

Apply the exponential function to the function $\ln \circ u'$. By a version of the chain rule for an absolutely continuous function (or Itô's Lemma applied to the deterministic Itô process $\ln u'$, interpreting s as time),

$$\begin{aligned} u'(s) &= \exp(\ln \circ u'(s)) \\ &= \exp(\ln \circ u'(r)) - \int_r^s \exp(\ln \circ u'(t)) R_A^*(u)(t)\, dt \\ &= u'(r) - \int_r^s u'(t) R_A^*(u)(t)\, dt \end{aligned}$$

so that $-u' R_A^*(u)$ is a density of u'. Conversely, if $-u' R_A^*(u)$ is a density of u', then a similar calculation using the logarithmic transformation shows that $R_A^*(u)$ is a density of $-\ln \circ u'$. □

Theorem 1. *Let u be a strictly increasing risk averse utility function on I. The following statements are equivalent:*

1. *u exhibits decreasing (increasing) absolute risk aversion*
2. *u is differentiable with $u' > 0$, and the cumulative absolute risk aversion function*

$$x \mapsto -\ln u'(x)$$

   is concave (convex)
3. *u has an absolute risk aversion density $R_A^*(u)$ which is decreasing (increasing).*

In the case of increasing absolute risk aversion, the equivalence of (1) and (2) in Proposition 1 is from Lemma 8.4.2 of Dubins and Savage [3, 4]. Pratt [6] and Dybvig and Lippman [5] proved the equivalence of (1) and (3) under the assumption that u is twice continuously differentiable.

Theorem 1 implies that if a strictly increasing risk averse utility function u exhibits monotone (decreasing or increasing) absolute risk aversion, then it is not only differentiable but continuously differentiable. It is twice differentiable everywhere except possibly at a countable number of points. If it is twice differentiable everywhere, then it is twice continuously differentiable. All this follows from the fact that the cumulative absolute risk aversion function $-\ln \circ u'$ is concave or convex. A concave or convex function is differentiable everywhere except possibly at a countable number of points, and if it is differentiable everywhere, then it is continuously differentiable.

It follows from the theorem that a strictly increasing utility function u is risk averse and exhibits decreasing (increasing) absolute risk aversion if and only if it has the form

$$u(s) = u(r) + \int_r^s \exp(-k(x))\,dx$$

for $r, s \in I$, for some increasing concave (convex) function k (the cumulative absolute risk aversion function of u). If we fix $r \in I$ and set $u(r) = 0$, then this equation establishes a bijection between increasing concave (convex) functions k and strictly increasing risk averse utility functions u that exhibit decreasing (increasing) absolute risk aversion and have value zero at r.

Pratt [6] wrote that "..., convenient utility functions for which [the coefficient of absolute risk aversion] is decreasing are not so very easy to find." The equation above represents a way of finding all such utility functions.

Theorem 1 assumes that the utility function is risk averse. This assumption is perfectly reasonable from an economic point of view, but it is somewhat stronger than what is actually needed. What is needed is to know, when proving that (1) implies (2), that the derivative u' cannot be discontinuous at every point. Risk aversion implies that u' is discontinuous at most at countably many points. This remark will be important when proving Theorem 2 on the basis of Theorem 1.

The rest of this section is devoted to the proof of Theorem 1.

Proof of Theorem 1. It follows directly from Proposition 2 and Proposition 1 that (1) of the theorem holds if and only if u is differentiable with $u' > 0$ and the function

$$x \mapsto \ln u'(x) - \ln u'(x+h)$$

is continuous and decreasing (increasing) for all $h \geq 0$.

(1) implies (2): First, we observe that u' must be continuous. This is seen as follows. Suppose there is a point $x \in I$ where u' is discontinuous. Then $\ln \circ u'$ is discontinuous at x, and since $x \mapsto \ln u'(x) - \ln u'(x+h)$ is continuous for all $h \geq 0$, u' is discontinuous at $x+h$ for all $h \geq 0$. However, since u is risk averse, u' is decreasing, which implies that it is continuous (in fact, differentiable) almost everywhere. This is a contradiction, so u' is indeed continuous.

Let $n \in \mathbb{N}$, set $h = 2^{-n}$, and set

$$\mathcal{D}_n = \left\{ \frac{a}{2^n} \in I : a \in \mathbb{Z} \right\}$$

The function

$$x \mapsto \ln u'(x) - \ln u'(x+h)$$

is decreasing on

$$\{x \in \mathcal{D}_n : x + h \in \mathcal{D}_n\}$$

Hence, the function $-\ln \circ u'$ is concave (convex) on $\mathcal{D}_n$. It follows that it is concave (convex) on the set

$$\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}_n = \left\{ \frac{a}{2^n} \in I : n \in \mathbb{N} \text{ and } a \in \mathbb{Z} \right\}$$

Since $\mathcal{D}$ is dense in I and $-\ln u'$ is continuous, this function is concave (convex) on I.

(2) implies (3): Since $-\ln \circ u'$ is concave (convex), it is absolutely continuous with density equal to its right derivative $(-\ln \circ u')'_+$ from the right, as shown by Rockafellar [7, Corollary 24.2.1]). So, u has the absolute risk aversion density $R_R^*(u) = (-\ln \circ u')'_+$, which is decreasing (increasing).

(3) implies (1): Since

$$\ln u'(x) - \ln u'(x+h) = \int_x^{x+h} R_A^*(u)(t)\, dt$$

u is differentiable with $u' > 0$, and $R_A^*(u)$ is decreasing (increasing), the function

$$x \mapsto \ln u'(x) - \ln u'(x+h)$$

is continuous and decreasing (increasing) for all $h \geq 0$. □

17.3 Monotone relative risk aversion

Let $\hat{u}$ be a strictly increasing utility function on $(0, \infty)$.

We say that $\hat{u}$ exhibits decreasing relative risk aversion (respectively, increasing relative risk aversion) if it implies less risk averse (respectively, more risk averse) behavior at higher wealth levels, in the sense both initial wealth and the random additions to wealth under consideration are scaled up by the same factor of proportionality. Decreasing or increasing relative risk aversion are referred to jointly as monotone relative risk aversion.

To express this formally, for every $\hat{d} > 0$, define a utility function $\hat{u}_{\{\hat{d}\}}$ on $(0, \infty)$ by

$$\hat{u}_{\{\hat{d}\}}(\hat{x}) = \hat{u}(\hat{d}\hat{x})$$

Say that $\hat{u}$ exhibits

- *decreasing relative risk aversion* if for all $\hat{d} \geq 1$, $\hat{u}$ is more risk averse than $\hat{u}_{\{\hat{d}\}}$
- *increasing relative risk aversion* if for all $\hat{d} \geq 1$, $\hat{u}_{\{\hat{d}\}}$ is more risk averse than $\hat{u}$
- *constant relative risk aversion* if $\hat{u}$ exhibits both increasing and decreasing relative risk aversion.

The logarithmic transformation maps relative risk aversion into absolute risk aversion, in the following sense.

Let u be the utility function defined on $\mathbb{R}$ by $u = \hat{u} \circ \exp$. Then $\hat{u} = u \circ \ln$. If $\hat{x} > 0$ and $\hat{d} \geq 1$, set $x = \ln \hat{x}$ and $d = \ln \hat{d}$. Then $d \geq 0$, and it is easily verified that

$$u_{[d]} \circ \ln = \hat{u}_{\{\hat{d}\}}$$

Consequently, $\hat{u}$ exhibits increasing (respectively, decreasing, constant) relative risk aversion if and only if u exhibits increasing (respectively, decreasing, constant) relative risk aversion.

Proposition 4. *Let $\hat{u}$ be a strictly increasing utility function on $(0, \infty)$. Suppose $\hat{u}$ exhibits increasing or decreasing relative risk aversion. Then $\hat{u}$ is differentiable with $\hat{u}' > 0$.*

Proof.. Set $u = \hat{u} \circ \exp$. Then u exhibits increasing or decreasing absolute risk aversion on $\mathbb{R}$, and hence, by Proposition 1, it is differentiable with $u' > 0$. But then $\hat{u} = u \circ \ln$ is differentiable with $\hat{u}'(\hat{x}) = u'(\ln \hat{x})/\hat{x} > 0$. □

If $\hat{u}$ is a differentiable utility function on $(0, \infty)$ with $\hat{u}' > 0$, then the *cumulative relative risk aversion function* of $\hat{u}$ is the function $-\ln \circ \hat{u}' \circ \exp$, defined on $\mathbb{R}$.

If $\hat{u}$ is differentiable with $\hat{u}' > 0$, and if the cumulative relative risk aversion function $-\ln \circ \hat{u}' \circ \exp$ is absolutely continuous, then a *relative risk aversion density* for $\hat{u}$ is a function $R_R^*(\hat{u})$, defined on $(0, \infty)$, such that $R_R^*(\hat{u}) \circ \exp$ is a density of $-\ln \circ \hat{u}' \circ \exp$. In other words,

$$\int_r^s |R_R^*(\hat{u})(\exp x)| \, dx < 0$$

and

$$\ln \circ \hat{u}' \circ \exp(r) - \ln \circ \hat{u}' \circ \exp(s) = \int_r^s R_R^*(\hat{u})(\exp x) \, dx$$

for all $r, s \in \mathbb{R}$.

If $\hat{u}$ has a relative risk aversion density $R_R^*(\hat{u})$, then $R_R^*(\hat{u})$ is unique almost everywhere.

Provided that $\hat{u}$ is differentiable with $\hat{u}' > 0$, the cumulative relative risk aversion function $-\ln \circ \hat{u}' \circ \exp$ is absolutely continuous if and only if the cumulative absolute risk aversion function $-\ln \circ \hat{u}'$ is absolutely continuous; and $R_R^*(\hat{u})$ is a relative risk aversion density of $\hat{u}$ if and only if it is a density of $\hat{u}'$. This follows from the identity,

$$\int_r^s R_R^*(\hat{u})(\exp x) \, dx = \int_{\exp r}^{\exp s} R_R^*(\hat{u})(y) \, dy$$

If $R^*_R(\hat{u})$ exists, then $-\ln \hat{u}' \circ \exp$ is differentiable at almost every x, with

$$(-\ln \circ \hat{u}' \circ \exp)'(x) = R^*_R(\hat{u})(\exp x)$$

and, hence,

$$R^*_R(\hat{u})(\hat{x}) = -\frac{\hat{x}\hat{u}''(\hat{x})}{\hat{u}'(\hat{x})}$$

at almost every $\hat{x} > 0$. So, the usual coefficient of relative risk aversion is almost everywhere well defined and equal to $R^*_R(\hat{u})$. Conversely, if $\hat{u}$ is twice continuously differentiable, then $R^*_R(\hat{u})$ exists and is equal to the usual coefficient of relative risk aversion almost everywhere.

If $\hat{u}$ is a differentiable utility function defined on $(0, \infty)$, with $\hat{u}' > 0$, set $u = \hat{u} \circ \exp$. Then the cumulative absolute risk aversion function of u and the cumulative relative risk aversion function of $\hat{u}$ are related by

$$-\ln \circ u'(x) = -\ln \circ \hat{u}' \circ \exp(x) - x$$

Proposition 5. *Let $\hat{u}$ be a differentiable utility function defined on $(0, \infty)$ with $\hat{u}' > 0$, and set $u = \hat{u} \circ \exp$. Then the cumulative relative risk aversion function of $\hat{u}$ is absolutely continuous if and only if the cumulative absolute risk aversion function of u is absolutely continuous; and $R^*_R(\hat{u})$ is a relative risk aversion density of $\hat{u}$ if and only if $R^*_A(u) = R^*_R(\hat{u}) \circ \exp -1$ is an absolute risk aversion density of u.*

Proof.. The functions $R^*_A(u)$ and $R^*_R(\hat{u})$ are related by the equation in the proposition if and only if

$$\int_r^s R^*_A(u)(x)\, dx = \int_r^s R^*_R(\hat{u})(\exp x)\, dx - (s - r)$$

for all $r, s \in I$. Since

$$-\ln \circ u'(x) = -\ln \circ \hat{u}' \circ \exp(x) - x$$

it follows that $R^*_A(u)$ is a density of $-\ln \circ u'$ if and only if $R^*_R(u) \circ \exp$ is a density of $-\ln \circ \hat{u}'$. □

Theorem 2. *Let $\hat{u}$ be a strictly increasing risk averse utility function on $(0, \infty)$ The following statements are equivalent:*

1. *$\hat{u}$ exhibits decreasing (increasing) relative risk aversion*
2. *$\hat{u}$ is differentiable with $\hat{u}' > 0$, and the cumulative relative risk aversion function*

$$x \mapsto -\ln \circ \hat{u}' \circ \exp(x)$$

   is concave (convex)
3. *$\hat{u}$ has a relative risk aversion density $R^*_R(\hat{u})$ which is decreasing (increasing).*

Proof.. The theorem follows from Theorem 1 applied to the function $u = \hat{u} \circ \exp$. Note that u is not necessarily risk averse, as assumed in Theorem 1. This does not matter, because $\hat{u}'$ and hence u' will be continuous almost everywhere, which is all that is needed in the proof of Theorem 1. □

Theorem 2 implies that if a strictly increasing risk averse utility function $\hat{u}$ exhibits monotone (decreasing or increasing) relative risk aversion, then it is not only differentiable but continuously differentiable. It is twice differentiable everywhere except possibly at a countable number of points. If it is twice differentiable everywhere, then it is twice continuously differentiable. All this follows from the fact that the cumulative relative risk aversion function $-\ln \circ \hat{u}' \circ \exp$ is concave or convex. A concave or convex function is differentiable everywhere except possibly at a countable number of points, and if it is differentiable everywhere, then it is continuously differentiable.

It follows from the theorem that a strictly increasing utility function $\hat{u}$ on $(0, \infty)$ is risk averse and exhibits decreasing (increasing) relative risk aversion if and only if it has the form

$$\hat{u}(\hat{s}) = \hat{u}(\hat{r}) + \int_{\hat{r}}^{\hat{s}} \exp(-h \circ \ln(\hat{x}))\, d\hat{x}$$

for $\hat{r}, \hat{s} \in (0, \infty)$, for some increasing concave (convex) function h on $\mathbb{R}$ (the cumulative relative risk aversion function of $\hat{u}$). If we fix $\hat{r} > 0$ and set $\hat{u}(\hat{r}) = 0$, then this equation establishes a bijection between increasing concave (convex) functions h and strictly increasing risk averse utility functions with decreasing (increasing) relative risk aversion and value zero at $\hat{r}$.

17.4 Appendix: Proof of Proposition 1

The proof of Proposition 1 relies on the following lemma.

To state the lemma, we need to be careful in defining what we mean by a concave function on a subset of the real line which is not necessarily an interval. Generally, if $C \subset \mathbb{R}$ is a set which is not necessarily an interval, and if k is a real valued function defined on C, then we say that k is *concave* if

$$k(tx + (1-t)y) \geq tk(x) + (1-t)k(y)$$

whenever $0 \leq t \leq 1$ *and all the points x, y and $tx + (1-t)y$ are in C.*

Lemma 1. *Let $C \subset \mathbb{R}$ be a set which has no smallest and no largest element, and let $h : C \to \mathbb{R}$ and $k : h(C) \to \mathbb{R}$ be strictly increasing concave functions.*

1. *If $k \circ h$ is continuous at a point x in C then so is h*
2. *If C and $h(C)$ are intervals, and if $k \circ h$ is differentiable at a point x in C then so is h*

Proof.. (1): Assume, to the contrary, that $k \circ h$ is continuous at x but h is discontinuous at x. Set

$$\underline{h} = \sup\{h(y) : y \in C, y < x\}$$
$$\overline{h} = \inf\{h(y) : y \in C, y > x\}$$
$$\underline{k} = \sup\{k(z) : z \in h(C), z < h(x)\} = \sup\{k \circ h(y) : y \in C, y < x\}$$

and

$$\overline{k} = \inf\{k(z) : z \in h(C), z > h(x)\} = \inf\{k \circ h(y) : y \in C, y > x\}$$

Then $\underline{k} = \overline{k}$ but $\underline{h} < \overline{h}$. Let $z, z' \in h(C)$ be such that $z < \underline{h}$ and $\overline{h} < z'$. Because k is concave, the graph of k above z' is on or below the line through $(z, k(z))$ and $(z', k(z'))$. Since this is true of all such z and z', the graph of k above $(\overline{h}, \overline{k})$ is on or below the line through $(\underline{h}, \underline{k})$ and $(\overline{h}, \overline{k})$. But since $\underline{k} = \overline{k}$, this line is horizontal. The point $(z', k(z'))$ on the graph of k approaches the point $(\overline{h}, \overline{k})$ as $z' \to \overline{h}$ from above through $h(C)$. But then, since k is strictly increasing, its graph cannot possibly stay on or below the horizontal line through $(\underline{h}, \underline{k})$ and $(\overline{h}, \overline{k})$, a contradiction.

(2): If h has a kink at x, then k can only make it worse, so $k \circ h$ will also have a kink at x. □

It is easily seen that if u and v are strictly increasing utility functions defined on an interval I, then u is more risk averse than v if and only if there exists a strictly increasing concave function $k : v(I) \to \mathbb{R}$ such that $u = k \circ v$. This is true even if v is not continuous, so that the set $v(I)$ where k is defined is not an interval.

Proof of Proposition 1. Assume that u exhibits increasing absolute risk aversion.

Since u is strictly increasing, it is almost everywhere differentiable, by Billingsley [2, Theorem 31.2]), and continuous at almost every point in I.

First, we show that u is continuous. Let x be a point in I where u is continuous. We will show that u is continuous at $x + d$ for every $d \geq 0$. It then follows that u is continuous everywhere in I.

