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Dariush Khezrimotlagh · Yao Chen

# Decision Making and Performance Evaluation Using Data Envelopment Analysis

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# Decision Making and Performance Evaluation Using Data Envelopment Analysis

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*To those who respect mathematical logic*

# Preface

This book is written as a self-teaching tool to assist every teacher, student, mathematician, or nonmathematician for educating themselves or others, and to support their understanding of the elementary concepts of assessing the performance of a set of homogenous firms, as well as how to correctly adapt mathematics to these concepts step by step.

The book can be used as a reference for undergraduate or graduate level courses in mathematics, economics, management, business, statistics, or industrial engineering. There is no precourse required to start reading the book, except a basic and elementary knowledge of mathematics and computer.

In order to prevent prejudgment about the meaning of efficiency and productivity, some new phrases are introduced and, after elucidating each phrase in detail, they are reintroduced for industry-wide accuracy.

In the first chapter, several concepts are introduced along with a simple example to measure the performance of a set of homogenous firms with one single input factor and one single output factor. The concepts are logically discussed to pave the way to introduce different aspects and to establish the corresponding mathematical foundation for further complex questions with respect to performance evaluation. For each aspect, a figure is provided to assist learning the concepts. In order to avoid any confusion based on previous literature, the concepts are introduced with new phrases and terms.

In Chap. 2, the discussions in Chap. 1 are continued, and the mathematical background of the requirement axioms is gradually built to evaluate performance of homogenous firms. Two important phrases are introduced, namely, "doing the job well" and "doing the job right." A philosophical argument to clarify the introduced concepts is provided. The overall elementary concepts to evaluate the performance of firms are covered in this chapter.

Chapter 3 starts with an example of two input factors. The introduced concepts in Chaps. 1 and 2 are adjusted for the example. The important information to introduce relative scores for firms, when the number of input factors is more than one, is provided. If there are no assumptions to introduce unit of measurement for each input factor, a linear combination of input factors is not meaningful. The relative scores of firms can be different when the introduced weights or prices for input factors are changed.

In Chap. 4, the discussion in Chap. 2 is continued and optimization approaches and linear programming models are introduced, without prerequirement knowledge in Operations Research. Linear programming problems are solved by Microsoft Excel Solver software and Microsoft Visual Basic for Applications (VBA) without preknowledge about the software. The readers will gradually develop their mathematics and computer skills as they follow the provided instructions to solve models. The concepts are also adapted for an example of two output factors.

In Chap. 5, an example of eight international airports in Persia is proposed, and the introduced concepts in the previous chapters are adjusted for four input factors and three output factors. The mathematical foundations, linear programming models, and the computer programming are demonstrated in detail. The aim of this chapter is to prepare the readers to generalize the concepts of doing the job right and doing the job well for a set of homogenous firms with multiple input factors and multiple output factors.

Chapter 6 discusses several views about performance evaluation of firms. A delta neighborhood of a firm is introduced to find a bridge between the concepts of "doing the job right" and "doing the job well." The prime and dual linear programming models are introduced. At the end of this chapter, readers are completely prepared to carry out the performance evaluation of a set of homogenous firms with multiple input factors and multiple output factors.

Chapter 7 adapts the introduced concepts in the previous chapters with the literature of Data Envelopment Analysis (DEA). The philosophical background is also discussed to elucidate the required information to measure the performance of a set of homogenous firms. It is also discussed that the measured technical efficiency scores by conventional DEA models do not allow us to rank Decision-Making Units (DMUs). In other words, the technical efficiency means "doing the job right" and as discussed in the first four chapters, the meaning of "doing the job right" is different from "doing the job well." At the end of this chapter, readers' knowledge to apply a DEA model and to interpret the results is significantly increased.

In Chap. 8, mathematical properties to describe the natural relationships between the DEA frontier and the ratio of output to input factors are illustrated. It is shown that the deficiency of ratio analysis fails to identify all types of dominating units unlike DEA. The conventional DEA models are discussed in detail, and the computer programming to solve these models is presented step by step.

A production-planning problem is discussed in Chap. 9. Two planning ideas with a centralized decision-making unit are discussed to arrange new input and output plans for all specific units when demand deviations can be predicted. The detained mathematical information is also demonstrated as well as solving the models with Microsoft Excel Solver and Visual Basic for Application (VBA) software.

Chapter 10 illustrates a context-dependent DEA model. The performance of firms can be influenced by the context and a product can appear attractive in comparison within the context of less attractive alternatives. An example of twenty-three Tokyo public libraries is considered and the attractiveness score of each library is measured. It is illustrated that when DMUs in a particular level of measured frontiers are observed as having the same performance, the attractiveness measure allows discrimination of the "equal performance" based upon the third option or the same particular evaluation context. The computer programming is also developed to run the models with just one click.

In Chap. 11, radial and a nonradial DEA Malmquist efficiency indexes are introduced to measure the performance of DMUs in different time periods. The Malmquist indexes are decomposed into three components and each component is illustrated. An example from the computer industry, a set of Fortune Global 500 Computer and Office Equipment companies from 1991 to 1997, is also considered and the computer instructions to solve the models are provided. The radial and nonradial models are also applied to calculate the efficiency change and the impact of economic development plans on efficiency changes of three Chinese major industries: (1) textiles, (2) chemicals, and (3) metallurgy during five-year-plan in four different periods, from 1966 to 1985.

In the last chapter, the introduced mathematical model in Chap. 6 is mathematically improved to fairly rank firms in various conditions. The chapter is finished with several outcomes of the model, which definitively resolves several arguments.

At the end of this book, the user can independently follow the literature of Operations Research, comprehend the specifications of linear programming, and solve a linear programming model using VBA.

We gratefully thank our friends who have supported us to write this book with their hope and love during the last 7 months. Without their love and support, it would have been very difficult to finish the book in this short period.

This is the first edition of this book and it is possible there might be some typos. Please do not hesitate to contact the authors at the following email address, if there are any points, correction, or questions, khezrimotlagh@gmail.com.