Given $d \geq 0$, since $u_{[d]}$ is more risk averse than u, there is a strictly increasing concave function $k_d : u(I) \to \mathbb{R}$ such that $u_{[d]} = k_d \circ u$ on $u(I)$.

We shall show that k_d is continuous at $u(x)$. Assume it is not. For every $e \geq 0$, $u_{[d+e]}$ is more risk averse than $u_{[d]}$, so there is a strictly increasing concave function $k_{d,e} : u_{[d]}(I) \to \mathbb{R}$ such that $u_{[d+e]} = k_{d,e} \circ u_{[d]}$ on $u_{[d]}(I)$. Since

$$k_{[d+e]} \circ u = u_{[d+e]} = k_{d,e} \circ u_{[d]} = k_{d,e} \circ k_d \circ u$$

it follows that

$$k_{[d+e]} = k_{d,e} \circ k_d$$

Since, by assumption, k_d is not continuous at $u(x)$, it follows from (1) of Lemma 1 that $k_{[d+e]}$ is not continuous at $u(x)$. Since $k_{[d+e]} \circ u = u_{[d+e]}$, this implies that $u_{[d+e]}$ is not continuous at x, which means that u is not continouos at $x+d+e$. This is true of all $e \geq 0$, contradicting the fact that u is continuous almost everywhere.

Hence, k_d is continuous at $u(x)$. This is true of all $d \geq 0$, so u is continuous $x + d$ for all $d \geq 0$. Since u is continuous almost everywhere, it follows that it is continuous everywhere.

Second, we show that u is differentiable with $u' > 0$. Let x be a point in I where u is differentiable.

Given $d \geq 0$, there is, as above, a strictly increasing concave function $k_d : u(I) \to \mathbb{R}$ such that $u_{[d]} = k_d \circ u$ on $u(I)$. Since u is continuous, $u(I)$ is an interval.

Since k_d is concave, if $u'(x) = 0$, then $u_{[d]}$ is differentiable at x with $u'_{[d]}(x) = 0$, and so u is differentiable at $x+d$ with $u'(x+d) = 0$. This would be true of all $d \geq 0$, contradicting the fact that u is strictly increasing. Hence, $u'(x) > 0$.

We shall show that k_d is differentiable at $u(x)$. Assume it is not. For every $e \geq 0$, there is, as above, a strictly increasing concave function $k_{d,e} : u_{[d]}(I) \to \mathbb{R}$ such that $u_{[d+e]} = k_{d,e} \circ u_{[d]}$ on I. Since, as above,

$$k_{[d+e]} = k_{d,e} \circ k_d$$

it follows from (2) of Lemma 1 that $k_{[d+e]}$ is not differentiable at $u(x)$. Since $k_{[d+e]} \circ u = u_{[d+e]}$ and u is differentiable at x, this implies that $u_{[d+e]}$ is not differentiable at x. Hence, u is not differentiable at $x + d + e$. This is true of all $e \geq 0$, contradicting the fact that u is differentiable almost everywhere.

So, k_d is differentiable at $u(x)$. This is true of all $d \geq 0$, so u is differentiable at $x + d$ with $u'(x + d) > 0$, all such c. Since u is differentiable almost everywhere, it follows that it is differentiable everywhere.

The proof for the case of decreasing absolute risk aversion is identical, except that $d \geq 0$ is replaced by $d \leq 0$ and $e \geq 0$ is replaced by $e \leq 0$. □

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18

Will Democracy Engender Equality?*

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Summary. Many suppose that democracy is an ethos which requires, inter alia, a degree of economic equality among citizens. In contrast, we conceive of democracy as ruthless electoral competition between groups of citizens with different interests, who are organized into parties. We inquire whether such competition, which we assume to be concerned with distributive matters, will engender economic equality in the long run. Society is modeled as OLG, and each generation competes politically over educational finance and tax policy; the policy space is infinite dimensional. A political equilibrium concept is proposed which determines the membership of two parties endogenously, and their proposed policies in political competition. One party wins the election (stochastically). This process determines the evolution of the distribution of human capital. We show that, whether the limit distribution of human capital is an equal one depends upon the nature of intra-party bargaining and the degree of inequality in the original distribution.

Key words: Political equilibrium, Human capital, Educational finance, Inequality.

JEL Classification Numbers: D72, H2.

* This article reports on a longer project, to be published as *Democracy, Education, and Equality* (Cambridge University Press). Here, I motivate the problem, explain one of the models, and state one of the principal results. Proofs of theorems are omitted, but will be available in the aforementioned monograph. Many individuals have provided valuable comments and help on this project, including Ignacio Ortuño-Ortin, Herbert Scarf, Roger E. Howe, Karine Van der Straeten, John Geanokoplos, and Colin Stewart. . A preliminary version of this work was delivered as the Graz-Schumpeter Lectures at the University of Graz in May 2003, and I am grateful to the Graz Schumpeter Society for their hospitality and comments. The general topic of this article has been, I believe, of keen interest to Birgit Grodal, a dear friend and staunch supporter of my unconventional work in economic theory since I embarked upon that path in the late 1970s. I will always be deeply grateful for her friendship, support and encouragement, and I am honored to contribute this paper to her festschrift.

18.1 Introduction: Democracy and equality

The identification of democracy with equality is not simply a practice of many political theorists, but of masses of people. Perhaps the most important aspect of political transformation in the world in the last half century has been the toppling of authoritarian regimes, and their replacement with democracies. Just as socialism was a powerful movement in the first half of the twentieth century – by 1950, fully one-third of the world's peoples lived under regimes that described themselves as socialist – so democracy has been the massively appealing political doctrine in, let us say, the period since 1960. But just as it was an error of socialists to identify socialism with All Good Things, so now it is an error of democrats to identify democracy with All Good Things.[2] The most common example of this fallacy is when some say that regime X cannot be a democracy, because it sustains Bad Thing Y (oppression of women, abrogation of civil rights, massive inequality). But if democracy is defined as a set of competitive political institutions, rather than as an ethos, then the correct approach is to study what those institutions entail. Indeed, do those institutions entail equality, or any other 'good thing'?

My topic, in this article, is the democratic struggle over income distribution and public education; in particular, I will be concerned with understanding whether democracy will bring about, through decisions that are made by the polity concerning the targeting of education on the disadvantaged, a distribution of human capital that is egalitarian. My premise is that education is the main instrument through which equality of human capacities could be rendered more equal than they currently are, and I want to understand whether the kind of political competition that democracy organizes will in fact engender that outcome.

Why be concerned with equality, and if equality, why with equality in the distribution of human capital, and not of income? The view of justice that motivates the present study is one that advocates equality of opportunity. The principal obstacle to equality of opportunity, with which I am here concerned, is the inequality of advantage that is conferred on children in a generation by their disparate family backgrounds, and more precisely, by the cultural and social differences of those backgrounds, rather than by biological (genetic) differentiation. As I said, I believe that education is the main instrument our societies have to rectify those inequalities of family advantage, although we still do not know exactly the extent to which it can do so. I take an optimistic view, in the present work, that the educational technology could, with sufficient resources, rectify those inequalities completely. My interest is in whether democratic processes will indeed engender the educational investments in disadvantaged children needed to do so.

The ethical aspiration, then, is to further equality of opportunities for life chances. I believe equalizing incomes is insufficient for this purpose, because 'human capital,' the kind of knowledge created by education, is indeed an important *direct* input into

[2] There are many people who identify democracy with justice and equality. For instance, Adolfo Perez Esquivel, a Nobel Peace Prize laureate, recently said, "The vote does not define democracy. Democracy *means* justice and equality. [Italics added –JER] " (*The Daily Journal* [Caracas], July 12, 2001).

an individual's welfare and possession of self-esteem. So I will model parents as caring about the human capital their children will come to have, and the supposition is that they so care not only because their child's human capital will produce an income for him, but also because its possession is a requirement of the child's future self-realization (for my more complete views on equality of opportunity, see Roemer, 1998).

Because I wish to focus upon social rather than genetic disadvantage, I will assume that all children have equal talents, where by talent I mean the genetic capacity to learn, and develop skill. This is, of course, an unrealistic assumption. Even if in reality we can never hope to equalize opportunities fully, to the extent that inequality of talent may prevent that,[3] it is important to know whether democratic politics can be expected to erase that part of disadvantage which is social, and could, under an optimistic assumption, be rectified through education.

18.2 The politico-economic environment

We will model a society which reproduces itself over many generations. At the initial date, there are households led by adults (parents), who are characterized by a distribution of human capital, that is, capacities to produce income. Each parent has one child. The human capital the child will have, when next period he has become an adult, is a monotone increasing function of his parent's level of human capital and the amount that is invested in his education. This relationship is deterministic, and is described by the educational production function for all children. Thus, more investment is needed to bring a child from a poor (low human capital) family up to a given level of human capital than a child from a richer family. All parents have the same utility function: a parent cares about her household's consumption (which will be her after-tax income), and the human capital her child will come to have, as an adult. We will, for simplicity, assume that adults do not value leisure. Thus, income will be produced inelastically with respect to taxation.

We will assume that educational finance is purely public. The polity of adults, at each date, must make four political decisions: how much to tax themselves, how to partition the tax revenues between a redistributive budget for households' current consumption and the educational (investment) budget, how to allocate the budget for redistribution among adults, and how to target the educational budget as investment in particular children, according to their type (that is, their parents' human capital). Once these political decisions are implemented, a distribution of human capital is determined for the next generation. When the present children become adults, characterized by that distribution of human capital, they face the same four political decisions. We wish to study the asymptotic distribution of human capital of this dynamic process.

[3] Some may find this a strange locution, but as I believe that a low degree of talent is a circumstance beyond a person's control, I therefore think that full equality of opportunity requires complete rectification of inequalities of condition that are due to unequal talent. My skepticism in this sentence refers to the possibility of achieving that result.

In the society we have described, a child is characterized by the family into which he is born, for his capacity to transform educational investment into future earning power is determined by his family background, proxied for by his parent's human capital. We imagine that the transmission of 'culture' to the child is indicated by the parent's human capital endowment. The child's capacity successfully to absorb educational investment, and transform it into human capital, is a circumstance beyond his control, and so a society of this kind that wished rapidly to *equalize opportunities* for all children would compensate children from poorer families with more educational investment. Equality of opportunity would be achieved when all adults come to have the same human capital, for that means, as children in the previous generation, the compensation for disadvantageous circumstances was complete. In the real world, equality of opportunity does not require equalizing outcomes in this way, because people remain responsible for some aspect of their condition (the effort they expend, for example), even after the necessary compensation for disadvantage has been made. But in our model there is no such element of personal responsibility, and so, if we take equality of opportunity as our conception of justice, then justice will have been achieved exactly when the wage-earning capacities of all adults are equal.

We will stipulate a democratic process for solving society's political problems, at each generation, and our focus will be on that process. We employ an extension of the PUNE (party-unanimity Nash equilibrium) concept of democratic political equilibrium that takes as data the distribution of preferences of the polity over a given policy space, and produces as output an endogenous partition of the polity into two political parties, a policy proposal by each party, and a probability that each party will win the election.[4] We suppose that an election occurs, and the policy of the victorious party is implemented. Our procedure will be to begin with a distribution of adult human capital at date 0, which will determine the distribution of adult preferences at date 0, and thus initialize this stochastic dynamic process.

Although I have described the political choice as consisting of four independent decisions, we will in fact model the political problem as one on an infinite dimensional policy space. That policy space, denoted T, will consist of pairs of functions (ψ, r) where $\psi(h)$ is the after-tax household income of an adult with human capital h, and $r(h)$ is the public educational investment in a child from a family where the parent has human capital h. These functions will be restricted only to be continuous, to jointly satisfy a budget constraint, and to satisfy an upper and lower bound on their derivatives, when the derivatives exist.

The present analysis therefore marks a technical advance over analyses in political economy that rely on equilibrium concepts that are non-vacuous only when policy spaces are unidimensional. The advance, I believe, is not merely technical. It is surely artificial to restrict a democratic polity's choice of policies to ones with simple mathematical properties, such as linearity. Characterizing the political equilibrium with no such restrictions means that we are able to model the democratic

[4] PUNE is defined and studied in Roemer (2001). The analogous concept in that book is called 'PUNE with endogenous parties.'

struggle as *ruthlessly competitive*: no holds, in the sense of unmotivated restrictions on the nature of policy proposals, are barred.

Many authors, over the past decade, have studied the relationship between education and equality in democracies.[5] Some of this work models the problem in an overlapping generations framework, and some of it endogenizes the political decision concerning the funding of public education. In all the work of which I am aware in which policy is endogenized, the policy space is unidimensional; taxes are proportional income or wealth taxes. Public education is always distributed equally to all students; what variation there is in the amount of education an individual receives is either due to the existence of private supplements, or to variations in time spent in school across wealth levels, where that choice is made by the individual. In almost all these papers, public education has an equalizing effect on incomes, at least in the long-run. (That effect may not hold at all times, because children from richer families may choose to spend more time in school than children from poorer families.) This is not surprising, if the same amount is invested in all children. Generally, more inequality in the initial distribution of income/human capital leads to higher taxation, more public education, and hence more growth.

The general characteristics which distinguish existing work from my study are that its authors postulate more heterogeneity among citizens than I do, and they model the political process more simply. In the papers mentioned, citizens differ not only in their levels of human capital and/or income, but sometimes in their preferences, often in their (randomly realized) talents, and sometimes in the neighborhoods in which they live. Sometimes the democracy is incomplete, and citizens differ in their voting rights. On the other hand, the political model is typically extremely simple: majority vote over a proportional tax rate. The present study complements this work: it abstracts from heterogeneity in the composition of the citizenry, with the sole exception of the differentiation in human-capital endowments, and articulates to a much greater degree the political mechanism. In particular, it is a general characteristic of policies of educational finance, in my model, that students from richer families receive more public investment in their education, a feature which is at least true of American democracy, among the advanced countries, and appears ubiquitous in democracies at low levels of economic development. In the dynamic models of the existing literature, to my knowledge, this does not occur.

18.3 The model

A. Preferences

We begin by stating the preferences of the adults, who are the political actors. A parent's utility function is

[5] See, among others, Glomm and Ravikumar (1992), Saint-Paul and Verdier (1992), Zhang (1996), Durlauf (1996), Gradstein and Justman (1996), Bénabou (1996), Turrini (1998), Fernandez and Rogerson (1998), Cardak (1999), Bourgignon and Verdier (2000), and Glomm and Ravikumar (2003).

$$u(x, h') = \log x + \gamma \log h'$$

where x is the family's consumption and h' is the human capital the child will (come to) have as an adult. We measure the child's human capital by his earning power, and a person with human capital h earns h in a unit of time. (Thus, we will also speak of human capital as the wage.) Now I wrote above that adults derive welfare directly from their human capital, and so one might wish to add a term to the utility function showing that the parent cares about her own human capital, as well as her income. But that term would be a constant at the time the parent is a decision maker, as the adult's human capital is unchangeable, and so it would be gratuitous to include it in the utility function.

We furthermore assume that these utility functions are von Neumann- Morgenstern, and are unit comparable, and so it makes sense to add them up.

B. Technology

The educational technology is

$$h' = \alpha h^b r^c$$

where h is the parent's human capital, h' is human capital the child will come to have as an adult, and r is investment in the child's education. We think of the input of parental human capital as operating through household culture, and perhaps social and professional connections of the parent, as well. (High wage parents can find high wage jobs for their children.) I assume that all children possess the same degree of talent, as I wish to study what democracy would produce, absent differentiation in natural talents.

Depending on whether $b + c$ is less than, equal to , or greater than one, we say the returns to scale in education are decreasing, constant, or increasing.

With this educational technology, returns to education are purely private: my child's wage depends only on what is invested in my child. In later work,[6] I modify the production function to include an endogenous-growth element, modeling the idea that there are positive externalities in education, so that educational investment has a public-good aspect.

C. The policy space

Political parties will choose, as their platform, a pair of functions

$$\psi : H \to \mathbf{R}_+, \quad r : H \to \mathbf{R}_+$$

$\psi(h)$ is the after-tax income that an h-family will receive, and $r(h)$ is the public educational investment in children from h-families. Let the distribution of parental human capital, which is the same as pre-tax income, at a given date be given by a probability measure $\mathbf{F}$ on the non-negative real numbers with mean μ. We assume that the support of $\mathbf{F}$ is the non-negative real line, and that $\mathbf{F}$ is equivalent to Lebesgue measure.[7] Then feasibility requires:

[6] See the forthcoming book alluded to in the opening footnote.

[7] Two measures are equivalent if their null sets are the same.

$$\int (\psi(h) + r(h))d\mathbf{F}(h) = \mu. \tag{18.1}$$

We insist as well that ψ and r be continuous functions. Denote the *total resource bundle* going to an h family by

$$X(h) = \psi(h) + r(h);$$

we restrict policies by requiring that the derivative of X , when it exists, be bounded above and below:[8]

$$\psi'(h) + r'(h) \leq 1 \tag{18.2}$$

$$\psi'(h) + r'(h) \geq 0 \tag{18.3}$$

Inequality (18.2) says that there cannot be intervals where there is redistribution from the relatively poor to the relatively rich, in the sense that the total resource bundle increases faster than pre-tax income. Inequality (18.3) says the redistribution from the relatively rich to the relatively poor cannot be excessive, in the sense that the total resource bundle decrease with pre-tax income.

Consider the *laissez-faire* policy, in which there is no taxation, and parents are assumed to finance education from private income. In that case, $X(h) = h$, and so $X'(h) = 1$. Thus, (18.2) says that the tax regime cannot be more regressive than laissez-faire. In particular, laissez-faire is feasible.