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# Contents

# Abbreviations and Symbols

## Abbreviations

| | |
|---|---|
| ADD | Additive Model |
| CCR | Charnes, Cooper and Rhodes |
| CRS | Constant Returns to Scale |
| CRS-PPS | Constant Returns to Scale Production Possibility Set |
| CE | Cost Efficiency |
| DEA | Data Envelopment Analysis |
| DRS | Decreasing Returns to Scale |
| $\delta$-KAM | Delta-Kourosh and Arash Model |
| ERM | Enhanced Russell Measure |
| EPA | Environmental Protection Agency |
| FDH | Free Disposal Hull |
| IRS | Increasing Returns to Scale |
| IO | Input Orientation |
| ICAO | International Civil Aviation Organization |
| KAM | Kourosh and Arash Method |
| NDRS | Non-Decreasing Returns to Scale |
| NIRS | Non-Increasing Returns to Scale |
| NO | Non-Orientation |
| Non-TE | Non-Technical Efficiency |
| Non-VRS-TE | Non-Variable Returns to Scale Technical Efficiency |
| OO | Output Orientation |
| PPS | Production Possibility Set |
| PE | Profit Efficiency |
| RS | Returns to Scale |
| RE | Revenue Efficiency |
| SBM | Slack-Based Measure |
| TE | Technical Efficiency |
| VRS | Variable Returns to Scale |

VRS-PPS Variable Returns to Scale Production Possibility Set
VRS-TE Variable Returns to Scale Technical Efficiency

## Symbols

$n$ number of firms
$m$ number of input factors
$p$ number of output factors
$i$ index of firms
$j$ index of input factors
$k$ index of output factors
$l$ index of specific firm whose performance is being assessed
$F_i$ firm number $i$
$x_{ij}$ non-negative observed amount of $j$th input factor of $F_i$
$y_{ik}$ non-negative observed amount of $k$th output factor of $F_i$
$\lambda_i$ multipliers used for computing linear combinations of firms' factors
$s_{ij}^-$ non-negative slack or potential reduction of $j$th input factor of $F_i$
$s_{ik}^+$ non-negative slack or potential increase of $k$th output factor of $F_i$
$W_{ij}^-$ positive specified weight or price for $j$th input factor of $F_i$
$W_{ik}^+$ positive specified weight or price for $k$th output factor of $F_i$
$\lambda_i^*$ optimal multipliers to identify the reference sets for $F_l$
$s_{ij}^{-*}$ optimal slack to identify an excess utilization of jth input factor of $F_i$
$s_{ik}^{+*}$ optimal slack to identify a shortage utilization of kth output factor of $F_i$
$\delta$ non-negative real number
$\Delta$ vector of non-negative real numbers with $m + p$ components
$x_{ij}^*$ target for jth input factor of $F_i y_{ik}^*$
target for kth output factor of $F_i$
$KA_\Delta^*$ KAM optimal score regarding the value of delta
$\mathbb{R}_+$ the positive real numbers set
$T_C$ PPS with CRS technology
$T_C^{+\delta}$ Extension of PPS with CRS technology regarding the value of delta
$T_V$ PPS with VRS technology
$T_V^{+\delta}$ Extension of PPS with VRS technology regarding the value of delta

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# Author Biography

**Yao Chen** is Professor of Operations Management, Manning School of Business, University of Massachusetts at Lowell. She is also a distinguished Professor in College of Auditing and Evaluation, Nanjing Audit University, China. Her current research interests include efficiency and productivity issues of information systems, information technology's impact on operations performance, and methodology development of data envelopment analysis. Her researches are published in journals such as *European Journal of Operational Research*, *OMEGA*, *Computers and Operations Research*, *Annals of Operations Research*, *International Journal of Production Economics*, and others.

**Dariush Khezrimotlagh** is Assistant Professor of Computer Science and Mathematics at Penn State University, Harrisburg, Pennsylvania. He was a visiting assistant professor at the College of Arts and Science at Miami University, Oxford, Ohio. He was previously an assistant professor of Statistics at the University of Malaya, Malaysia, and visiting researcher at University Technology Malaysia, Center for Industrial and Applied Mathematics (UTM-CIAM). He served as the director of the Studies and Information Center, Commercial Aviation Organization in Iran, and has more than 20 years of experience as an educator. He earned his MS in Pure Mathematics and his PhD in Applied Mathematics, specializing in operations research and data envelopment analysis (DEA). His most recent research areas include DEA, performance evaluation, benchmarking, ranking, and statistics.

# Chapter 1
# The Gemstone Example

## 1.1 Introduction

In this chapter, a simple example is provided to explain several concepts which are important to measure the performance of a set of homogenous factories, organizations, firms and so on. No computer knowledge is needed in this chapter, but a very elementary knowledge of mathematics will be required, such as sketching a line in a plane. The section is started with a question about a jewelry company which would like to evaluate 17 candidates according to the results of a task. The concepts are logically discussed to pave the way to introduce different aspects of the company and to establish the corresponding mathematical foundation for further complex questions on measuring the performance of firms. Several figures are provided to elucidate the concepts. Of course, the provided concepts are not new and have been mentioned in the literature of Economics, Management, Engineering and Operations Research; however, in order to avoid any confusion, these concepts are re-introduced and the pros and cons of each concept are generally illustrated. At the end of this chapter, readers will have learned the elementary requirements and concepts of measuring the performance evaluation of firms.

## 1.2 The Gemstone Example

Suppose that 17 eligible candidates, labeled by A–Q, are being considered to be dealers for a jewelry company specializing in unique gemstones. In order to find the best dealers, the company sends the candidates to Sri Lanka to test their dealing skills. In one determining task, the company gives the same amount of money to each candidate to buy a gemstone and sell it without re-cutting/re-polishing. Each candidate spent a different amount of time and earned a different amount of money.



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Table 1.1 represents the used time in minutes and the earned money in $ for each candidate. Which one of the candidates completed this task well/better?

Generally, there is not a grading system to claim how much time should be used to buy or sell a gemstone and earn an expected amount of money. Different gemstones have different values according to their color, cut, clarity and carats. However, one may assume that candidates who used lesser amount of time in this task have done the job well, without paying attention to the possible shortage amount of the earned money. Another may assume that candidates who earned higher amount of money have done the job well, without paying attention to the possible excess amounts of the used time. In both these views, a part of information is neglected and the decisions are not strong enough. Thus, the candidates in this task are discriminated by finding who has used the least time to earn the highest amount of money, according to the following definition.