Thus, our policy space is

$$T = \{(\psi, r)|(\psi, r) \text{ continuous and (18.1), (18.2), and (18.3) hold}\}$$

Further comment on the constraints that characterize T is warranted. The present analysis (like all analyses) makes some assumptions that are not formally modeled. One is that policies must be continuous. Another is that the derivatives on X must lie in the interval [0,1]. I think that these assumptions are justified by the norms of advanced democracy: those norms put restrictions on what kinds of policy it is ethically acceptable for a political party to propose. Continuity is such a restriction: it is, if you will, an extension of the norm of horizontal equity, which says that likes should be similarly treated by the state. (If a policy is discontinuous, then types that are almost identical are treated very differently.) Similarly, one observes, in no advanced democracy, a fiscal policy which *is seen* to violate the condition on the derivatives of X.[9] To tax at a marginal rate of greater than one would be viewed as unfair to the rich, and to tax at a rate of less than zero would be viewed as expropriation of the poor.

The analogue of assumptions (18.2) and (18.3) in the standard model of unidimensional affine taxation is that the (constant) marginal tax rate lie between zero and

[8] To be mathematically precise, we need only require that the the function X be non-decreasing and satisfy the Lipschitz condition $X(h^2) - X(h^1) \leq h^2 - h^1$ for all $h^2 > h^1$.

[9] It is a more delicate question to ask whether , after the incidence of taxation and benefits are accounted for, this condition is violated.

one. This assumption is traditional, although it must – there too– be considered a social norm, in the sense of being an underived restriction on the policy space.

The policy space T is infinite dimensional. It contains very complicated functions. We will see that the piece-wise linear functions play a special role: indeed, equilibrium policies will always be piece-wise linear.

We can now write the indirect utility function of an adult on policies:

$$\begin{aligned} v(\psi, r, h) &= \log \psi(h) + \gamma \log \alpha h^b r(h)^c \\ &= \log \psi(h) + \gamma(\log \alpha h^b) + \gamma c \log r(h) \\ &\tilde{=} \log \psi(h) + \gamma c \log r(h) \end{aligned} \tag{18.4}$$

where, in the last line, I drop a term that is gratuitous, because the decision maker's human capital h is fixed. Thus, parental utility is a Cobb-Douglas function of after-tax income and educational investment in her child.

D. Probability of victory

Suppose the polity is partitioned into two parties, called L and R. Let t^L and t^R be policies proposed by the two parties. Define the set of voter types who prefer the first policy to the second by

$$H(t^L, t^R) = \{h | v(t^L, h) > v(t^R, h)\}$$

and the set of indifferent types as

$$I(t^L, t^R) = \{h | v(t^L, h) = v(t^R, h)\}.$$

We say that the set of voters who are *expected to vote for policy* t^L *is*

$$H(t^L, t^R) \cup (I(t^L, t^R) \cap L) \equiv \Omega(t^L, t^R);$$

that is, the types expected to vote for L are those who prefer L's policy plus the indifferent voters who are members of L. The complement is expected to vote for t^R. Thus, the expected fraction of the vote that L will receive is $\mathbf{F}(\Omega(t^L, t^R))$.

There is, however, presumed to be a stochastic element in elections; there may be scandals during the campaign, or some issues that we have not modeled may be important to voters. We attempt to capture this in a simple way by saying that the realized fraction of the vote that L will receive is

$$\mathbf{F}(\Omega(t^L, t^R)) + Z,$$

where Z is a random variable uniformly distributed on an interval $[-\beta, \beta]$. Consequently the *probability* that L will win the election is

$$\text{prob}[\mathbf{F}(\Omega(t^L, t^R)) + Z > 0.5] \equiv \pi(t^L, t^R),$$

which is easily computed to be given by the formula:

$$\pi(t^L, t^R) = \begin{cases} \dfrac{\beta + F(\Omega) - 0.5}{2\beta}, \text{ if } \beta > 0.5 - F(\Omega) > -\beta \\ 1, \text{ if } 0.5 - F(\Omega) \leq -\beta \\ 0, \text{ if } 0.5 - F(\Omega) \geq \beta \end{cases}.$$

More general formulations of the probability-of-victory function would be acceptable for our purposes, but this one will do. Note that π is weakly monotone increasing in the size of $\Omega(t^L, t^R)$.

18.4 Two concepts of political equilibrium

Space constraints prohibit my motivating the definition of party –unanimity Nash equilibrium (PUNE). The reader is referred to Roemer (2001, chapters 8 and 13). The basic idea is that, within a party, there are three factions – Opportunists, Militants, and Reformists. A PUNE is a Nash equilibrium between parties, in which each party's proposal is an outcome of bargaining among the party's factions, facing the policy proposed by the opposing party.

A PUNE for this model is:

(P1) a partition $L \cup R = \mathbf{R}_+, \quad L = [0, h^*), \quad R = [h^*, \infty)$

(P2) party utility functions

$$V^L(\psi, r) = \int_0^{h^*} (\log \psi(h) + \gamma c \log r(h)) d\mathbf{F}(h),$$

$$V^R(\psi, r) = \int_{h^*}^{\infty} (\log \psi(h) + \gamma c \log r(h)) d\mathbf{F}(h),$$

(P3) a pair of policies $(\psi^L, r^L), (\psi^R, r^R)$ such that:

(a) there is no $(\psi, r) \in T$, such that $V^L(\psi, r) \geq V^L(\psi^L, r^L)$ and $\pi((\psi, r), (\psi^R, r^R)) \geq \pi((\psi^L, r^L), (\psi^R, r^R))$, with at least one inequality strict, and

(b) there is no $(\psi, r) \in T$, such that $V^R(\psi, r) \geq V^R(\psi^R, r^R)$ and $\pi((\psi^L, r^L), (\psi, r)) \leq \pi((\psi^L, r^L), (\psi^R, r^R))$, with at least one inequality strict;

(P4)
$$h \in L \Rightarrow v(\psi^L, r^L; h) \geq v(\psi^R, r^R; h),$$
$$h \in R \Rightarrow v(\psi^L, r^L; h) \leq v(\psi^R, r^R; h).$$

We call the two parties *Left* and *Right,* because they represent the bottom and top of the human-capital distribution, respectively.[10]

[10] For the general definition of PUNE, see Roemer (2001). Note that I restrict PUNEs to be equilibria in which the two parties comprise intervals on the space of human capital endowments. There are surely other equilibria of a more general form.

In the above definition, I employ the fact that the Reformist factions, whose payoff functions are the expected average utility of their respective parties' members, are gratuitous. The set of PUNEs does not change with their inclusion.

Our first observation will allow us to simplify our problem. It is the following:

Proposition 1. *In any PUNE, $r^L(h) = \gamma c \psi^L(h)$ for almost all $h \in L$ and $r^R(h) = \gamma c \psi^R(h)$ for almost all $h \in R$.*

Because of Proposition 1, it suffices to study the total resource functions

$$X(h) = \psi(h) + r(h);$$

for once a party proposes a total resource function, its decomposition into the two functions r and ψ is immediate from the proposition.

Thus, we have reduced the problem of studying policies that are pairs of functions on H to the problem of studying singletons of functions on H. (Clearly, this is the purchase of the Cobb-Douglas assumption.) These are the continuous functions X that are non-decreasing, have slopes no larger than unity, and integrate to μ. We denote this space of functions by T^*.

It is not surprising that it is very difficult to characterize PUNEs on this large policy space. We will introduce another concept, the so-called *quasi*-PUNE. Quasi-PUNEs will be relatively easy to characterize, and we will argue that the quasi-PUNE notion is itself a satisfactory concept of equilibrium.

A *quasi*-PUNE is:

(Q1) a level of human capital h^* and a partition $L = [0, h^*)$, $R = [h^*, \infty)$ of the polity;

(Q2) party utility functions

$$V^L(X) = \int_0^{h^*} \log X(h) d\mathbf{F}(h),$$

$$V^R(X) = \int_{h^*}^{\infty} \log X(h) d\mathbf{F}(h),$$

(Q3) a pair of total resource functions X^L, X^R in T^* such that

(a) there exists no policy $X \in T^*$ such that $X(h) \geq X^R(h)$ for $h \leq h^*$ and $V^L(X) \geq V^L(X^L)$ and $\pi(X, X^R) \geq \pi(X^L, X^R)$, with at least one strict inequality;

(b) there exists no policy $X \in T^*$ such $X(h) \geq X^L(h)$ for all $h \geq h^*$ and $V^R(X) \geq V^R(X^R)$ and $\pi(X^L, X) \leq \pi(X^L, X^R)$, with at least one strict inequality;

(Q4) for all

$$h \in L, v(X^L, h) \geq v(X^R, h)$$
$$h \in R, v(X^R, h) \geq v(X^L, h).$$

As in the defintion of PUNE, we can as well include Reformist factions in both parties, which will not change the set of quasi-PUNEs.

The only difference between PUNEs and quasi-PUNEs is that an extra constraint on the solution is required of the quasi-PUNE: this appears in (Q3a) and (Q3b). Consider, for example, (Q3a). A candidate X for a deviation from X^L must satisfy the additional requirement

"that $X(h) \geq X^R(h)$ for $h \leq h^*$" ;

this means that a policy X^L is an acceptable response to X^R only if there is no policy *that preserves the loyalty of all L members* and is an improvement of the payoffs for at least one of the Militant and Opportunist factions. A similar statement holds for acceptable Right deviations, in (Q3b).

We may justify this additional requirement as follows. In its bargaining with the Opportunists (and the Reformists), the Militants insist not only that the average welfare of the party's members (constituents) not fall, under a proposed policy deviation, but that the party deliver at least as much to *each* of its members as the opposition proposes to do. Thus the Militants are concerned both with the global welfare of their membership (average utility) and with their local welfare (serve every member at least as well as the opposition proposes to do). Thus, we may think of quasi-PUNEs as PUNEs in which the Militants are *even more militant* with regard to protecting the interests of constituents (members).

Proposition 1 continues to hold for quasi-PUNEs (and so we have dispensed with the statement of the definition in terms of the original policy space T).

It is immediately clear that:

Proposition 2. *Every* PUNE *is a quasi*-PUNE.

Simply observe that there are additional restrictions on deviations in the definition of quasi-PUNE, so the set of quasi-PUNEs must include the PUNEs.

18.5 Characterization of quasi-PUNEs

Our strategy will be, henceforth, to study quasi-PUNEs.

We next provide a complete characterization of quasi-PUNEs of a non-trivial kind – to wit, those in which both parties win with positive probability.

Proposition 3. *Let* (h^*, X^L, X^R) *be a quasi*-PUNE *at which both parties win with positive probability. Then there exists a number* $\bar{y}$ *such that*

$$\begin{aligned} &X^L \text{ solves } \max_{X \in T^*} V^L(X) \\ &\text{subject to} \\ &h \leq h^* \Rightarrow \log X(h) \geq \log X^R(h) \text{ (L1)} \\ &\log X(h^*) \geq \bar{y} \text{ (L2)} \end{aligned} \qquad \text{(Q}^*\text{3a)}$$

$$\begin{aligned}&X^R \text{ solves } \max_{X\in T^*} V^R(X)\\&\text{subject to}\\&h \geq h^* \Rightarrow \log X(h) \geq \log X^L(h) \text{ (R1)}\\&\log X(h^*) \geq \bar{y} \text{ (R2)}\end{aligned} \tag{Q*3b}$$

and constraints (L2) *and* (R2) *are binding. Conversely, if there is a number* $\bar{y}$ *such that* X^L *and* X^R *satisfy* (Q*3a) *and* (Q*3b) *and both constraints* (L2) *and* (R2) *bind, and both parties win with positive probability at this pair of policies, then* (h^*, X^L, X^R) *is a quasi*-PUNE.

We will proceed by solving even simpler programs than (Q*3ab), namely:

$$\begin{aligned}&\max_{X\in T^*} \int_0^{h^*} \log X(h) d\mathbf{F}(h)\\&s.t.\\&\log X(h^*) \geq \bar{y},\\&\text{and the constraint binds;}\end{aligned} \tag{SL}$$

and

$$\begin{aligned}&\max_{X\in T^*} \int_{h^*}^{\infty} \log X(h) d\mathbf{F}(h)\\&s.t.\\&\log X(h^*) \geq \bar{y}\\&\text{and the constraint binds.}\end{aligned} \tag{SR}$$

After solving (SL) and (SR) we will note that the other constraints (L1)of (Q*3a) and (R1) of (Q*3b) hold, and furthermore, that (Q4) holds, and so solutions of (SL) and (SR) are indeed quasi-PUNEs.

We now note that (SL) and (SR) are concave problems: they involve maximizing a concave functional (the integral of the logarithm) on a convex set of functions. (Just note that if two functions are in T^*, so is their convex combination, and furthermore, the constraints in (SL) and (SR) are convex constraints.) So, in principle, these are solvable programs.

At this juncture, we make an important technical observation. (SL) and (SR) are concave optimization problems, and no fixed point methods need be employed to solve them. The fortuitous aspect of program (SL) is that *it does not refer to the policy* X^R; likewise, program (SR) does not refer to the policy X^L. Thus the *interaction* between policies, which makes solving for Nash equilibrium an intrinsically difficult problem, has disappeared with the formulation of the quasi-PUNE concept. We have, that is to say, replaced the very difficult problem of finding fixed points in an infinite dimensional space (the problem of finding PUNEs) with solving two optimization problems. It is this 'trick' that makes our project tractable.

We solve (SL) and (SR) in two steps. In step 1, we characterize the set of ordered pairs $(h^*, \bar{y})$ such that, at the solutions to (SL) and (SR), the constraints in those programs are binding, as required. In step 2, we solve the programs for those values of $(h^*, \bar{y})$.

Step 1 is conceptually simple. Fix a type h^*. First we solve the program:

$$\max_{X \in T^*} \int_0^{h^*} \log X(h) d\mathbf{F}(h); \tag{SL1}$$

call its solution X^L , and define $y^L(h^*) = \log X^L(h^*)$. Next we solve the program:

$$\max_{X \in T^*} \log X(h^*), \tag{SL2}$$

call its solution X^* and its value $y^*(h^*) = \log X^*(h^*)$. Then it follows that the values of $\bar{y}$ for which (SL) possesses a solution, and the constraint binds, are exactly

$$y^L(h^*) \le \bar{y} \le y^*(h^*).$$

In like manner, there is an interval $[y^R(h^*), y^*(h^*)]$ in which $\bar{y}$ must lie for (SR) to possess a solution in which its constraint binds. We find $y^R(h^*)$ by solving the program:

$$\max_{X \in T} \int_{h^*}^{\infty} \log X(h) d\mathbf{F}(h); \tag{SR1}$$

then set $y^R(h^*) = \log X^R(h^*)$, where X^R is the solution.

We have:

Proposition 4. *Both* (SL) *and* (SR) *possess solutions in which both constraints bind if and only if:*

$$\max[y^L(h^*), y^R(h^*)] \le \bar{y} \le y^*(h^*). \tag{18.8}$$

When we solve (SL1) , (SL2), and (SR1), we observe that the interval defined in (18.8) is non-empty for all $h^* > 0$. However, there will be an open interval C in the positive reals of values of h^* for which the probability of victory for both parties at these quasi-PUNEs is positive. If the random variable Z which defines electoral uncertainty is uniformly distributed on the interval $[-\beta, \beta]$, then C is the interval $\{h^* | -\beta < \frac{1}{2} - F(h^*) < \beta\}$. We are, in any case, not interested in quasi-PUNEs where one party has a zero probability of winning: we can assume that Opportunists have sufficient power to veto these.

In sum, there are quasi-PUNEs for every possible 'pivotal' type $h^* \in C$. Every point in the set

$$\Gamma = \{(h^*, \bar{y}) | h^* \in C \text{ and } \max[y^L(h^*), y^R(h^*)] \le \bar{y} \le y^*(h^*)\} \tag{18.9}$$

characterizes a quasi-PUNE with both parties' winning with positive probability.

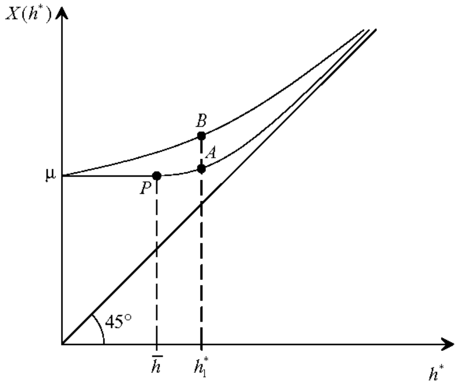

Fig. 18.1.

Hence Γ is the 2-manifold of quasi-PUNEs that interests us. Γ is graphed in Figure 18.1. The points on the lower envelope of Γ are the points where

$$\bar{y} = \max[y^L(h^*), y^R(h^*)];$$

these correspond to equilibria where the Militants are as strong as they can possibly be in their respective parties, subject to the existence of a quasi-PUNE. At all points on this envelope, at least one Militant faction is proposing its ideal policy, the policy in T^* that maximizes the average welfare of its members. There is a point on the lower envelope of Γ, labeled P in Figure 18.1, at which both parties play the ideal policy of their constituents, in the sense that the equilibrium policy X^L maximizes V^L and X^R maximizes V^R. We denote the pivot at his equilibrium by $\bar{h}$. At points on the upper envelope of Γ we have $\bar{y} = y^*(h^*)$. At such points, it immediately follows that *both parties propose the same policy,* the ideal policy of the *pivotal* voter, h^*. These are equilibria where the Opportunists in both parties are as powerful as they can possibly be. Note the apparent similarity to the Hotelling-Downs model, where with opportunist politics, both parties play the *median* ideal policy.

The lower and upper envelopes of Γ then, correspond to equilibria with *ideological* and *opportunist* politics, respectively. These two boundaries play an important role in the analysis to follow.