**Definition 1.1** A candidate, who has done the job well in comparison with another candidate, should have used lesser amount of time and earned greater amount of money in comparison with the other.

One simple measure for finding such an aim is to calculate the ratio of the earned money to the used time given by the following equation:

$$\frac{\text{Earned (\$)}}{\text{Time (min)}} \tag{1.1}$$

**Definition 1.2** A candidate has done the job well in comparison with another candidate, if it has a greater value of Eq. 1.1.

In order to have a greater value of Eq. 1.1, the four following types can be illustrated.

**Type 1** *If only the denominator in Eq.* 1.1 *decreases*, (that is, the used time is decreased while the earned money is fixed), *the ratio increases*.

For instance, candidates B and I earned $1500, whereas B used 380 min and I used 800 min, that is, B used 420 min less than I to earn the same amount of money.

**Table 1.1** The record data for candidates of gemstones

| Candidate | Time (min) | Earned ($) | Candidate | Time (min) | Earned ($) |
|---|---|---|---|---|---|
| A | 400 | 1600 | J | 1300 | 1780 |
| B | 380 | 1500 | K | 700 | 900 |
| C | 420 | 1580 | L | 1100 | 1300 |
| D | 450 | 1550 | M | 200 | 200 |
| E | 500 | 1620 | N | 1800 | 1800 |
| F | 220 | 700 | O | 1800 | 1600 |
| G | 500 | 1200 | P | 1300 | 700 |
| H | 600 | 1400 | Q | 400 | 200 |
| I | 800 | 1500 | | | |

Therefore, the scores of B and I by Eq. 1.1 are 3.95 ($/min.) and 1.88 ($/min.), respectively, which shows the score of B is greater than the score of I, and therefore, B has *done the job well* in comparison with I.

**Type 2** *If only the numerator in Eq.* 1.1 *increases*, that is, the earned money is increased while the used time is fixed, *the ratio increases*.

For example, both candidates E and G used 500 min, but E earned $1620 and G earned $1200, that is, E earned $420 more than G by using the same amount of time. Therefore, the scores of E and G by Eq. 1.1 are 3.24 ($/min.) and 2.40 ($/min.), respectively, which shows that E has a greater score, and thus, has done the job well in comparison with G.

**Type 3** *If the rate of increasing the numerator in Eq.* 1.1 *is greater than the rate of increasing the denominator, the ratio increases*.

For instance, M uses 200 min to earn $200, but F uses 220 min (20 min extra than M, that is, the rate of increasing the denominator is almost 0.091) and earn $700 ($500 extra than M, that is, the rate of increasing the numerator is almost 0.714). Therefore, the values of Eq. 1.1 for M and F are 200/200 (1 $/min.), 700/220 (= 3.18 $/min.), respectively, which shows the score of F is greater than the score of M, and this means F has done the job well in comparison with M.

**Type 4** *If the rate of decreasing the numerator in Eq.* 1.1 *is greater than the rate of decreasing the denominator, the ratio increases*.

For instance, N uses 1800 min to earn $1800, but J uses 1300 min (500 min less than N, that is, the rate of decreasing the denominator is almost negative 0.384) and earn $1780 ($20 less than N, that is, the rate of decreasing the numerator is almost negative 0.011). Note that $-0.011 > -0.384$. Therefore, the value of Eq. 1.1 for N is 1800/1800 (= 1 $/min.), whereas it is 1780/1300 (=1.37 $/min.) for J, which shows that the score of J is greater than the score of N, and this means J has done the job well in comparison with N.

Table 1.2 illustrates the results of calculating Eq. 1.1 for each candidate. From the table, A has done the job well in comparison with other candidates, followed

**Table 1.2** The scores of candidates

| Candidate | Time (min) | Candidate | Time (min) |
|---|---|---|---|
| A | 4 | J | 1.37 |
| B | 3.95 | K | 1.29 |
| C | 3.76 | L | 1.18 |
| D | 3.44 | M | 1 |
| E | 3.24 | N | 1 |
| F | 3.18 | O | 0.89 |
| G | 2.4 | P | 0.54 |
| H | 2.33 | Q | 0.5 |
| I | 1.88 | | |

by B–Q. Indeed, candidate A only uses 400 min to earn $1600, and has the best score, whereas candidate Q with using 400 min only earned $200, and has the worst score among other candidates.

## 1.3 Factors and the Cartesian Coordinate Plane

The used time in Table 1.1 is a *factor* that a lesser amount of this factor has more value while the earned money is fixed (Type 1), whereas the earned money in Table 1.1 is a factor that a higher amount of this factor has more value while the used time is fixed (Type 2). Although a lesser amount of the used time is desirable, both the used time and the earned money can simultaneously increase as long as the rate of increasing the used time is less than the rate of increasing the earned money (Type 3). Moreover, a greater amount of the earned money is desirable; however, a company can simultaneously decrease both the used time and the earned money as long as the rate of decreasing the used time is greater than the rate of decreasing the earned money (Type 4). If any of these four types are neglected the discrimination power decreases.

**Definition 1.3** When a company desires a lesser (greater) amount for a factor, the factor is called *input* (*output*) factor.

From definition 1.3, Eq. 1.1 can be rewritten as follows:

$$\frac{\text{Output}}{\text{Input}} \tag{1.2}$$

Each candidate in Table 1.1 has two factors which can be shown with a pair (input, output) and depicted in a Cartesian coordinate plane similar to Euclidean geometry.

For example, the value of input factor for candidate A is 400 and the value of output factor is 1600, which can be shown with the pair (400, 1600), and depicted as a point in a plane. Figure 1.1 depicts the point A (400, 1600).

The horizontal axis in Fig. 1.1 shows the values of input factor in minutes and the vertical axis shows the values of output factor in dollars. The dotted lines elucidate how the pair (400, 1600) is depicted in the Cartesian coordinate plane.