The second step of our program involves solving programs (SL) and (SR) for points $(h^*, \bar{y}) \in \Gamma$. The solutions are given in the next proposition.

Proposition 5. *Let* $(h^*, \bar{y}) \in \Gamma$. *Then:*

a) *The solution to* (SL) *is defined by:*

$$X^L(h) = \begin{cases} \hat{X}_0^L, & 0 \le h \le h_L \\ \hat{X}_0^L + (h - h_L), & h_L \le h \le h^* \\ e^{\bar{y}}, & h > h^* \end{cases}$$

where $(\hat{X}_0^L, h_L)$ *is the simultaneous solution of the two equations:*

$$\log(\hat{X}_0^L + (h^* - h_L)) = \bar{y}, \tag{18.10a}$$

$$\hat{X}_0^L + \int_{h_L}^{h^*} (h - h_L) d\mathbf{F}(h) + (1 - F(h^*))(h^* - h_L) = \mu. \tag{18.10b}$$

We have $\hat{X}_0^L > 0$.

b) *The solution of* (SR) *is defined by:*

$$X^R(h) = \begin{cases} \hat{X}_0^R + h, & 0 \leq h \leq h_R \\ \hat{X}_0^R + h_R, & h > h_R \end{cases}$$

where $(\hat{X}_0^R, h_R)$ *is the simultaneous solution of*:

$$\log(\hat{X}_0^R + h^*) = \bar{y}, \tag{18.10c}$$

$$\hat{X}_0^R + \int_0^{h_R} h d\mathbf{F}(h) + (1 - F(h_R))h_R = \mu. \tag{18.10d}$$

We have $\hat{X}_0^R > 0$.

Proposition 5 is more easily comprehended by studying Figure 18.2, where I graph the solutions X^L and X^R at a point $(h^*, \bar{y}) \in \Gamma$. We indeed see that the constraints (L1) and (R1) in (Q3a) and (Q3b) hold, as promised, and so indeed the solutions to programs (SL) and (SR) are quasi-PUNEs.

Note that both policies are fairly simple piece-wise linear policies. The Left proposes to give more to the poor and less to the rich than does the Right, but Left and Right coincide in what they propose for a 'middle class' of voters. The Left proposes a 100% marginal tax rate for all types less than h_L and all types greater than h^*, and a zero marginal tax rate for voters in the middle class. The Right proposes a zero marginal tax rate for all voters less than h_R, and a 100% marginal tax rate after that. These extreme marginal tax rates are a consequence of the assumption that labor supply is inelastic.

There is one important fact about these two policies : the graphs of both policies cut the vertical axis *above the origin*, that is, both Left and Right give a positive resource transfer to the type with zero wage capacity.

18.6 Equilibrium dynamics

I take the case of laissez-faire as the benchmark. Under laissez-faire, $X(h) = h$ for all h, and each parent allocates her income between consumption and investment in her child's education according to the partition of Proposition 1. It is not hard to see that if $b + c = 1$, then , under laissez-faire, the coefficient of variation of human capital (CVH) remains constant forever. (Simply observe that the ratio of human capitals in any two dynasties remains forever constant.) However, if $b + c < 1$, then

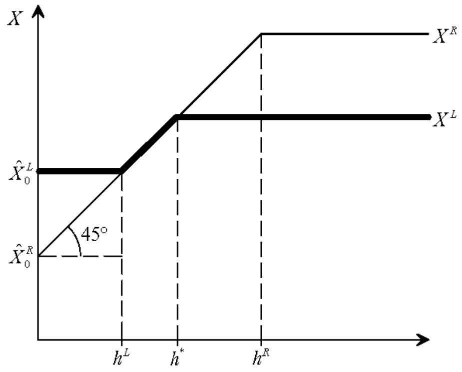

Fig. 18.2. Left (bold) and right policies in a quasi-PUNE

the coefficient of variation tends to zero – in this case we say 'human capital becomes equally distributed, asymptotically.' Our strategy will therefore be to study the CVH, under democracy, when $b + c = 1$, for the benchmark of laissez-faire provides a very clear statement. Were we to assume $b + c < 1$, then to compare laissez-faire to democracy, we would have to study relative speeds of convergence to equality of human capital, a more sensitive undertaking.

We begin by observing:

Proposition 6. *Let* (X^L, X^R) *be any quasi-PUNE. Let* $(h^1, X(h^1))$ *and* $(h^2, X(h^2))$ *be two points on the graph of either policy* (i.e., X can be either X^L or X^R). *Then the line containing these two points intersects the ordinate axis above the origin.* (*Formally,* $\frac{X(h_1)}{h_1} > \frac{X(h_1)-X(h_2)}{h_1-h_2}$.)

This proposition is clear from examining Figure 18.2. It follows from observing that both policies themselves cut the vertical axis above the origin,[11] and that their slopes are never greater than one.

Let X be an equilibrium policy in a quasi-PUNE, and let $h_2 > h_1$. Then the educational investment is $r(h) = \frac{\gamma c X(h)}{1+\gamma c}$. Let the chord through the points $(h_1, X(h_1))$ and $(h_2, X(h_2))$ have the equation *x* = *mh*+*d;* we know that m and d are both positive, by Proposition 5. Then we can write the wage of the son of h_i as:

$$h_i' = \alpha h_i^b \left(\frac{\gamma c}{1 + \gamma c} (m h_i + d) \right)^c$$

and so the ratio of wages of the two dynasties at the first generation is:

$$\frac{h_2'}{h_1'} = \left(\frac{h_2}{h_1} \right)^b \left(\frac{m h_2 + d}{m h_1 + d} \right)^c < \left(\frac{h_2}{h_1} \right)^{b+c},$$

where the strict inequality follows from the fact that $d > 0$. Thus, with decreasing returns to scale, the ratio of the human capitals in any two dynasties approaches unity,

[11] See the last paragraph of Section 4.

and so the coefficient of variation of the distribution of human capital approaches zero, regardless which party wins which election at each date. And even with constant returns to scale, the wage ratios of any two dynasties decrease monotonically over time. *Whether they decrease to unity or not* will determine whether or not wages tend to equality.

There is a two dimensional manifold of quasi-PUNES at each date; we have provided no theory of which quasi-PUNE will be realized. We must therefore arbitrarily specify intertemporal sequences of quasi-PUNEs to study. Our dynamic analysis will investigate two such sequences. Fix a value of h^*; we shall study two sequences of quasi-PUNEs, each of which 'descends' from h^*, in the sense that, at every date, the pivot (who defines the partition of the polity into the two parties) is always the descendent of h^*. One might refer to the parent possessing human capital h^* at the initial date as 'Eve.' We denote the human capital of Eve's descendent at date t as is $S_t(h^*)$.

Our first intertemporal sequence of quasi-PUNEs is denoted $\{A^t(h^*)\}$; here, the pivot is $S_t(h^*)$ at date t and the quasi-PUNE lies on the lower boundary of the manifold illustrated in Figure 18.1. The other sequence, denoted $\{B^t(h^*)\}$, again has the pivot as the tth descendent of Eve, but it lies on the upper boundary of the manifold. Recall that the A sequence is one where politics are *ideological,* in the sense that the Militants are as powerful as they can be in the intra-party bargaining game, and the B sequence is one in which politics are Opportunist.

We have:

Theorem. *Let* $b + c = 1$, $\mathbf{F}$ *be the original distribution of human capital, and let* $h^* \in C$.

A. *The limit* CVH *induced by the sequence of* PUNEs $\{B^t(h^*)\}$ *is positive.*
B. *If* $\int \log h \, d\mathbf{F}(h) < \log h^*$ *then the limit* CVH *induced by the sequence of* PUNEs $\{A^t(h^*)\}$ *is positive.*
C. *[conjecture] If* $\int \log h \, d\mathbf{F}(h) > \log h^*$ *then with positive probability the limit* CVH *induced by the sequence of* PUNEs $\{A^t(h^*)\}$ *is zero.*

Part C is a conjecture, based upon simulations. If it is true, then we only get convergence to equality of the distribution of human capital if politics are sufficiently partisan, and the original distribution of human capital is sufficiently skewed. The intuition is that, even with ideological politics, if the Left party is too large , then its policies are so moderate that they do not induce convergence to equality.

An instructive case is the one where h^* is the median of the distribution **F.** This implies that both parties of are equal size, and each has a 50% chance of winning at every election. By Jensen's inequality, the condition $\int \log h \, d\mathbf{F}(h) > \log h^*$ implies that $h^* < \int h d\mathbf{F}(h) = \mu$ – that is, that the median is less than the mean, a familiar condition of skewness. So, fixing the pivot at the median implies, from statement C, that convergence to equality, even with highly ideological politics, occurs only if the original distribution is *strongly skewed* – because the condition $\int \log h \, d\mathbf{F}(h) > \log h^*$ is stronger than the condition that the median be less than the mean.

18.7 Conclusion

Our summary is that, if and only if democratic politics are sufficiently ideological, as opposed to opportunist, and the Left party is not too large, then the coefficient of variation of human capital will tend to zero over time if $b+c=1$; and if $b+c<1$, it will do so more rapidly under democracy than under laissez-faire. This statement has many qualifications, by virtue of the deviations of reality from the model. The first is that, in reality, children are differentially talented, and so the production function we have used is inaccurate. If we modify it by inserting a term allowing for stochastic variation of talent, then our dynamic theorem becomes transformed as follows: If the condition of statement C of Theorem 1 holds, and if politics are ideological, then the human capital of the distant descendents of any Eve is with positive probability independent of Eve's human capital, whereas if politics are opportunist, then there is permanent *persistence* of the effect of Eve's human capital on her descendents'.

There are many other ways in which the present model is too simple. The price of modeling 'ruthless political competition' – that is, constructing a model of political equilibrium on a large policy space – has been to simplify drastically the economic side of the model.

At the very least, we have modeled the complexity of democracy, which here takes the form of a continuum of possible equilibria; members of that continuum are characterized by specifying the relative bargaining powers of the factions within the two parties. A PUNE or quasi-PUNE can be viewed as a Nash equilibrium between parties, where each party's strategy is the outcome of a bargaining process between its internal factions. The multiplicity of equilibria is inherited from the variations in bargaining powers of the factions within the parties.[12] We have not attempted to derive endogenously what the bargaining powers of those factions are. We have, however, shown that these parameters are of key importance in predicting the dynamics of human capital in democracies. Only when Opportunists are weak within parties, and the partition of citizens into parties is such that the Left party is not too moderate will democracy engender equality of human capitals in the long run, and then only with positive probability.

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[12] For discussion of this point, see Roemer (2001, chapter 8).

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19

Consumption Externalities, Rental Markets and Purchase Clubs*

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Summary. A premise of general equilibrium theory is that private goods are rival. Nevertheless, many private goods are shared, e.g., through borrowing, through co-ownership, or simply because one person's consumption affects another person's wellbeing. I analyze consumption externalities from the perspective of club theory, and argue that, provided consumption externalities are limited in scope, they can be internalized through membership fees to groups. Two important applications are to rental markets and "purchase clubs," in which members share the goods that they have individually purchased.

Key words: Consumption externalities, Clubs, Purchase clubs, Rental markets.

JEL Classification Numbers: D11, D62.

19.1 Introduction

Microeconomic theory generally treats private goods as rival in the sense that preferences depend only on an agent's own consumption. Nevertheless, consumption externalities are pervasive. If someone plants a tree to shade his garden, he may block his neighbor's view. If the neighbor buys a dog that barks, the whole neighborhood may suffer.

The term "externality" generally involves an assumption about market institutions. If the commodity space is defined so that the "externalities" are chosen voluntarily, they are no longer externalities. A better term might be "shared consumption." There are many formal and informal institutions that facilitate such sharing, or allow

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the coordination of consumption decisions. On the informal side, agents often benefit from their friends' purchases of books, CD's, or magazine subscriptions. An implicit condition of friendship may be reciprocity in such purchases. On the more formal side, shared living arrangements often impose restrictions that prohibit or require certain consumption or activities ("no smoking"), and those restrictions may be reflected in the price. Higher fees may entitle the member to more privileges or relieve him of obligations. Other formal institutions for sharing the consumption of private goods include rental markets and lending libraries. Rental prices may be higher at establishments where the rental good is less often in use, so that scheduling is easier. Rental fees may be seasonal.

Shared consumption can be handled in various ways in general equilibrium theory. Some sharing arrangements, such as rental markets, can be handled by reinterpreting traded goods in the theory of Arrow and Debreu. The treatment of Grodal and Vind (2001) and Vind and Grodal (2005) (see also Vind, 1983) extends that theory to allow each agent's preferences to depend on all agents' consumptions. The description of the economy includes a "coordination function" whereby agents are allowed to make joint decisions. Whether an equilibrium is efficient depends on the extent to which preferences are interdependent, and on the coordination function. Gersbach and Haller (2001) study economies where preferences are independent within households, and allow households to make joint consumption decisions using a combined budget.

This paper shows that shared private goods can also be handled through club theory. The essential idea in club theory is that, by forming a group called a "club," members share the services that the club provides, and also share the "externalities" conferred by the attributes or activities of the club's members. An important difference between the club approach and the approaches mentioned above is in the commodity space. In the club model, each agent purchases a vector of private goods, and also purchases memberships in clubs. When purchasing memberships in clubs, members anticipate the full suite of externalities, which are therefore internalized, since each member has the option not to join. The club model has wide-ranging applicability, comprising educational opportunities, firms, schools, social activities, academic departments, and many other human activities that take place in groups. See Ellickson, Grodal, Scotchmer and Zame (EGSZ 1999, 2001, 2003), Cole and Prescott (1999) and Prescott and Townsend (forthcoming). For how these ideas relate to local public goods, see Scotchmer (2002).

The question for club theory, just as for more traditional general equilibrium theory, is how to model the sharing of private goods, and how to provide for pricing. Different goods involve different protocols for sharing. Goods like power tools, ski equipment and sometimes cars are used only occasionally by each user. As long as the transactions costs are not exorbitant, it is more efficient to keep the good in use than to let it sit idle. Nevertheless, sharing may be inconvenient. If the good cannot be used simultaneously, then there must be a protocol for resolving conflicts or scheduling use. We would expect prices to reflect the priority that a member gets, or the overall inconvenience of the use, as measured, for example, by the ratio of total use to total goods.

For some goods there is rather little inconvenience due to sharing. Computer software and entertainment products (music, movies) can be used simultaneously when installed simultaneously on different users' computers. The only inconvenience is in keeping the sharing group small enough to avoid detection, since simultaneous use will typically violate the seller's intellectual property rights.

For shared consumption that generates pleasure for one person and discomfort for another, such as playing Beatles tunes at midnight or smoking cigarettes, the protocol of sharing might be to prohibit use at certain hours or in certain places.

In Section 2, I reprise the clubs model of EGSZ (1999, 2003), and show how it can accommodate consumption externalities. In Section 3, I extend that model to accommodate proprietary goods. This extension allows us to address policy concerns raised by the sharing of proprietary computer software and entertainment products. Section 4 shows that club theory leads to a useful model of rental markets, with peak and off-peak pricing, and prices that reflect the inconvenience of competing with other users.

19.2 Consumption externalities in groups

The most convenient clubs model for this purpose is that of EGSZ (1999, as extended in 2003). The commodity space includes private goods and memberships in clubs. In order to define the memberships, we must first define the types of groups that can form as clubs. A primitive of the economy is an exogenous set of *group types*. To define memberships in the group types, we need a set of *membership characteristics,* designated by elements of Ω, which is an abstract, finite set. In addition, the groups can have *activities,* designated by elements of Γ, which is also an abstract, finite set.

A *group type* is a triple (π, γ, y) consisting of a *profile* $\pi : \Omega \to \mathbf{Z}_+ = \{0, 1, \ldots\}$, an activity $\gamma \in \Gamma$, and a vector of private goods $y \in \mathbb{R}^N$. The negative elements of y represent net inputs, and the positive elements represent net outputs. For $\omega \in \Omega$, $\pi(\omega)$ is the number of members having the membership characteristic ω. A membership characteristic specifies the role in the group type that the membership entails (such as teacher or student), as well as the personal qualities required for the membership such as intelligence and personal habits, or (as discussed here) contributions of goods. In Sections 3 and 4, the membership characteristics are respectively contributions of a proprietary good that will be shared by members or the usage of a rental good. In those cases, the membership characteristics are described by real numbers, but in general no such structure is imposed on Ω.

We take as given a finite set of possible group types $\mathcal{G} = \{(\pi, \gamma, y)\}$.

A *membership* is an opening in a particular group type for an agent assuming a particular membership characteristic; i.e., $(\omega, (\pi, \gamma, y))$ such that $(\pi, \gamma, y) \in \mathcal{G}$ and $\pi(\omega) \geq 1$. We write $\mathcal{M}$ for the (finite) set of memberships. Each agent may choose many memberships in groups or none. A membership *list* is a function $\ell : \mathcal{M} \to \{0, 1, \ldots\}$, where $\ell((\omega, (\pi, \gamma, y)))$ specifies the number of memberships of type $(\omega, (\pi, \gamma, y))$. The list may include memberships in firms ("jobs"), in schools (as "student" or "teacher"), in living groups, and (as below) in groups that share private goods. It is due to membership in groups that agents may be subject to externalities or confer externalities.

The set of agents is a nonatomic measure space $(A, \mathcal{F}, \lambda)$. That is, A is a set, $\mathcal{F}$ is a σ-algebra of subsets of A and λ is a non-atomic measure on $\mathcal{F}$ with $\lambda(A) < \infty$. A complete description of an agent $a \in A$ consists of a consumption set, an endowment of private goods and a utility function.

The agents' *consumption sets* X_a, $a \in A$, take on a different role in this model than in, for example, the Arrow-Debreu model of general equilibrium. An agent's consumption set constrains the memberships available to him, *e.g.,* according to his innate abilities (not everyone can have a job as a professional basketball player), or by specifying collateral consumption of private goods or other memberships that must also be consumed in order to "learn" a membership characteristic. (A computer programmer must learn his skill either by owning a computer and programming books or by choosing a student membership in a programming school.) The consumption set would also prevent the agent from choosing memberships that are inconsistent, such as being simultaneously a sumo wrestler and a member of a ballet club. The consumption of private goods is always restricted to be nonnegative, but may additionally be restricted by the list of memberships. For technical convenience, there is a bound M on how many memberships an agent can consume.

Agent a's *endowment* is $(e_a, 0) \in X_a \subset \mathbb{R}^N_+ \times \mathbb{R}^{\mathcal{M}}$. Agents are endowed with private goods but not with group memberships. Agent a's *utility function* $u_a : X_a \to \mathbb{R}$ is defined over private goods consumptions and lists of group memberships. The utility function is continuous and strictly increasing in the private goods. A *state* of an economy is a measurable mapping

$$(x, \mu) : A \to \mathbb{R}^N \times \mathbb{R}^{\mathcal{M}}$$

A state specifies choices of private goods and a list of group memberships for each agent.

Feasibility of a state of the economy entails consistent matching of agents, that every agent's consumption is in his consumption set, and that the aggregate resource constraint is satisfied. Consistent matching is a type of coordination of the agents' membership choices. It means that there are no partially filled groups. If an agent chooses a membership in a group of a certain type, then other agents must choose the complementary memberships to fill the group.

Consistent matching is expressed in terms of an *aggregate membership vector* $\bar{\mu} \in \mathbb{R}^{\mathcal{M}}$, representing the total number of memberships of each type chosen by the agents collectively. We say that an aggregate membership vector $\bar{\mu} \in \mathbb{R}^{\mathcal{M}}$ is *consistent* if for every qrouptype $(\pi, \gamma, y) \in \mathcal{G}$, there is a real number $\alpha(\pi, \gamma, y)$, representing the "number" (measure) of groups of type (π, γ, y), such that

$$\bar{\mu}(\omega, (\pi, \gamma, y)) = \alpha(\pi, \gamma, y)\pi(\omega)$$

for each $\omega \in \Omega$. Thus, associated with a feasible state of the economy is a collection $\{\alpha(\pi, \gamma, y) | (\pi, \gamma, y) \in \mathcal{G}\}$ which describes the measures of the groups of various types. For each $(\pi, \gamma, y) \in \mathcal{G}$, either $\alpha(\pi, \gamma, y) = 0$ (no such groups) or $\alpha(\pi, \gamma, y) > 0$ (a positive measure of such groups).

The state (x, μ) is *feasible* if it satisfies the following requirements:

(i) **Individual feasibility** $(x_a, \mu_a) \in X_a$ for each $a \in A$
(ii) **Material balance**

$$\int_A x_a \, d\lambda(a) \leq \int_A e_a \, d\lambda(a) + \int_A \sum_{(\omega,(\pi,\gamma,y))\in\mathcal{M}} \mu_a(\omega,(\pi,\gamma,y)) \frac{y}{|\pi|} \, d\lambda(a)$$

(iii) **Consistency** The aggregate vector of memberships $\int_A \mu_a \, d\lambda(a)$ is consistent.

Both private goods and group memberships are priced, so prices (p, q) lie in $\mathbb{R}^N_+ \times \mathbb{R}^{\mathcal{M}}$. The vector of prices for private goods is p, and the vector of prices for group memberships is q. Prices of group memberships may be positive, negative or zero. Membership prices have different interpretations in different examples. They may be required to pay for the infrastructure of the group or its activities, to remunerate a member for his opportunity cost of membership, in particular, wages, or may, when negative, compensate him for a membership that other members value, but he himself dislikes. In Section 3 below, a negative price might mean that the member of a purchase club is partially reimbursed by other members for the purchases he contributes.

At prices (p, q), we say that a list $(x, \ell) \in X_a$ is *budget feasible* for $a \in A$ if

$$(p, q) \cdot (x, \ell) \leq p \cdot e_a$$

A *group equilibrium* consists of a feasible state (x, μ) and prices $(p, q) \in \mathbb{R}^N_+ \times \mathbb{R}^{\mathcal{M}}, p \neq 0$ such that

(a) **Budget feasibility:** For almost all $a \in A$, (x_a, μ_a) is budget feasible.
(b) **Optimization:** For almost all $a \in A$, if $(x, \ell) \in X_a$ and $u_a(x, \ell) > u_a(x_a, \mu_a)$, then (x, ℓ) is not budget feasible.
(c) **Budget balance for group types:** For each $(\pi, \gamma, y) \in \mathcal{G}$:

$$\sum_{\omega\in\Omega} \pi(\omega) \, q(\omega, (\pi, \gamma, y)) + p \cdot y = 0$$

Thus, at an equilibrium individuals optimize subject to their budget constraints and the sum of membership prices in a given group type is exactly equal to the net cost or surplus generated by the use or production of private goods, $p \cdot y$.

By a simple extension of their (1999) arguments, EGSZ (2003) assert the first welfare theorem for this model as well as existence and core/competitive equivalence.[2] The (2003) paper differs from the (1999) paper in that private goods may be produced and membership characteristics can be acquired. There is a concept

[2] In club economies there are considerations beyond those known for exchange economies, in going from quasi-equilibrium to equilibrium; see example 3 of Gilles and Scotchmer (1997) and example 3.2 of EGSZ (1999). These difficulties are not emphasized here because this paper focuses on characterizations of equilibrium, and not existence.

of learning, which may be facilitated by memberships in schools or consumption of certain private goods.

The internalization of consumption externalities can be described by this model using the following adaptions, illustrated by the examples that follow.

- Using consumption sets, consumption of private goods can be restricted in a way that depends on group memberships.
- The activity vector γ can specify how private goods, modeled in y as inputs, are shared.
- The membership characteristic can obligate the member for certain purchases that must be shared with other members, with the terms of sharing specified by γ.
- The membership characteristic can entitle the member to certain specified usage of the shared good.

Example 1. Suppose that a group of friends share a house. The household is a type of club, (π, γ, y), and the house is an input in y. If the house seeks to limit the negative externalities that arise from late-night parties or playing Beatles tunes, the activity γ can consist of a commitment not to do those things. If it is a no-smoking house, the consumption set may prohibit agents from both belonging to such a house and consuming cigarettes.

Example 2. Suppose that members of the household share tasks. Someone must be the cook, someone must bring sports equipment, and someone must do the members' collective homework. These commitments could be built into the membership characteristics ω. Some characteristics, like being the cook, could be acquired skills. The cookbooks can be an input in y, but alternatively, the cook can be required to bring them. The prices will be different in these two arrangements. The homework membership may require innate abilities and also learned skills. Whether the homework membership is feasible for a certain student is reflected in his consumption set. Similarly, the person who contributes the sports equipment must presumably invest in it, and his membership price should reflect this investment. If different members bring different sports equipment, their personal characteristics will reflect their contributions. The activity γ must specify the organizational arrangements under which they decide how to ration the sports equipment.

Example 3. The friends may band together for the dedicated purpose of sharing music or software CD's, in order to avoid purchasing duplicate copies. This is a purchase club, described in Section 3.

Example 4. Suppose that the shared good is partially rival, in the sense that intense use creates inconvenience. This is described in Section 4. A membership price will then reflect the overall usage, modeled in γ, and will also reflect the member's usage, modeled in ω. The membership price may also reflect time. Other things equal, the rental price of a sailboat on a balmy Saturday in August might be higher than on a random Tuesday in January, when the price must be low to keep the sailboat in use. The vector y represents the input vector of shared goods themselves.

19.3 Proprietary pricing and purchase clubs

Copyright owners have argued for many years that their profits are undermined when users share. Their calculation of the loss usually involves the assumption that every unauthorized user would otherwise purchase a legitimate copy at the prevailing price. Both common sense and the economics literature challenge this view. What is argued in the literature (Besen and Kirby, 1989; Varian, 2000; Bakos et al., 1999) is that proprietors will anticipate the sharing behavior, and set different prices if the good is sold to individual users than if sold to users who are expected to share it. These papers argue, somewhat provocatively, that sharing may actually *increase* the proprietor's profit. We revisit this question, using a variant of the club model that allows for proprietary pricing.

We begin with an example to show what the club model adds to previous discussions of purchase clubs. Whether sharing enhances profit depends on the groups that form. However the whole point of club theory is that group formation is an equilibrium phenomenon. Instead of taking group formation as exogenous, the club perspective recognizes that groups will form in a way that is collectively efficient – efficient for the buyers, that is. Group formation that is efficient for the buyers is probably not efficient for the sellers. Indeed, this is more or less what the theorem in Section 3.2 shows. In the example of Section 3.1, the sellers' profits may be enhanced if group formation is, for example, random, but profit will not be enhanced if group formation is systematic in some way that serves the interests of the buyers. The theorem shows that the profit available to the sellers is exactly the same with sharing of purchases as without, provided the purchase groups form efficiently in equilibrium, and the selling price can depend on the size of the group.

This result would not survive in the form given if the shared goods involved marginal costs of supply, as sharing would then reduce industry costs, and the proprietor would presumably share in the benefits. This is the focus of the related work by Besen and Kirby (1989) and Varian (2000).

19.3.1 Purchase clubs: an example

We use this example to show two things: that if the groups form exogenously prior to the setting of prices, then group formation may indeed increase the proprietors' profit, but if the groups can re-form conditional on the prices, that result is nullified. The example suggests the theorem of the next section, which is that the maximum profit, anticipating group formation, is the same as if consumers do not share the proprietary goods.

We will consider purchase clubs that share CD's of two kinds, classical and jazz. Assume that for each CD, half the population has willingness to pay (WTP) equal to a and the other half has WTP x, $x < a/2$. Our benchmark will be the profitability of selling to single buyers. The most profitable price is $p = a$, so half the agents buy. We compare this benchmark with a situation where groups of size 2 can share CD's.

Suppose that there are four types of consumers, with different willingnesses to pay for the two CDs: $\{v_1, v_2, v_3, v_4\} = \{(a, a), (a, x), (x, a), (x, x)\}$. (No such structure is used in Section 3.2.)

Suppose first that each taste vector occurs in 1/4 of the population, and that the agents are randomly and exogenously matched into groups. The groups occur with the following frequencies:

Group	Frequency	WTP classical WTP jazz	Group	Frequency	WTP classical WTP jazz
(v_1, v_1)	$1/16$	$a + a$ $a + a$	(v_1, v_3)	1/8	$a + x$ $a + a$
(v_2, v_2)	$1/16$	$a + a$ $x + x$	(v_1, v_4)	1/8	$a + x$ $a + x$
(v_3, v_3)	1/16	$x + x$ $a + a$	(v_2, v_3)	1/8	$a + x$ $a + x$
(v_4, v_4)	1/16	$x + x$ $x + x$	(v_2, v_4)	1/8	$a + x$ $x + x$
(v_1, v_2)	1/8	$a + a$ $a + x$	(v_3, v_4)	1/8	$x + x$ $a + x$

The most profitable price is $p_c = p_j = (a + x)$, which entails selling to 3/4 of the groups. It is not optimal to charge price $2a$ for either CD, because only 1/4 of the groups would purchase it.

But at prices $p_c = p_j = (a + x)$, groups will want to re-form. To see that the randomly matched groups cannot be optimal for all agents, notice that, if they re-form into homogeneous groups ($(v_1, v_1), (v_2, v_2)$, etc.) there is more total consumers' surplus to divide. With random matching, only 1/4 of the groups receive positive consumers' surplus for a given CD, those with 2 agents who have WTP equal to a. If agents reorganize into homogeneous groups, then for each CD, half the groups have two agents with WTP equal to a for a given CD. But while total consumers' surplus goes up, profit goes down, since only half the groups buy each CD rather than 3/4 of them.

But once the groups are reorganized into homogeneous groups, the proprietors can do better by charging $p_c = p_j = 2a$. With these prices, proprietors earn the same profit as selling to individual agents. Although this is not obvious, they cannot do better. There are no prices they could choose which would generate more profit once the consumers reorganize into the best groups conditional on those prices. That is the theorem in the next section.

The optimum in this example entails groups with the same tastes. But in the model below there is no reason that tastes should be duplicated in the population. It is not the homogeneity of tastes that erases any profit advantage to selling to groups, but rather the fact that groups form endogenously in a way that is collectively efficient for the members, conditional on the proprietary prices.

19.3.2 Purchase clubs: a Theorem

In developing the example, it was convenient to describe the members of groups by their tastes. However membership prices cannot depend on tastes, as tastes are unobservable. The membership characteristics will be the contributions of proprietary goods.

We will make the notation easier by supposing that the only group types are purchase clubs. (Otherwise we must distinguish memberships in purchase clubs from other types of memberships.) Suppose there are C goods that can be purchased and shared. Proprietors market these goods at prices $r = (r_1, ..., r_C) > 0$, anticipating the groups that will form. In the example, $C = 2$, jazz and classical.

The set Ω will serve various purposes in the model that follows. Most importantly, the elements $\omega \in \Omega$ will represent the contributions that a member might make to a group. For convenience, let

$$\Omega = \left\{ z \in Z_+^C \mid z \leq (Mk, Mk....Mk) \right\}$$

for a given $k > 1$ where M is the maximum number of memberships in an agent's consumption set.

A *purchase club type* (π, γ, y) is a club type such that the membership characteristics $\omega \in \Omega$ are interpreted as contributions, and $\sum_{\omega \in \Omega} \pi(\omega)\omega$ is the vector of proprietary goods shared by members of the group. (Recall that, in general, the expression $\sum_{\omega \in \Omega} \pi(\omega)\omega$ has no meaning, as the characteristic ω need not be a number.) If a member chooses a membership for which $\omega > 0$, then he contributes at least one shared good, and may be paid in equilibrium by members who choose $\omega = 0$.

To isolate the points of interest, we make some special assumptions. We shall assume there is a single private good, the numeraire, and shall refer to equilibrium as $(x, \mu), q$. We assume that the set Γ is a singleton which specifies that members will share their purchases. We thus suppress the activity $\gamma \in \Gamma$ in the description of the group type. We also suppress y in the description of a group type, since the shared goods are described in the membership characteristics. Thus, a group type is only described by the profile π indicating how many members contribute each vector ω of shared goods. We will also assume that there is an exogenous bound k on the size of sharing groups. This is the k in the definition of Ω. (In the example, $k = 2$). Since γ is a singleton and $y = 0$, the sets of possible group types and memberships are

$$\mathcal{G} = \{\pi : \Omega \to Z_+ \mid |\pi| \leq k\}$$
$$\mathcal{M} = \{(\omega, \pi) \mid \omega \in \Omega, \pi \in \mathcal{G}\}$$

The *contributions* of an agent who consumes a list ℓ of memberships are

$$\sum_{(\omega,\pi) \in \mathcal{M}} \ell(\omega, \pi)\omega.$$

We will use the notation ω^ℓ to refer to the *consumption* of an agent (distinct from the *contributions* of the agent) if he consumes a list ℓ:

$$\omega^{\ell} = \sum_{(\omega,\pi)\in\mathcal{M}} \ell(\omega,\pi) \left(\sum_{\omega\in\Omega} \pi(\omega)\omega\right) \tag{19.1}$$

The group equilibrium defined in Section 2 must be altered to account for the cost of contributing proprietary goods, which is the second term in (19.2). At prices (p, q) and r, we say that a list $(x, \ell) \in X_a$ is *budget feasible* for $a \in A$ if

$$(p,q)\cdot(x,\ell) + \sum_{(\omega,\pi)\in\mathcal{M}} \ell(\omega,\pi)\, r\cdot\omega \quad \le p\cdot e_a \tag{19.2}$$

A *purchase-club equilibrium* at prices r consists of a feasible state (x, μ) and prices $(p, q) \in \mathbb{R}^N_+ \times \mathbb{R}^{\mathcal{M}}, p \neq 0$, such that (a), (b) and (c) of group equilibrium hold, with budget feasibility defined as (19.2). This equilibrium can be understood as involving absentee proprietors who collect the profit on the proprietary goods.

Utility functions $u_a : X_a \to \mathbb{R}$, are defined by

$$u_a(x,\ell) = U_a(x,\omega^{\ell}) \tag{19.3}$$

where $U_a : \tilde{X}_a \to \mathbb{R}$ represents utility as a function of the goods themselves, and

$$X_a = \mathbb{R}_+ \times \left\{\ell \in Z^{\mathcal{M}}_+ : |\ell| \le M\right\}$$
$$\tilde{X}_a = \left\{(x,\omega^{\ell}) | (x,\ell) \in X_a\right\}$$

A1: Preferences can be defined as in (19.3), where for all $a \in A$, (i) U_a is strictly increasing in its first argument, and (ii) if $z \notin \Omega$, then there exists $z' \in \Omega$, $z' \le z$, such that $U_a(x,\ z') \ge U_a(x,\ z)$ for all $x \ge 0$.

Part (ii) of this assumption is satisfied if consumers can reduce their consumption of a shared good from a number larger than Mk to a smaller number, without reducing utility. If members of each group can use the shared good simultaneously, as in the example where they were assumed to install digital music separately on all their computers, one unit of each shared good is sufficient.

The next three claims characterize a purchase-club equilibrium. Claim 1 describes prices such that, in equilibrium, agents are indifferent as to which membership they have in a group type that is used in equilibrium. Members who contribute shared goods pay low prices (perhaps negative prices), and members who contribute no shared goods pay high prices, to just an extent that they are indifferent.