By the same technique to show candidate A as a point in the plane, other candidates B–Q can also be depicted in the plane. Figure 1.2 illustrates candidates A–Q in the Cartesian coordinate plane.

From the figure, why is the score of A, greater than the scores of other candidates? Why is the score of B greater than the score of F? Why is the score of J less than the score of D? Why does F have a score less than D? Why does H have a greater score than J, M and N? How are the measured scores of candidates in Table 1.2 clarified geometrically?

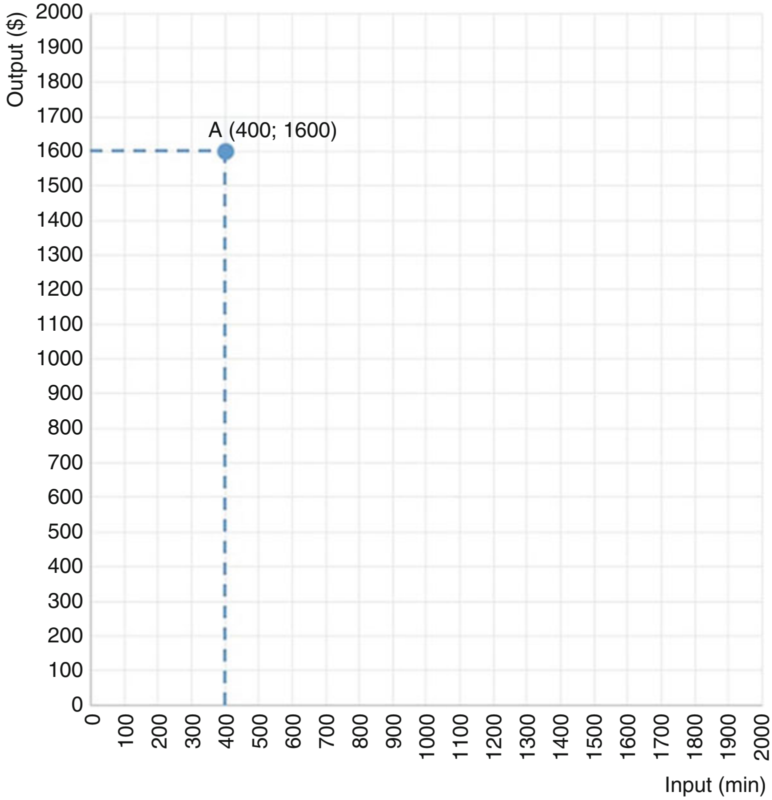


**Fig. 1.1** Depicting the candidate A

For the first step to answer the previous questions, let's compare volunteers M (200, 200) and F (220, 700). As it is explained in Type 3 in the first section, the rate of increasing the value of input factor from 200 to 220 is much less than the rate of increasing the value of output factor from 200 to 700.

Figure 1.3 depicts that F by using 10% extra minutes earned $250% more than M. Therefore, the company knows that the performance of F is better than M, due to the fact that, increasing 10% of the input factor is reasonable while it causes an increase of 250% the output factor, which means the ratio of output/input increases.

For another view, let us sketch the lines which pass origin (0, 0) and F, and origin and M. As can be seen in Fig. 1.4, the scores of F and M in Table 1.2 are the same as the slopes of the sketched lines in Fig. 1.4 which passes F and M, respectively. Indeed, 1 min increase on the value of input factor in the line which passes origin and M, yields only an one dollar increase on the value of output factor, whereas

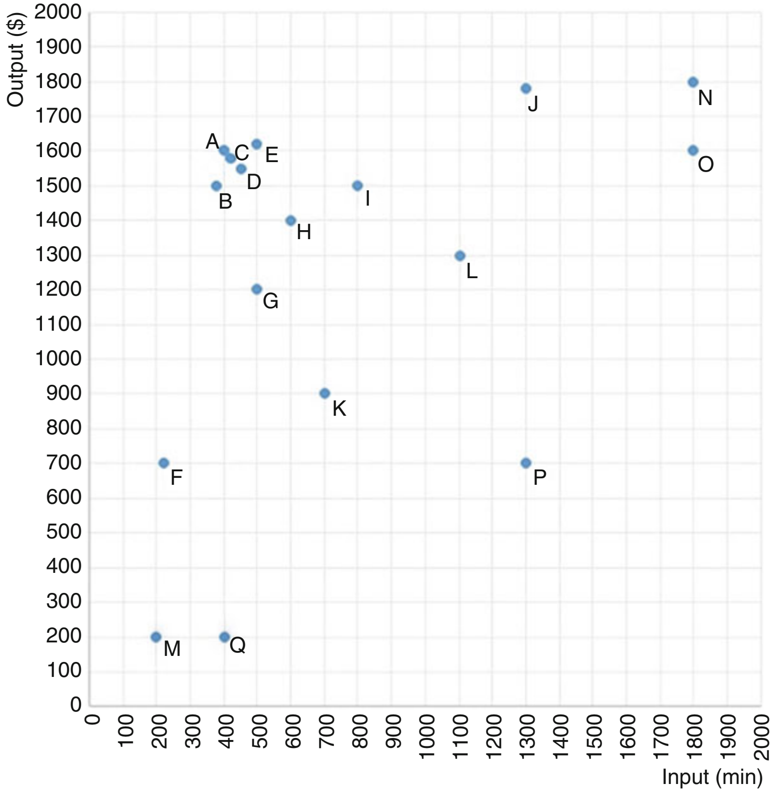


**Fig. 1.2** Depicting the candidates A–Q

1 min increase on the value of input factor in the line, which passes origin and F, yields 3.18 dollars increase on the value of output factor. This shows the worth of the used time by F via M.

Figure 1.5 also shows the line which passes (0, 0) and J (1300, 1780), and the line which passes (0, 0) and N (1800, 1800). The slopes of the lines in Fig. 1.5 are also the same as the scores of J and N in Table 1.2, and represent the worth of the used time by J via N. As depicted in Fig. 1.6, if N decreases 28% of the used time from 1800 to 1300, it only loses 1% of the earned money from 1800 to 1780. In other words, the rate of decreasing the value of output factor (= −1%) is much greater than the rate of decreasing the value of input factor (= −28%) (Type 4), and therefore, the performance of J is better than N. Note that, N used 500 min more than J in this task to earn only $20 more, whereas 500 min are equal or more than the used time that A, B, C, D, E, F, M and Q used to earn $1600, $1500, $1580, $1550, $1620, $700, $200 and $200, respectively.