Claim 1 *Suppose that* A1 *holds. Let* $(x, \mu), q$ *be a purchase-club equilibrium at prices* $r > 0$. *Then*

(i) For each group type π, q *satisfies* (19.4) *for at least one membership* (ω, π).

$$q(\omega,\pi) \le \frac{r}{|\pi|}\cdot\sum_{\omega\in\Omega} \pi(\omega)\omega - r\cdot\omega \tag{19.4}$$

(ii) If the group type π *is used in equilibrium* $(\alpha(\pi) > 0)$ *and* $(\pi(\omega) > 0)$, *then* q *satisfies (19.4) with equality.*

(iii) If the group type π is used in equilibrium and $\pi(\omega_1), \pi(\omega_2) > 0$, then for all $a \in A$

$$q(\omega_1, \pi) + r \cdot \omega_1 = q(\omega_2, \pi) + r \cdot \omega_2 = \frac{r}{|\pi|} \cdot \sum_{\omega \in \Omega} \pi(\omega)\omega$$

Proof.

(i) If (19.4) holds with equality, budget balance is satisfied. (Multiply both sides of (19.4) by $\pi(\omega)$ and sum on ω.) If (19.4) does not hold with equality, then by budget balance, (19.4) holds as an inequality for at least one membership in a given group type π.

(ii) Suppose to the contrary that for a given π such that $\alpha(\pi) > 0$,

$$q(\omega_1, \pi) > \frac{r}{|\pi|} \cdot \sum_{\omega \in \Omega} \pi(\omega)\omega - r \cdot \omega_1$$
$$q(\omega_2, \pi) < \frac{r}{|\pi|} \cdot \sum_{\omega \in \Omega} \pi(\omega)\omega - r \cdot \omega_2$$

where $\pi(\omega_1), \pi(\omega_2) > 0$. Using (19.2), an agent's total payments when he chooses a membership are the cost of the contributions plus the membership fee, $q(\omega, \pi) + r \cdot \omega$. Since all memberships in a given group type π give access to the same shared goods $\sum_{\omega \in \Omega} \pi(\omega)\omega$, and, by A1, agents care only about their consumption (x, ω^{ℓ}), every agent is better off with the cheaper membership (ω_2, π) than with the more expensive (ω_1, π). This is a contradiction, since $\pi(\omega_1) > 0$.

(iii) follows from (ii). □

Claim 2 *Suppose that* A1 *holds. Let* $(x, \mu), q$ *be a purchase-club equilibrium at prices* $r > 0$. *Let* $\{\omega^{\mu_a} | a \in A\}$ *be the consumptions of shared goods defined by* (19.1). *Then*

(i) $\omega^{\mu_a} \in \Omega$ for almost every $a \in A$

(ii) If the group type π is used in equilibrium ($\alpha(\pi) > 0$), then $|\pi| = k$.

(iii) For almost every $a \in A$, the consumption of private goods satisfies

$$x_a = e_a - \frac{r}{k} \cdot \omega^{\mu_a} \tag{19.5}$$

Proof. Using Claim 1(ii) and budget feasibility, equilibrium consumption (x_a, μ_a) satisfies the following for almost every $a \in A$:

$$\begin{aligned} x_a &= e_a - \sum_{(\omega,\pi) \in \mathcal{M}} \mu_a(\omega, \pi) \left[r \cdot \omega + q(\omega, \pi) \right] \\ &= e_a - \sum_{(\omega,\pi) \in \mathcal{M}} \mu_a(\omega, \pi) \frac{r}{|\pi|} \cdot \sum_{\omega \in \Omega} \pi(\omega)\omega \\ &\leq e_a - \frac{r}{k} \cdot \sum_{(\omega,\pi) \in \mathcal{M}} \mu_a(\omega, \pi) \sum_{\omega \in \Omega} \pi(\omega)\omega = e_a - \frac{r}{k} \cdot \omega^{\mu_a}. \end{aligned} \tag{19.6}$$

Before proving the claim, we observe that, given $\omega \in \Omega$, and provided x satisfies $e_a - (r/k) \cdot \omega \geq x$, we can find a list ℓ such that $(x, \ell) \in X_a$ is budget-feasible and $\omega^\ell = \omega$. Choose a group type with $|\pi| = k$ and $\omega = \sum_{\hat{\omega} \in \Omega} \pi(\hat{\omega})\hat{\omega}$, and choose the list ℓ with a single membership in that group type such that (19.4) holds. Then

$$
\begin{aligned}
0 \leq x \leq e_a - \frac{r}{k} \cdot \omega^\ell &= e_a - \sum_{(\omega,\pi) \in \mathcal{M}} \ell(\omega, \pi) \frac{r}{k} \cdot \sum_{\omega \in \Omega} \pi(\omega)\omega \\
&= e_a - \sum_{(\omega,\pi) \in \mathcal{M}} \ell(\omega, \pi) \frac{r}{|\pi|} \cdot \sum_{\omega \in \Omega} \pi(\omega)\omega \\
&\leq e_a - \sum_{(\omega,\pi) \in \mathcal{M}} \ell(\omega, \pi) \left[r \cdot \omega + q(\omega, \pi) \right]
\end{aligned}
$$

Thus, (x, ℓ) is budget feasible.

(i) Suppose $\omega^{\mu_a} \notin \Omega$ for a set of agents of positive measure, say $\bar{A} \subseteq A$. For each $a \in \bar{A}$, by A1 there exists $\omega' \in \Omega$, $\omega' \leq \omega^{\mu_a}$ such that $U_a(x_a, \omega^{\mu_a}) \leq U_a(x_a, \omega')$. Using (19.6), and since $\omega'_i < \omega_i^{\mu_a}$ for at least one proprietary good i, $0 \leq x_a \leq e_a - (r/k) \cdot \omega^{\mu_a} < e_a - (r/k) \cdot \omega'$. Then, as observed above, there is a list ℓ such that $\omega^\ell = \omega'$ and $(x_a, \ell) \in X_a$ is budget feasible. Further, there is a budget-feasible $(x, \ell) \in X_a$ such that $e_a - \frac{r}{k} \cdot \omega^\ell \geq x > x_a$. But then $U_a(x_a, \omega^{\mu_a}) \leq U_a(x_a, \omega^\ell) < U_a(x, \omega^\ell)$, which contradicts equilibrium.

(ii) Suppose that there is a group type π such that $|\pi| < k$ and $\alpha(\pi) > 0$. Let $\bar{A} \subseteq A$ be the set of agents with memberships in this group type π. Let $a \in \bar{A}$. Using (i), we can assume without loss of generality that $\omega^{\mu_a} \in \Omega$. Since (19.6) holds as a strict inequality, $e_a - \frac{r}{k} \cdot \omega^{\mu_a} > x_a \geq 0$. As observed above, there is a list ℓ with a single membership in a group type π, $|\pi| = k$, such that $\omega^\ell = \omega^{\mu_a}$ and (x_a, ℓ) is budget feasible for a. For this ℓ there is a budget-feasible (x, ℓ), $e_a - (r/k) \cdot \omega^\ell \geq x > x_a$, such that $U_a(x, \omega^\ell) = U_a(x, \omega^{\mu_a}) > U_a(x_a, \omega^{\mu_a})$, which contradicts equilibrium.

(iii) But if $|\pi| = k$, then, using Claim 1(ii), (19.5) holds because

$$
x_a = e_a - \sum_{(\omega,\pi) \in \mathcal{M}} \mu_a(\omega, \pi) \left[r \cdot \omega + q(\omega, \pi) \right] = e_a - (r/k) \cdot \omega^{\mu_a}. \qquad \square
$$

Our objective is to compare the group equilibrium at prices r to a market in which proprietors sell to individual agents at prices r/k. To study the market with individual buyers, we define the agents' demand sets. For each $a \in A$ and $r > 0$

$$
\begin{aligned}
D^a(r) = \{ f \in \Omega \mid e_a - r \cdot f \geq 0 \text{ and for all } \omega \in \Omega, \text{ either} \\
U_a(e_a - r \cdot f,\ f) \geq U_a(e_a - r \cdot \omega,\ \omega) \text{ or } e_a - r \cdot \omega < 0 \}
\end{aligned} \tag{19.7}
$$

By A1, there is no loss of generality in restricting to demand vectors in Ω.

Aggregate demand is the integral of a selection from individual demand sets. A *demand selection at prices* r is an integrable function $f : A \to \Omega$ such that $f(a) \in D^a(r)$ for each $a \in A$. The *aggregate demand correspondence* is

$$D(r) = \left\{ \int_A f(a) d\lambda(a) | f \text{ is a demand selection at prices } r \right\}$$

Claim 3 *Suppose that* A1 *holds. Let* $(x, \mu), q$ *be a purchase-club equilibrium at prices* $r > 0$*, and let* $\{\omega^{\mu_a}\}_{a \in A}$ *be the associated consumptions of shared goods. Then (8) holds for almost every agent* $a \in A$:

$$U_a\left(x_a, \omega^{\mu_a}\right) \geq U_a\left(e_a - \frac{r}{k} \cdot \omega,\ \omega\right) \tag{19.8}$$
$$\textit{for all } \omega \in \Omega \ \textit{ such that } e_a - \frac{r}{k} \cdot \omega \geq 0$$

Proof. Suppose (8) does not hold for a set of agents of positive measure, $\bar{A} \subseteq A$. We will find budget-feasible consumptions $\{(\tilde{x}_a, \tilde{\mu}_a)\}_{a \in A}$ for which

$$\begin{aligned} U_a\left(\tilde{x}_a, \omega^{\tilde{\mu}_a}\right) &\geq U_a\left(x_a, \omega^{\mu_a}\right) \text{ for all } a \in A \\ U_a\left(\tilde{x}_a, \omega^{\tilde{\mu}_a}\right) &> U_a\left(x_a, \omega^{\mu_a}\right) \text{ for all } a \in \bar{A}. \end{aligned} \tag{19.9}$$

This contradicts equilibrium.

We construct the state $(\tilde{x}, \tilde{\mu})$ from a demand selection f at prices r/k. Using the definition of f and Claim 1(i)(iii), the following holds for all $a \in A$ and holds strictly for almost all $a \in \bar{A}$.

$$U_a\left(e_a - \frac{r}{k} \cdot f(a),\ f(a)\right) \geq U_a\left(e_a - \frac{r}{k} \cdot \omega^{\mu_a},\ \omega^{\mu_a}\right) = U_a\left(x_a, \omega^{\mu_a}\right) \tag{19.10}$$

Therefore we can complete the proof by constructing $(\tilde{x}, \tilde{\mu})$ such that for each $a \in A$, $\omega^{\tilde{\mu}_a} = f(a)$, $\tilde{x}_a = e_a - (r/k) \cdot f(a)$. Since there are a finite number of possible demands (those in Ω), we can match consumers with the same demands into size-k groups. For each $\bar{\omega} \in \Omega$, let $A^{\bar{\omega}} \equiv \{a \in A \mid f(a) = \bar{\omega}\}$. If $A^{\bar{\omega}}$ has positive measure, assign each $a \in A^{\bar{\omega}}$ to a single membership in a group type $\pi^{\bar{\omega}} \in \mathcal{G}$ defined such that $\bar{\omega} = \sum_{\omega \in \Omega} \pi^{\bar{\omega}}(\omega)\omega = f(a)$ and $|\pi^{\bar{\omega}}| = k$. Since (19.9) holds for $(\tilde{x}, \tilde{\mu})$, and the consumptions are budget-feasible, the state (x, μ) cannot be an equilibrium. □

The inequality (8) characterizes agents' consumption of shared goods in a group equilibrium. Combined with (19.5), it becomes

$$U_a\left(e_a - \frac{r}{k} \cdot \omega^{\mu_a},\ \omega^{\mu_a}\right) \geq U_a\left(e_a - \frac{r}{k} \cdot \omega,\ \omega\right) \tag{19.11}$$
$$\text{for all } \omega \in \Omega \ \text{ such that } e_a - \frac{r}{k} \cdot \omega \geq 0$$

which looks very much like the definition of the demand correspondence (19.7) for individual purchases at prices r/k. This is the basis of the argument that follows, which says that proprietors have the same profit opportunities in both market circumstances.

A complication, however, is that neither the individual demand correspondence nor group equilibrium is necessarily unique. Consumers may be indifferent between these equilibria, but the proprietors will not be. Assuming that the proprietors price above marginal cost, they prefer more sales to fewer. Similarly, the several group equilibria at prices r will generate the same total utility for agents, but will generate different total profit for the proprietors. We must define a notion of equivalence that accounts for the problem of multiple equilibria.

We show that, despite the multiple equilibria, the profit possibilities are the same whether the proprietors sell to individuals or to groups. Aggregate sales in the group equilibrium can be defined as

$$\omega(x,\mu) = \int_A \sum_{(\omega,\pi)\in\mathcal{M}} \mu_a(\omega,\pi)\,\omega \;\, d\lambda(a) \in Z_+^C$$

If z represents an aggregate demand vector at prices r/k and $\omega(x,\mu)$ represents aggregate sales to members of groups in a group equilibrium $(x,\mu),q$ at prices r, then the profits in the two situations are the same if (19.12) holds. The following proposition says that there is always an equivalence of that type.

$$z = k\omega\,(x,\mu) \tag{19.12}$$

Proposition 1. [Profit Equivalence] *Suppose that* A1 *holds.*

(i) Let $(x,\mu),q$ *be a purchase-club equilibrium at prices* $r > 0$. *Then* $k\omega(x,\mu) \in D(r/k)$.

(ii) Let $z \in D(r/k)$ *be an aggregate demand vector at prices* $r/k > 0$. *Then there exists a purchase-club equilibrium* $(x,\mu),q$ *at prices* r*, with aggregate purchases* $\omega(x,\mu) = z/k$.

Proof.
(i) If $\{\omega^{\mu_a}\}_{a\in A}$ are the consumptions associated with the group equilibrium, they satisfy

$$\frac{1}{k}\int_A \omega^{\mu_a}\, d\lambda(a) = \int_A \sum_{(\omega,\pi)\in\mathcal{M}} \mu_a(\omega,\pi)\,\omega \;\, d\lambda(a) = \omega(x,\mu) \tag{19.13}$$

This is because there are k agents consuming every purchased good. Since $\{\omega^{\mu_a}\}_{a\in A}$ satisfy (19.11), they are also a demand selection at prices r/k. Hence $k\omega(x,\mu) \in D(r/k)$.

(ii) The aggregate demand can be written $z = \int_A f(a)\, d\lambda(a)$ for a demand selection f at prices r/k. Construct an equilibrium $(x,\mu),q$ from the demand selection f as in the proof of Claim 1, using prices q described by (19.4) with equality. Budget feasibility for agents and budget balance for group types are satisfied by construction of q. We also need to show that for almost every $a \in A$,

$$U_a(x_a, \omega^{\mu_a}) \geq U_a(x, \omega^{\ell}) \text{ for every budget feasible } (x, \ell) \in X_a. \tag{19.14}$$

By construction of $(x, \mu), q$ and the definition of f, for each $a \in A$,

$$U_a(x_a, \omega^{\mu_a}) = U_a(e_a - \frac{r}{k} \cdot f(a), f(a)) \geq U_a(e_a - \frac{r}{k} \cdot \omega, \omega) \tag{19.15}$$
$$\text{for all } \omega \in \Omega \text{ such that } e_a - \frac{r}{k} \cdot \omega \geq 0.$$

But if $(x, \ell) \in X_a$ satisfies $\omega^{\ell} \in \Omega$, (19.15) implies (19.14). If (x, ℓ) is budget feasible, it satisfies $e_a - \frac{r}{k} \cdot \omega^{\ell} \geq x \geq 0$, so $U_a(x_a, \omega^{\mu_a}) \geq U_a(e_a - \frac{r}{k} \cdot \omega^{\ell}, \omega^{\ell}) \geq U_a(x, \omega^{\ell})$.

Suppose that $\omega^{\ell} \notin \Omega$, and $U_a(x_a, \omega^{\mu_a}) < U_a(x, \omega^{\ell})$ for some budget feasible (x, ℓ), so (19.14) does not hold. Using A1, choose a list ℓ' such that $\omega^{\ell'} \in \Omega, \omega^{\ell'} \leq \omega^{\ell}$ and $U_a(z, \omega^{\ell}) \leq U_a(z, \omega^{\ell'})$ for all $z \geq 0$. Since $\omega_i^{\ell'} < \omega_i^{\ell}$ for at least one proprietary good i, $e_a - (r/k) \cdot \omega^{\ell} < e_a - (r/k) \cdot \omega^{\ell'}$. Thus, if there is a budget feasible (x, ℓ), there is a budget-feasible $(x', \ell'), x < x'$. Hence, $U_a(x_a, \omega^{\mu_a}) < U_a(x, \omega^{\ell})$ $< U_a(x', \omega^{\ell'})$. But this is a contradiction, since we have already shown that, if $\omega^{\ell'} \in \Omega$ and (x', ℓ') is budget feasible, then $U_a(x_a, \omega^{\mu_a}) \geq U_a(x', \omega^{\ell'})$. Thus (19.14) holds, and $(x, \mu), q$ is a purchase-club equilibrium at prices r. □

19.4 Rental markets

Example 4 in Section 3 suggests that the club model can be interpreted as a rental market. Our objective here is to elaborate that example, and to show circumstances in which sharing groups are equivalent to how we would conceive of a rental market in ordinary general equilibrium theory.

The easiest way to think of rental markets is that there is an amortized cost of keeping the rental good continuously in use. The competitive price of using it will reflect this amortized cost. If this is all there is to it, then general equilibrium theory as conceived by Arrow and Debreu can account for rental markets, even if demand depends on time. If, for example, there are peak and off-peak demand periods (in the case of sailboats, balmy summer days and dark winter days), then we might think of rentals in the two periods as jointly produced, but different, goods. Price cannot equal "marginal cost" in both periods, since the price in the two periods will be different.