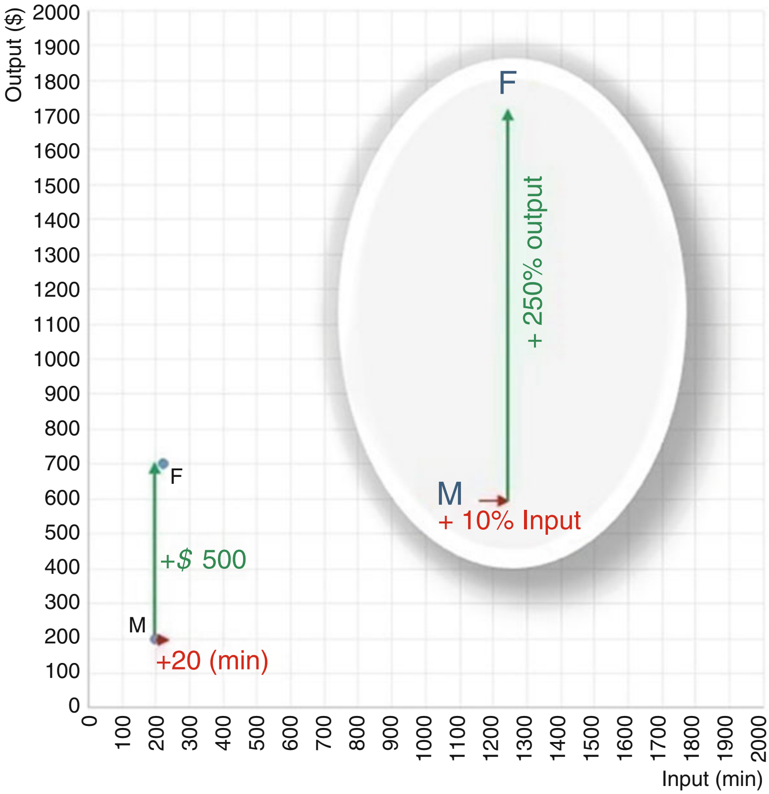


**Fig. 1.3** Comparing the job of F and M

## 1.4 The Partially/Wholly Dominant Concept

Figure 1.7 depicts the line which passes origin (0, 0) and K(700, 900), that is, $y = (9/7)\,x$. According to Table 1.2, the points in the right side of the line in Fig. 1.7, that is, L–Q, have the lesser scores than K, and the points in the left side of the line, that is, A–J, have the greater scores than K.

Indeed, the line which passes origin and K says that 1 min increasing on the value of input factor yields 1.29 dollars increasing on the value of output factor, whereas for the lines which pass the origin and the points in the right side of the line in Fig. 1.7, 1 min increasing on the value of input factor, yields lesser amount of dollars on increasing the value of output factor. Therefore, the weak performance of L–Q (the points in the right side of the line) can be shown by the performance of K. This phenomenon is read "*K partially dominates L-Q*".

Likewise, the weak performance of K can be shown by the stronger performance of A–J, which this phenomenon is read "*K is partially dominated by A–J*".

From the above illustration the following definition is suggested.

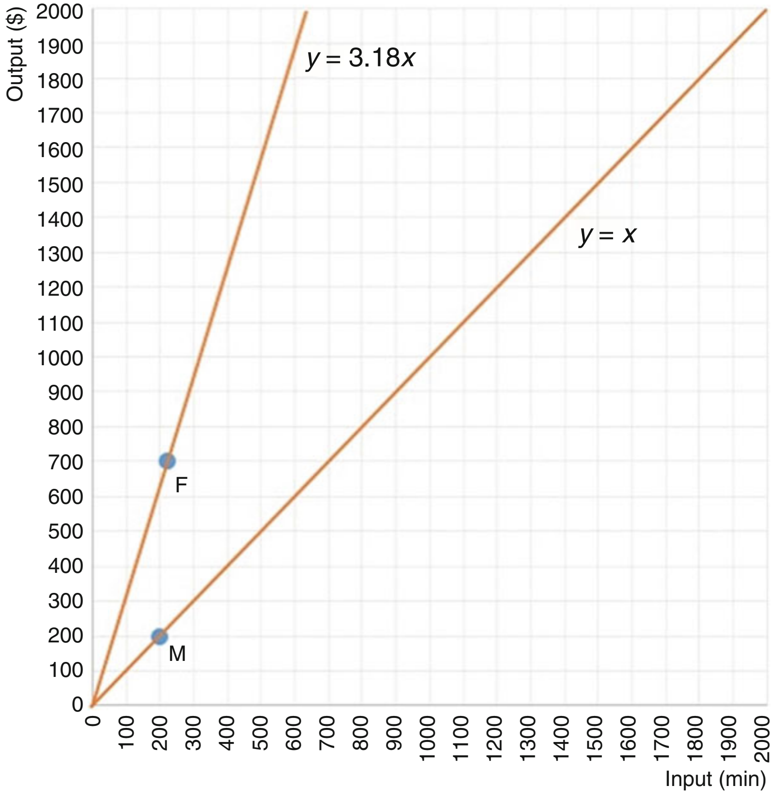


**Fig. 1.4** The lines which passes origin and F and M

**Definition 1.4** A candidate partially dominates another candidate, if the candidate has done the job well in comparison with the other candidate.

On the other hand, as can be seen in Fig. 1.7 as well as Table 1.1, A–Q used different values of input and output factors in comparison with K. For instance, the values of input and output factors of M are less than that of K, whereas the components of L are greater than the components of K. Every point in the plane can be shown with a pair $(x, y)$, where $x$ displays the value of input factor and $y$ displays the value of output factor. The components of K introduce four different areas by the following Eqs. 1.3, 1.4, 1.5, and 1.6, as Fig. 1.8 represents.

$$700 \leq x \quad \& \quad y \leq 900. \tag{1.3}$$

$$x < 700 \quad \& \quad y < 900. \tag{1.4}$$

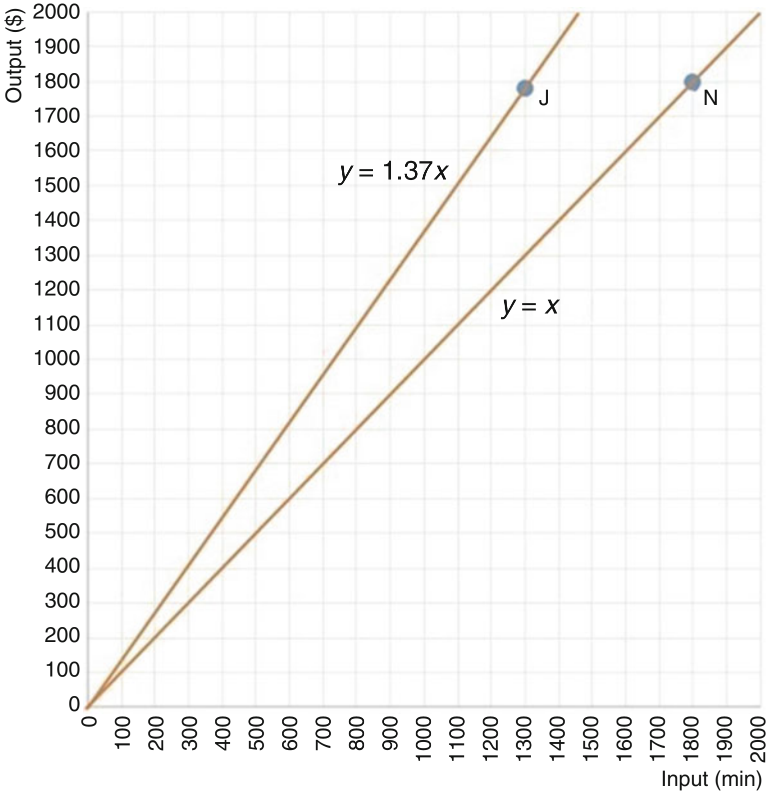


**Fig. 1.5** The lines which passes origin and J and N

$$700 < x \quad \& \quad 900 < y. \tag{1.5}$$

$$x \leq 700 \quad \& \quad 900 \leq y. \tag{1.6}$$

Equation 1.3 displays the points in the area in which the value of input factor is equal or greater than 700 and the value of output factor is equal or less than 900, which is the right bottom shaded rectangle, as shown in Fig. 1.8.

Indeed, when the first component of $(x, y)$, that is, $x$, is equal or greater than 700, the following inequality is satisfied, $700 \leq x$. Likewise, when the second component of $(x, y)$, that is, $y$, is equal or less than 900, the following inequality is satisfied, $y \leq 900$. A point $(x, y)$ which satisfies both inequalities in Eq. 1.3 belongs to the right bottom rectangle in Fig. 1.8. From Table 1.1, only P (1300, 700) satisfies the Eq. 1.3, because $x = 1300$ satisfies the first inequality, that is, $700 \leq x$, and $y = 700$ satisfies the second inequality, that is, $y \leq 900$.

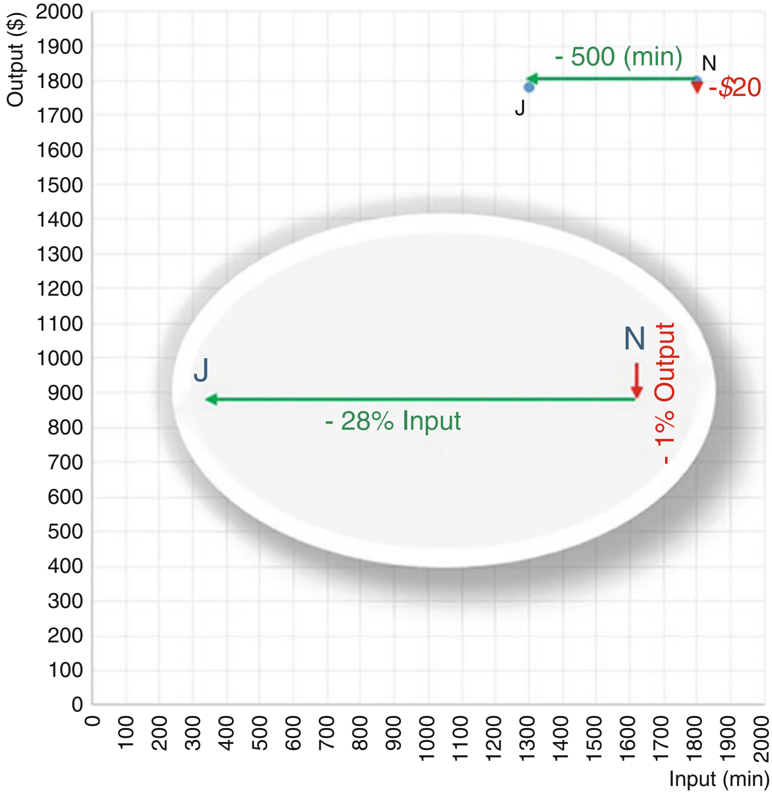


**Fig. 1.6** Comparing the job of J and N

On the other hand, Eq. 1.6 displays the points in the area that the value of input factor is not greater than 700 (which means, the value of input factor is equal or less than 700, that is, $x \leq 700$) and the value of output factor is not less than 900 (which means, the value of output factor is equal or greater than 900, that is $900 \leq y$). The area which is displayed by Eq. 1.6 is the shaded rectangle in the left top in Fig. 1.8. According to Table 1.1, the components of A–E, G and H satisfy the inequalities in Eq. 1.6. For instance, H (600, 1400) has less than 700 value of the input factor (that is, $600 \leq 700$) and greater than 900 value of the output factor (that is, $900 \leq 1400$).