We now show how the club model accommodates rental markets, allowing the quality of the rentals (in the sense of inconvenience due to congestion) to be endogenous, and differentiating prices according to peak and off-peak periods.

Pricing in the club model is more flexible than in a rental market. Prices in an ordinary rental market are linear on units of usage, although possibly different in peak and off-peak periods. We show conditions under which rental prices in a group equilibrium can also be interpreted as linear prices on usage.

Let elements of Ω represent *usage*. For fixed k, represent usage by

$$\Omega = \{(\omega_p, \omega_o) | \omega_p \in \{0, 1, 2, ...k\}, \omega_o \in \{0, 1, 2, ...k\}\},$$

where the membership characteristic $(\omega_p, \omega_o) \in \Omega$ represents the number of units of rental of each type, peak and off-peak. As in the model of the previous section, this model specializes Ω to be a space of numbers rather than an abstract space.

A *rental group type* is (π, γ, y), where y represents the rental goods bought in a competitive market, and $\gamma = (\gamma_p, \gamma_o) \in \Gamma$ specifies the total usage offered by the rental group at both peak and offpeak times. In particular, let $\Gamma = \{\{1, 2, ...\bar{\gamma}_p\} \times \{1, 2, ...\bar{\gamma}_o\}\}$. A *feasible* rental group type (π, γ, y) satisfies $\pi \in \Pi(\gamma_p, \gamma_o)$, where

$$\Pi(\gamma_p, \gamma_o) = \left\{ \pi : \Omega \to Z_+ \ \mid \sum_{(\omega_p, \omega_o) \in \Omega} \pi(\omega_p, \omega_o)\omega_p = \gamma_p, \right.$$
$$\left. \sum_{(\omega_p, \omega_o) \in \Omega} \pi(\omega_p, \omega_o)\omega_o = \gamma_o \right\}$$

In order to interpret total usage $(\gamma_p, \gamma_o) \in \Gamma$ as congestion in the two periods, assume that all group types have the same input vector y, say, one sailboat. We shall thus leave y out of the description of a group type, although it remains in the budget balance condition for each rental group type. The feasible set of group types and memberships are

$$\mathcal{G} = \{(\pi, \gamma) \mid \gamma = (\gamma_p, \gamma_o) \in \Gamma, \pi \in \Pi(\gamma_p, \gamma_o)\}$$
$$\mathcal{M} = \{((\omega_p, \omega_o), (\pi, \gamma)) \mid (\omega_p, \omega_o) \in \Omega, \ (\pi, \gamma) \in \mathcal{G}\}$$

Let $\omega^\ell = (\omega_p^\ell, \omega_o^\ell) : \Gamma \to Z_+ \times Z_+$ represent *usage* associated with ℓ. For each $(\gamma_p, \gamma_o) \in \Gamma$,

$$\omega_p^\ell(\gamma_p, \gamma_o) = \sum_{\pi \in \Pi(\gamma_p, \gamma_o)} \sum_{(\omega_p, \omega_o) \in \Omega} \ell((\omega_p, \omega_o), (\pi, \gamma)) \ \omega_p \tag{19.16}$$
$$\omega_o^\ell(\gamma_p, \gamma_o) = \sum_{\pi \in \Pi(\gamma_p, \gamma_o)} \sum_{(\omega_p, \omega_o) \in \Omega} \ell((\omega_p, \omega_o), (\pi, \gamma)) \ \omega_o$$

Consumption sets are

$$X_a = \left\{ x \in \mathbb{R}_+^N, \ \ell \in Z_+^{\mathcal{M}} | \, |\ell| \leq M \right\}$$
$$\tilde{X}_a = \left\{ \left(x, \omega^\ell\right) \mid (x, \ell) \in X_a \right\}.$$

The following assumption says that utility depends only on usage.

A2. For each $a \in A$, the utility function $u_a : X_a \to \mathbb{R}$ is defined by $u_a(x, \ell) = U_a(x, \omega^\ell)$ for a utility function $U_a : \tilde{X}_a \to \mathbb{R}$. U_a is increasing in its first argument.

We say that a group equilibrium $(x, \mu), (p, q)$ is *equivalent to equilibrium in a rental market* if there exist rental prices $(\theta_p, \theta_o) : \Gamma \to \mathbb{R}_+ \times \mathbb{R}_+$ such that the price of each list ℓ satisfies

$$(\theta_p, \theta_o) \cdot \left(\omega_p^\ell, \omega_o^\ell\right) = \ell \cdot q \tag{19.17}$$

The important feature of rental markets is that they impose a restriction on prices. The membership price $q((\omega_p, \omega_o), (\pi, \gamma))$ will reflect the member's peak usage and offpeak usage, as well as the congestion, but there is no prior restriction that the price $q((\omega_p, \omega_o), (\pi, \gamma))$ can be conceived as a linear price on usage, and that the linear price is the same as that of other users, scaled by usage. However:

Proposition 2. *Suppose that* A2 *holds. Let* $(x, \mu), (p, q')$ *be a group equilibrium. Then there is another group equilibrium* $(x, \mu), (p, q)$ *that is equivalent to an equilibrium in a rental market.*

Proof. We first define a distinguished set of group types that offer single-usage memberships, rather than selling usage in bulk. For each $\gamma \in \Gamma$, let (π^γ, γ) be a group type such that $\pi^\gamma(0,1) = \gamma_o$, $\pi^\gamma(1,0) = \gamma_p$, and $\pi^\gamma(\omega_p, \omega_o) = 0$ for $(\omega_p, \omega_o) \notin \{(0,1),(1,0)\}$. For example, a membership $((0,1),(\pi^\gamma, \gamma))$ is a single off-peak use. For each $(\omega, (\pi, \gamma)) \in \mathcal{M}$, let

$$q(\omega, (\pi, \gamma)) = \omega_p \;\; q'((1,0),(\pi^\gamma, \gamma)) + \omega_o \;\; q'((0,1),(\pi^\gamma, \gamma))$$

To define the prices in the rental market, for each $\gamma \in \Gamma$ let

$$\begin{aligned} \theta_p(\gamma) &= q'((1,0),(\pi^\gamma, \gamma)) = q((1,0),(\pi^\gamma, \gamma)) \\ \theta_o(\gamma) &= q'((0,1),(\pi^\gamma, \gamma)) = q((0,1),(\pi^\gamma, \gamma)) \end{aligned} \tag{19.18}$$

To prove the proposition, we only need to show that (x, μ),(p, q) is an equilibrium. We first show that almost all the agents are in their budget sets, and then show that they are optimizing.

We show that, for almost all $a \in A$, $p \cdot e_a = x_a + q \cdot \mu_a = x_a + q' \cdot \mu_a$. Construct a consistent list assignment $\tilde{\mu}$ with the same individual usage as in the equilibrium lists μ, but with single-usage memberships. That is, for each $a \in A$, $\gamma \in \Gamma$, let

$$\begin{aligned} &\tilde{\mu}_a((1,0),(\pi^\gamma, \gamma)) &&= \omega_p^{\mu_a}(\gamma) \\ &\tilde{\mu}_a((0,1),(\pi^\gamma, \gamma)) &&= \omega_o^{\mu_a}(\gamma) \\ &\tilde{\mu}_a((\omega_p, \omega_o),(\pi, \gamma)) = 0 && \text{for all other memberships} \end{aligned}$$

Then the following holds by construction for $a \in A$.

$$q \cdot \tilde{\mu}_a = q' \cdot \tilde{\mu}_a = (\theta_p, \theta_o) \cdot \left(\omega_p^{\mu_a}, \omega_o^{\mu_a}\right) \tag{19.19}$$

Since $u_a(x_a, \mu_a) = u_a(x_a, \tilde{\mu}_a)$ and $(x, \mu), (p, q')$ is an equilibrium, it holds that $p \cdot e_a = x_a + q' \cdot \mu_a \leq x_a + q' \cdot \tilde{\mu}_a$ for almost all $a \in A$, so that

$$q' \cdot \mu_a \leq q' \cdot \tilde{\mu}_a = q \cdot \tilde{\mu}_a \tag{19.20}$$

The assignment $\tilde{\mu}$ is consistent because μ is consistent. Since $\omega^{\mu_a}(\gamma) = \omega^{\tilde{\mu}_a}(\gamma)$ for each $\gamma \in \Gamma$ and every $a \in A$, the number of groups associated with $\tilde{\mu}$ is the same as the number associated μ, and each has the same cost $p \cdot y$. Since (p, q') balances the budget for each group type, μ and $\tilde{\mu}$ must generate the same revenue in aggregate.

$$\int_A q' \cdot \mu_a \, d\lambda(a) = \int_A q' \cdot \tilde{\mu}_a \, d\lambda(a) \tag{19.21}$$

But this proves that the inequality in (19.20) cannot be strict for a set of agents with positive measure. Hence we can conclude that for almost all $a \in A$,

$$p \cdot e_a = x_a + q' \cdot \mu_a = x_a + q' \cdot \tilde{\mu}_a = x_a + q \cdot \tilde{\mu}_a = x_a + q \cdot \mu_a.$$

We now show that agents are optimizing at the prices (p, q). Suppose that $(x, \ell) \in X_a$ and $u_a(x, \ell) > u_a(x_a, \mu_a)$. We show that (x, ℓ) is not budget feasible at prices (p, q). Construct a new list ℓ' from ℓ by substituting single-usage memberships in the same amount. For each membership $(\omega, (\pi, \gamma)) \in \mathcal{M}$, let $\ell'(\omega, (\pi, \gamma)) = 0$ except let $\ell'((0, 1), (\pi^\gamma, \gamma)) = \omega_o^\ell(\gamma)$, $\ell'((1, 0), (\pi^\gamma, \gamma)) = \omega_p^\ell(\gamma)$. Then the lists ℓ and ℓ' provide the same usage $\omega^\ell = \omega^{\ell'}$, and $u_a(x, \ell') = u_a(x, \ell) > u_a(x_a, \mu_a)$. Because $(x, \mu), (p, q')$ is an equilibrium, $p \cdot x + q' \cdot \ell > p \cdot e_a$ and $p \cdot x + q' \cdot \ell' > p \cdot e_a$. But since $q \cdot \ell' = q' \cdot \ell' = q \cdot \ell$, this implies that $p \cdot x + q \cdot \ell' > p \cdot e_a$ and $p \cdot x + q \cdot \ell > p \cdot e_a$. □

19.5 Conclusion

The club model can account for consumption externalities in various ways. By consumption externalities, we mean that each member of a club cares about the private-goods consumption of other members. The models above elaborate that idea by introducing different technologies of sharing, and showing how the technologies of sharing can be reflected in group types and membership characteristics.

The term "consumption externality" suggests that each agent makes a consumption decision without considering its impact on others. The club model forces him to consider the impact. Groups that want to avoid negative externalities that arise from private consumption decisions will have memberships that involve a commitment to avoid consumption of certain private goods. Groups that want to generate positive externalities due to private consumption decisions will have membership characteristics that require certain kinds of consumption. These commitments can be built into feasible consumption sets, which can constrain the consumption of private goods in a way that is linked to memberships in groups.

The technology of sharing private goods was more precise in what we called purchase clubs and rental clubs. In the case of automobiles, sailboats and other durable goods which cannot be used simultaneously by all members of a group, the terms of sharing must be specified in the group type and the membership characteristics. Nothing requires that congestible durable goods be shared in rental groups rather than purchase groups; in fact, there is no clear distinction between those two concepts. We only chose those terms to suggest familiar market institutions, and to give different names to models based on different sharing technologies. The key point is that, if users care about total congestion as well as their own usage, then membership prices must reflect both. And membership prices may also reflect the externality-producing private goods that a member brings as part of his membership.

In the purchase-club and rental-club models, we respectively treated proprietary pricing and congestion costs. Of course proprietary pricing and congestion can be combined in the same model: Goods that are purchased at proprietary prices can nevertheless be subject to congestion. A group equilibrium will be efficient for the users conditional on the proprietary prices, but this is a conditional notion of efficiency. Each copy of a proprietary good that is subject to congestion may be used "too much" in equilibrium, to conserve on paying the proprietary price.

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20

Core-Equivalence for the Nash Bargaining Solution*

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Summary. Core equivalence and shrinking of the core results are well known for economies. The present paper establishes counterparts for bargaining economies, a specific class of production economies (finite and infinite) representing standard two-person bargaining games and their continuum counterparts as coalition production economies. Thereby we get core equivalence of the Nash solution. The results reconfirm the Walrasian approach to Nash bargaining of Trockel (1996). Moreover we establish the same speed of convergence as is known from Debreu (1975) and Grodal (1975) for replicated pure exchange economies and for regular purely competitive sequences of economies, respectively.

Key words: Nash-solution, Core convergence, Equivalence theorem.

JEL Classification Numbers: C71, C78, D51.

20.1 Introduction

The Nash solution of two person bargaining games introduced by Nash (1953) has been characterized in manifold ways. Some of them hint to the fact that it reflects some kind of immunity against strategic exploitation. So does Rubinstein's (1982) alternating offer game whose subgame perfect equilibrium approaches the Nash solution the closer the less the discounting of future is by the strength of alternative future options. Also Trockel's (1996) Walrasian characterization of the Nash solution reflects the pressure of competition providing sufficient outside options. In the latter approach, following ideas of Shapley (1969), the two players' utilities are treated as the two commodities in an artificial economy with production and private property.

The Walrasian characterization establishes the vector λ in Shapley's λ-transfer value as a competitive price system thereby proving a conjecture of Shubik (1985).

In the present paper we relate the Nash solution with the Edgeworthian rather than the Walrasian version of perfect competition. To do so, we define an artificial

* This article is dedicated to Birgit Grodal, a friend since 30 years. Financial support of the DFG under grant #444 USA 111/2/03 is gratefully acknowledged.

coalition production economy (cf. Hildenbrand, 1974) representing a two person bargaining game.

In a similar way the Nash solution has been applied in Mayberry et al. (1953) to define a specific solution for a duopoly situation and comparing it with other solutions, among them the Edgeworth contract curve. The relation between these two solutions will be the object of our investigation in this paper.

Though it would not be necessary to be so restrictive we define a two person bargaining game as the closed subgraph of a continuously differentiable strictly decreasing concave function $f : [0,1] \longrightarrow [0,1]$ with $f(0) = 1$ and $f(1) = 0$.

$$S := subgraph f := \{(x_1, x_2) \in [0,1]^2 | x_2 \leq f(x_1)\}$$

The normalization reflects the fact that bargaining games are usually considered to be given only up to positive affine transformations. Smoothness makes life easier by admitting unique tangents.

The model S is general enough for our purpose of representation by a coalition production economy. In particular, S is the intersection of some strictly convex comprehensive set with the positive orthant of $\mathbb{R}^2$.

20.2 The basic model

Define for any S as described in Section 1 a two person coalition production economy $\mathcal{E}^S$ as follows:

$$\begin{aligned}
\mathcal{E}^S &:= ((e_i, \succsim_i, Y_i)_{i=1,2}, (\vartheta_{ij})_{i,j=1,2}) \text{ such that} \\
e_i &= (0,0), x = (x_1, x_2) \succsim_i x' = (x'_1, x'_2) \Leftrightarrow x_i \geq x'_i, i = 1,2 \\
\vartheta_{11} &= \vartheta_{22} = 1, \vartheta_{12} = \vartheta_{21} = 0, Y_1 = Y_2 = \left(\frac{1}{2}\right) S
\end{aligned}$$

The zero initial endowments reflect the idea that all available income in this economy comes from shares in production profits.

Each agent owns fully a production possibility set that is able to produce for any $x \in S$ the bundle $(\frac{1}{2})x$ without any input.

Both agents are interested in only one of the two goods called "agent $i's$ utility", $i = 1, 2$.

Without any exchange agent i would maximize his preference by producing and consuming one half unit of commodity i and zero units of commodity $3 - i, i = 1, 2$.

However, the agents would recognize immediately that they left some joint utility unused on the table.

Given exchange possibilities for the two commodities they would see that improvement would require exchange or, to put it differently, coordinated production (see Fig. 1).

The point $(\frac{1}{2}, \frac{1}{2})$ corresponds to the vector of initial endowments, the set $S_1 := S \cap (\{(\frac{1}{2}, \frac{1}{2})\} + \mathbb{R}^2_+)$ to the famous lens and the intersection of S_1 with the efficient

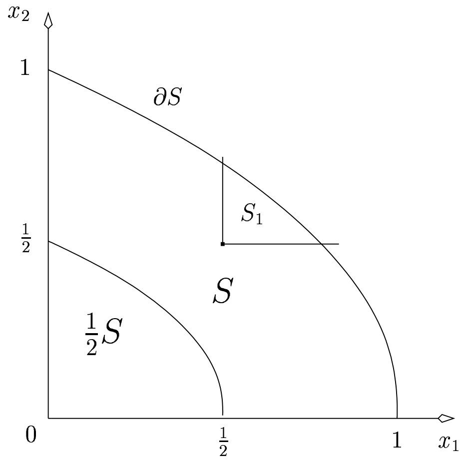

Fig. 1.

boundary of S, i.e. $S_1 \cap \partial S$, to the core in the Edgeworth Box. This is exactly what Mayberry and al. (1953, p. 144) call the *Edgeworth contract curve* in their similar setting.

The according notions of improvement and of the core are analogous to the ones used for *Coalitional Production Economies* by Hildenbrand (1974, p. 211).

$\tilde{Y} : \{\{1\}, \{2\}, \{1,2\}\} \Longrightarrow \mathbb{R}^2$ with $\tilde{Y}(\{1\}) = Y_1, \tilde{Y}(\{2\}) = Y_2, \tilde{Y}(\{1,2\}) = S$, is the production correspondence, which is additive, as $Y(\{1\} \cup \{2\}) = Y_1 + Y_2 = S$.