Equation 1.4 represents the points $(x, y)$ that the values of input and output factors of those points are less than the values of input and output factors of K, respectively. However, Eq. 1.5 represents the points $(x, y)$ which the values of their input and output factors are greater than the values of input and output factors of K, respectively. The left bottom and the right top rectangles in Fig. 1.8 are displayed by Eqs. 1.4 and 1.5, respectively.

As a result, the points which satisfy both inequalities in Eq. 1.3 are completely in the area which is partially dominated by K. The points which do not satisfy both inequalities in Eq. 1.3, that is, the points which satisfy the inequalities in Eq. 1.6, are

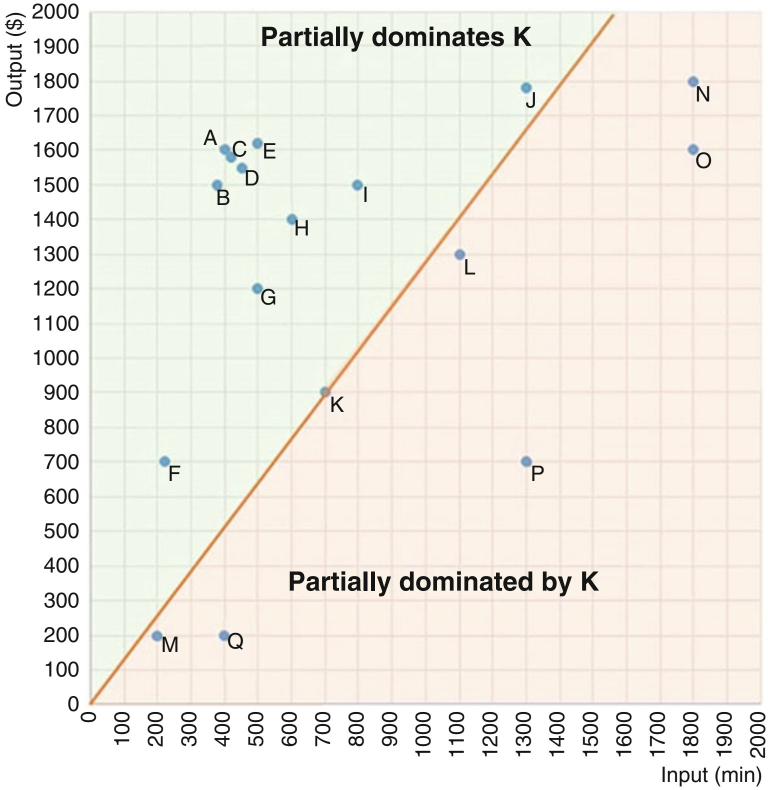


**Fig. 1.7** The performance of K

completely in the area that partially dominates K. However, the points which only satisfy one of the inequalities in Eq. 1.3 may partially dominate K or be partially dominated by K. Therefore, as Fig. 1.9 represents, 6 different areas can be introduced.

*Region (1)*: In the gemstone example, candidate P (1300, 700) does not get a higher score than K (700, 900), because the company can claim that P with 600 extra values of input factor has earned 200 lesser values of output factor in comparison with K, as shown in Fig. 1.10.

This phenomenon is read as '*K wholly dominates P*' or '*K dominates P*'. All the points in Region (1) satisfy the two following inequalities

$$700 \leq x \quad \& \quad y \leq 900.$$

In other words, K wholly dominates every point which has greater than 700 values of input factor and less than 900 values of output factor (Fig. 1.11).

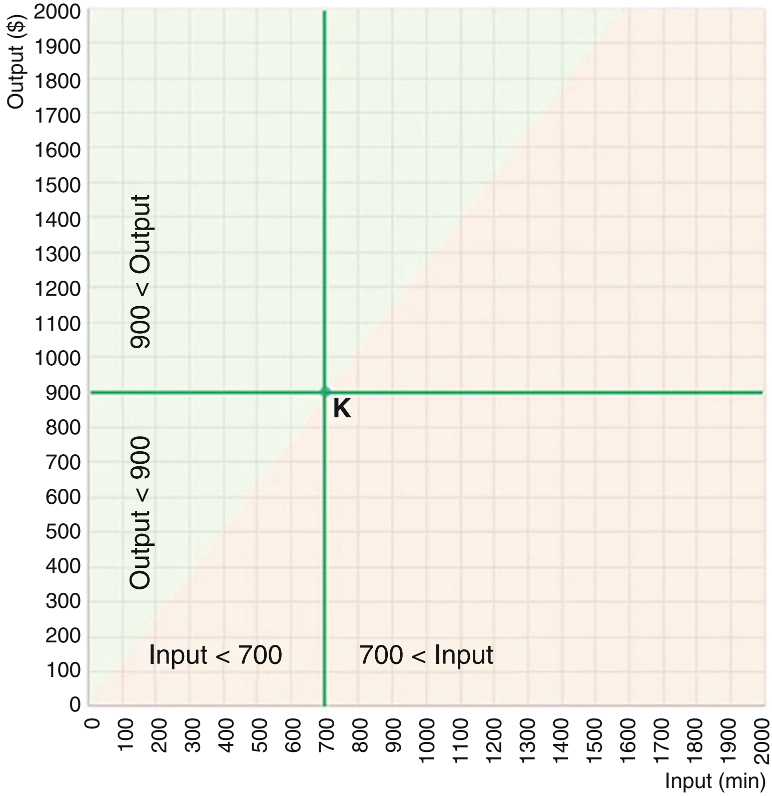


**Fig. 1.8** The performance of K

*Region (2)*: Likewise, candidate C (420, 1580) gets a higher score than K (700, 900), because the company can claim that C with 280 lesser values of input factor has earned 680 extra values of output factor in comparison with K, as shown in Fig. 1.12. This phenomenon is read as '*K is wholly dominated by C*' or '*K is dominated by C*'.

Indeed, K is wholly dominated by every point which has less than 700 values of input factor, and has greater than 900 values of output factor.

This means, K is wholly dominated by the points which satisfy the two following inequalities, as Fig. 1.13 illustrates

$$x \leq 700 \quad \& \quad 900 \leq y,$$

The following definition illustrates the concept of wholly dominant.