An allocation $x^i = ((x_1^i, x_2^i))_{i=1,2}$ for $\mathcal{E}^S$ is *T-attainable* for $T \in \{\{1\}, \{2\}, \{1,2\}\}$ if $\sum_{i \in T} x^i \in \tilde{Y}(T)$; it is called *attainable* if it is $\{1,2\}$-attainable.

An allocation (x^1, x^2) can be *improved upon* by a coalition $T \in \{\{1\}, \{2\}, \{1,2\}\}$ if there is a *T-attainable* allocation (y^1, y^2) such that $\forall i \in T : y^i \succ_i x^i$.

The *core* of $\mathcal{E}^S$ is the set of $\{1,2\}$-attainable allocations that cannot be improved upon.

The analogous definitions hold for all n-replicas $\mathcal{E}_n^S$ of $\mathcal{E}^S$, $n \in \mathbb{N}$.

Notice that our choice of $Y_i = (\frac{1}{2})S, i = 1,2$ ensures the utility allocation $(\frac{1}{2}, \frac{1}{2})$ for the two players in case of non-agreement. This differs from Nash's status quo or threat point $(0,0)$.

Formalizing an n-replica economy $\mathcal{E}_n^S$ is standard. All characteristics are replaced by n-tupels of identical copies of these characteristics. In particular $\mathcal{E}_n^S$ has $2n$ agents, n of each of the two types 1 and 2. And the total production possibility set for the grand coalition of all $2n$ agents is nS.

Although the use of strict convex preferences as in Debreu and Scarf (1963) is not available here a short moment of reflection shows that a major part of their arguments can be used in our case as well.

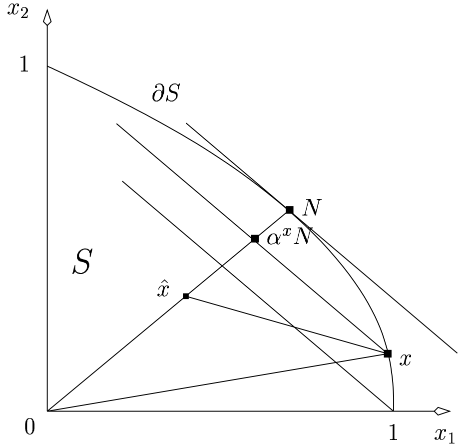

Fig. 2.

In the case of a continuum of agents a coalition production economy representing the bargaining game S is a map

$$\tilde{\mathcal{E}}^S\text{:}([0,1],\mathcal{B}[0,1],\lambda) \longrightarrow 2^{\mathbb{R}^2_+} \times \mathbb{R}^2_+ \times C^\circ(S) \times \{0,1\}^{[0,1]} : t \mapsto (\tilde{Y}(t), e_t, u_t, \vartheta_t),$$
$$\text{with } \tilde{Y}(t) := S, e_t = (0,0), u_t := 1_{[0,\frac{1}{2}]}(t) \cdot proj_1 + 1_{]\frac{1}{2},1]}(t) \cdot proj_2$$

for all $t \in [0,1]$. For each t the share function ϑ_t is given by $\vartheta_t : [0,1] \longrightarrow \{0,1\} : s \mapsto \vartheta_t(s) := \delta_{t,s}$(Kronecker delta).

An allocation $\tilde{x} : [0,1] \longrightarrow [0,1]^2$ is T-*attainable* for $T \in \mathcal{B}[0,1]$ if $\int \tilde{x}(t)\lambda(dt) \in \int_T \tilde{Y}(t)\lambda(dt) = \lambda(T)S$.

A $[0,1]$-attainable allocation $\tilde{x}$ can be *improved upon* by a coalition $T \subset \mathcal{B}[0,1]$ via a T-attainable allocation $\tilde{y}$, if λ $a.e.$ in $T : u_t(\tilde{y}(t)) > u_t(\tilde{x}(t))$. The *core of* $\tilde{\mathcal{E}}^S$ is the set of attainable allocations that cannot be improved upon.

20.3 Core equivalence

Consider S as illustrated in Figure 2.

It is well known that the vector (N_2, N_1) is normal to ∂S at the Nash solution N of S. For any $x \in \partial S$ with $x_1 > N_1$, and thus $x_2 < N_2$ we have:

$$(N_2, N_1) \cdot (x_1, x_2) \in [(N_2, N_1) \cdot (1,0), (N_2, N_1) \cdot (N_1, N_2)] = [N_2, 2N_1N_2].$$

Therefore we get

$$\begin{aligned}(N_2, N_1) \cdot (x_1, x_2) &= (N_2, N_1) \cdot (\alpha^x N_1, \alpha^x N_2) \\ &= 2\alpha^x N_1 N_2 \text{ for some } \alpha^x \in \left[\frac{1}{2N_1}, 1\right].\end{aligned}$$

The efficient point x can only be realized by the grand coalition $[0,1]$ if λ a. e. agent produces x. Clearly, x is allocated to the agents in such a way that each agent in $[0,\frac{1}{2}]$ receives $(2x_1,0)$ and each agent in $]\frac{1}{2},1]$ receives $(0,2x_2)$. The idea for the construction of a coalition C^x that improves upon x is as follows. Let all members of C^x produce some alternative point more favorable for type 2 agents, hence less favorable for type 1 agents. But choose the set C_1^x of type 1 agents in C^x so small that by allocating among them equally the whole production of good 1 each of them is even better off than before.

The type 2 agents improve by choosing that alternative point in such a way that the increase in production of good 2 per capita overcompensates the loss caused by the fact that only few type 1 agents are contributing to the production of good 2.

First we choose $\hat{x} := x + \beta^x(-x_1, x_2)$ in such a way that it is in the segment $[0,N]$. As $(-N_1,N_2)$ is steeper than $(-x_1,x_2)$ we know that $\hat{x} \in]0,\alpha^x N[$.

Now choose a small set C_1^x of type 1 agents of measure $\alpha_1^x > 0$ and a set C_2^x of type 2 agents of measure $\alpha_2^x > \alpha_1^x$ with $\alpha_2^x < \frac{1}{2}$.

Denote the union of these two sets C^x. So C^x has the measure $\alpha_1^x + \alpha_2^x$. Let each of the members of C^x produce $\hat{x} := ((1-\beta^x)x_1, (1+\beta^x)x_2)$.

Their total production is $(\alpha_1^x+\alpha_2^x)((1-\beta^x)x_1,(1+\beta^x)x_2)$.

We want to reallocate this by distributing equally the total amount of good i among type i agents, $i = 1,2$ in such a way that each agent gets exactly what he received from x. Hence we must have:

$$\int_{C^x} ((1-\beta^x)x_1,(1+\beta^x)x_2)d\lambda = (\alpha_1^x+\alpha_2^x)((1-\beta^x)x_1,(1+\beta^x)x_2)$$
$$= (\alpha_1^x\, 2x_1, \alpha_2^x\, 2x_2) = \int_{C_1^x} (2x_1,0)\, d\,\lambda + \int_{C_2^x} (0,2x_2)\, d\,\lambda.$$

Therefore:

$$(\alpha_1^x+\alpha_2^x)(1-\beta^x) = 2\alpha_1^x \text{ and } (\alpha_1^x+\alpha_2^x)(1+\beta^x) = 2\alpha_2^x \text{ hence :}$$
$$\beta^x = \frac{\alpha_2^x-\alpha_1^x}{\alpha_1^x+\alpha_2^x} \in]0,1[\text{ and } \alpha_2^x = \frac{1+\beta^x}{1-\beta^x}\alpha_1^x.$$

Among those α_1^x, α_2^x satisfying this equation we can indeed choose $\alpha_2^x < \frac{1}{2}$, as we did before.

Up to now all agents in C^x are indifferent between the original production and allocation and the new one.

Now assume that each member of C^x instead of $\hat{x}$ even produces $\alpha^x N > \hat{x}$. If $(\alpha^x N_1 - \hat{x}_1, 0)$ and $(0, \alpha^x N_2 - \hat{x}_2)$ are distributed equally among the type 1 and type 2 agents, respectively, while $\hat{x}$ is distributed as before, all members of C^x are better off than they were under the production of x. Hence C^x improves upon x.

In an analogous way one can show that any $x \in \partial S$ with $x_1 < N_1, x_2 > N_2$ can be improved upon.

It is obvious that N itself cannot be improved upon by any coalition. Also it is known from Trockel (1996) that N is the unique Walrasian allocation of $\tilde{\mathcal{E}}^S$ and is

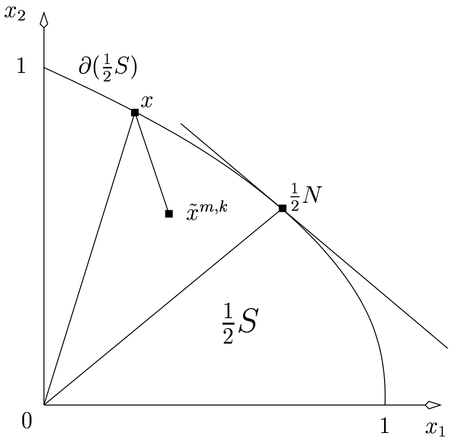

Fig. 3.

therefore in the core of the coalition production economy $\tilde{\mathcal{E}^S}$ (cf. Hildenbrand, 1974, p. 216). So we have established that $\{N\} = Core(\tilde{\mathcal{E}^S})$.

This result could have been derived alternatively via Proposition 2 and Theorem 1 in Hildenbrand (1974, p. 216) exploiting the fact that the unique Walrasian equilibrium is the only quasi-equilibrium in $\tilde{\mathcal{E}^S}$ and by Trockel (1996) coincides with the Nash solution N of S.

Our proof has the advantage to hint to the way one may get a Debreu-Scarf type convergence result for the core in our framework. This will be carried out in the next section.

20.4 An Edgeworth-Debreu-Scarf approach

In this section we are looking at the core of n-replicas $\mathcal{E}_n^S$ of the economy $\mathcal{E}^S$. Again it suffices to look at S. Notice that it does not make any difference whether in an n-replica economy every agent has the technology $Y = \frac{1}{2n}S$ and the total production set is S or wether each agent has $Y = (\frac{1}{2})S$ and total production is nS. We will assume that each agent in $\mathcal{E}_n^S$ owns a production possibility set $Y := \frac{1}{2}S$ as illustrated in Figure 3.

This time we assume w.l.o.g. that $x \in \partial(\frac{1}{2}S)$ and $x_1 < \frac{1}{2}N_1, x_2 > \frac{1}{2}N_2$. By choosing $n, m, k \in \mathbb{N}, k < m \leq n$ sufficiently large we can make the vector $\frac{m-k}{m+k}(x_1, -x_2)$ arbitrarily small and, thereby, position the point $\tilde{x}^{m,k} := (x_1, x_2) + \frac{m-k}{m+k}(x_1, -x_2)$ in $int(\frac{1}{2}S)$.

A coalition C_n^x in the n-replica economy $\mathcal{E}_n^S$ of $\mathcal{E}^S$ consisting of m agents of type 1 and k agents of type 2 can realize the allocation $(m+k)\tilde{x}^{m,k} =$
$((m+k)x_1 + (m-k)x_1, (m+k)x_2 - (m-k)x_2) = (2mx_1, 2kx_2)$.

This bundle can be reallocated to the members of C_n^x by giving to each of the m type 1 agents $(2x_1, 0)$ and to each of the k type 2 agents $(0, 2x_2)$. Clearly, everybody gets thereby the same as he received in the beginning when everybody produced x. Therefore, nobody improves! However, for $\eta > 0$ sufficiently small $\tilde{x}^{m,k} \in int\ \frac{1}{2}S$ implies that $\tilde{x}^{m,k} + \eta N \in int \frac{1}{2}S$. Now reallocation of that bundle among the members of C_n^x can be performed in such a way that each type 1 agent receives $(2x_1 + \frac{m+k}{m}\eta N_1, 0)$ and each type 2 agent gets $(0, 2x_2 + \frac{m+k}{k}\eta N_2)$. Therefore x for every agent can be improved upon by C_n^x via production of $\tilde{x}^{m,k} + \eta N$ by each of its members. Again, the only element of $\partial(\frac{1}{2}S)$ remaining in the core for all $n-$replications of $\mathcal{E}^S$ is the point $\frac{1}{2}N$, i.e. the Nash solution for $\frac{1}{2}S$.

Notice that any point $y \in \partial(\frac{1}{2}S)$ with $y_1 < x_1 < N_1$ can be improved upon by the same coalition C_n^x via $\tilde{y}^{m,n} + \eta N$ with the same η by a totally identical construction of $\tilde{y}^{m,n}$ from y.

The same is not true for $z \in \partial(\frac{1}{2}S)$ with $x_1 < z_1 < N_1$.

Here the $\frac{m-k}{m+k}(z_1, -z_2)$ may require a larger m and k to make $\frac{m-k}{m+k}(z_1, -z_2)$ small enough. We may for any $x \in \partial(\frac{1}{2}S), x_1 < N_1$ choose the m, k in the construction of $\tilde{x}^{m,k}$ in such a way that $\tilde{x}^{m,k}$ is on or arbitrary close to the segment $[0, \frac{1}{2}N]$. This fact will help us in the last section to derive the speed of convergence.

20.5 Speed of convergence

There are results in the literature on the speed of core convergence first by Debreu (1975) for replica exchange economies and then, more generally, by Grodal (1975) for competitive sequences of regular economies. They state that the distance between the core and the Walrasian allocations of the economies converges to zero like $1/n$, where n is the number of agents in the economy. The metric by which they measure the distance between the core and the set of Walrasian allocations is the Hausdorff distance based on the metric on the set of allocations induced by the sup-norm. In our context due to equal treatment in the core of replica economies this amounts to estimation of the maximal possible distance between two commodity bundles allocated to an arbitrary member of the replica economy by some core allocation and by the unique Walrasian allocation.

Also the restriction to regular economies that is essential in Debreu (1975) and Grodal (1975) is automatically satisfied in the present paper as the excess demand functions of our bargaining economies satisfy the weak axiom of revealed preferences (see Trockel, 1996).

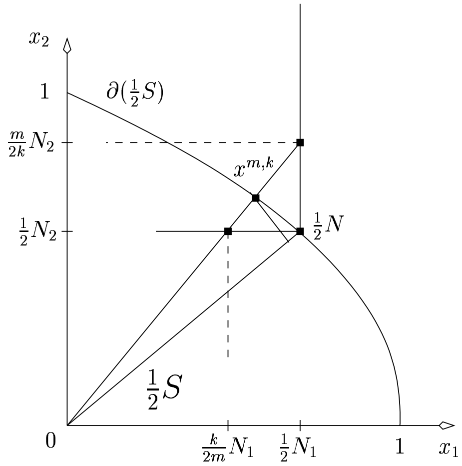

Fig. 4.

Keeping these remarks in mind we see that the result we will derive next is totally analogous to the results quoted above.

For given $\frac{1}{2}S$ and $\frac{1}{2}N$ we associate with any pair $k, m, 0 < k < m \leq n$ the points $\frac{1}{2}N^{m,k} := \frac{1}{2}(N_1, \frac{m}{k}N_2)$ and $\frac{1}{2}N_{m,k} := \frac{1}{2}(\frac{k}{m}N_1, N_2)$.

Denote the unique point in the intersection

$$\partial\left(\frac{1}{2}S\right) \cap \left[\frac{1}{2}N^{m,k}, \frac{1}{2}N_{m,k}\right] \text{ by } x^{m,k}.$$

This point can be improved upon by the coalition $C_n^{x^{m,k}}$ via $x^{m,k} + \frac{m-k}{m+k}(x_1^{m,k}, -x_2^{m,k}) + \eta N$ for sufficiently small $\eta > 0$, as we have proved in Section 4.

In fact for given m of all points $x^{m,k}$ the closest one to N is $x^{m,m-1} =: x^m$ (see Fig. 4). So the length of the path on $\partial(\frac{1}{2}S)$ connecting x^m with $\frac{1}{2}N$ is an upper bound of the size of the core of $\mathcal{E}_m^S$ (or more precisely of that part of the core to the left of $\frac{1}{2}N$).

This length is estimated from above by the number

$$\max\left(\left\|\frac{1}{2}N^m - \frac{1}{2}N\right\|, \left\|\frac{1}{2}N_m - \frac{1}{2}N\right\|\right)$$
$$\text{with } N^m := N^{m,m-1}, N_m := N_{m,m-1}.$$

But this number equals $\frac{1}{2}\max((\frac{m}{m-1} - 1)N_2, (1 - \frac{m-1}{m})N_1) \leq \frac{1}{2}\max(N_1, N_2) \cdot \frac{1}{m-1} = 0(\frac{1}{m})$. □

20.6 Concluding remarks

The present paper continues the idea of Trockel (1996) to approach cooperative games with methods from microeconomic theory. Considering sets of feasible utility allocations as production possibility sets representing the possible jointly "producable" utility allocations and tranformation rates as prices goes back to Shapley (cf. Shapley, 1969). See also Mayberry et al. (1953). The possibility to get Core Equivalence of Walrasian equilibria of the artificial bargaining economies and to derive an Edgeworth-Debreu-Scarf type convergence result makes bargaining economies besides Edgeworth-Boxes or Robinson-Crusoe economies another attractive class of economies for illustrative purposes. The identity of the Walrasian equilibrium of a finite bargaining economy $\mathcal{E}^S$ with the Nash solution of its underlying bargaining game S stresses the competitive feature of the Nash solution.

Moreover the Nash solution's coincidence with the Core of a large bargaining coalitional production economy with equal production possibilities for all agents reflects a different fairness aspect in addition to those represented by the axioms or by alternative characterizations.

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