**Definition 1.5** A candidate wholly dominates another candidate, if the candidate does not have greater values of input factors and lesser values of output factors in comparison with that of the other candidate.

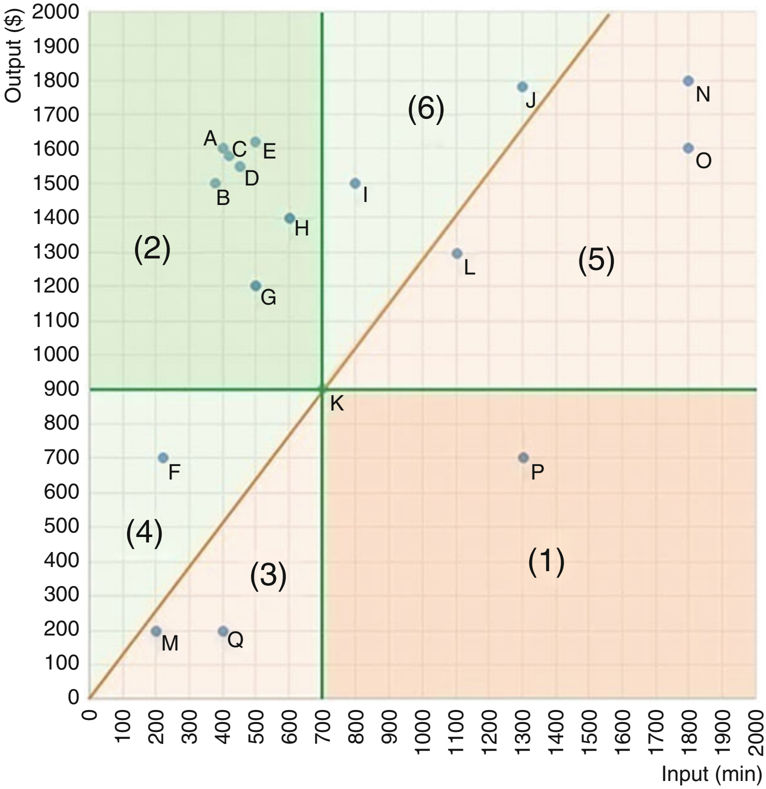


**Fig. 1.9** The performance of K

*Regions (3) and (4)*: The points in Regions (3) and (4) do not satisfy the inequalities in Eqs. 1.3 and 1.6, but satisfy the inequalities in Eq. 1.4.

In other words, the points in these regions in both factors have less values than the factors of K, that is, every point in Regions (3) and (4) satisfies the two following inequalities

$$x < 700 \quad \& \quad y < 900.$$

However, the points in Region (3) are partially dominated by K, whereas the points in Region (4) partially dominate K, as depicted in Fig. 1.7.

For instance, Q (400, 200) has used 300 values of input factor less than that of K, but earned 700 values of output factor less than that of K as well (Fig. 1.14), that is, the rate of decreasing the output factor ($\approx -0.78$) is not greater than the rate of decreasing the input factor ($\approx -0.43$), and hence Q has a score less than K (Type 4), and K has done the job well in comparison with Q.

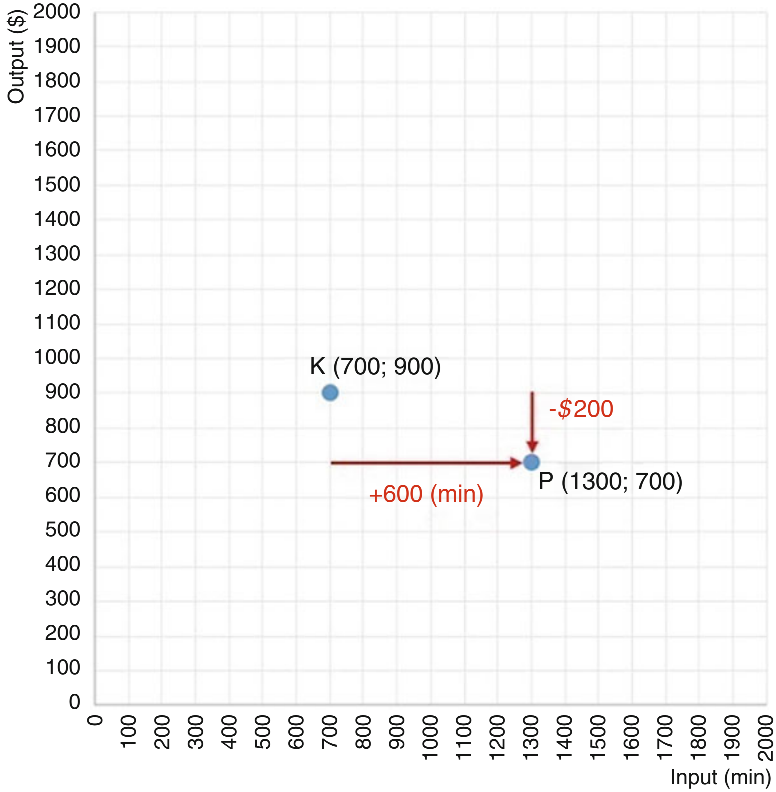


**Fig. 1.10** The performance of K via P

In contrast, F (220, 700) has used 480 values of input factor less than that of K and earned 200 values of output factor less than that of K (Fig. 1.15), that is, the rate of decreasing the output factor ($\approx -0.22$) is greater than the rate of decreasing the input factor ($\approx -0.69$), and hence the score of F is greater than the score of K, and F has done the job well in comparison with K.

Figure 1.16 illustrates the generated area by Eq. 1.4, and the points that partially dominate K as well as those are partially dominated by K. Therefore, the points in Regions (3) and (4), that is, the points which satisfy the inequalities in Eq. 1.4, that is, $x < 700$ & $y < 900$, can be discriminated by measuring the rate of decreasing each factor and comparing the rates according to Type 4.

Since both factors decrease, the rate of each factor is a negative number, hence the ratio of the rate of output factor over the rate of input factor for a point in Regions (3) and (4) is a non-negative number. If the ratio is greater than 1, the point is in Region (3) and if the ratio is between 0 and 1, the point is in Region (4).

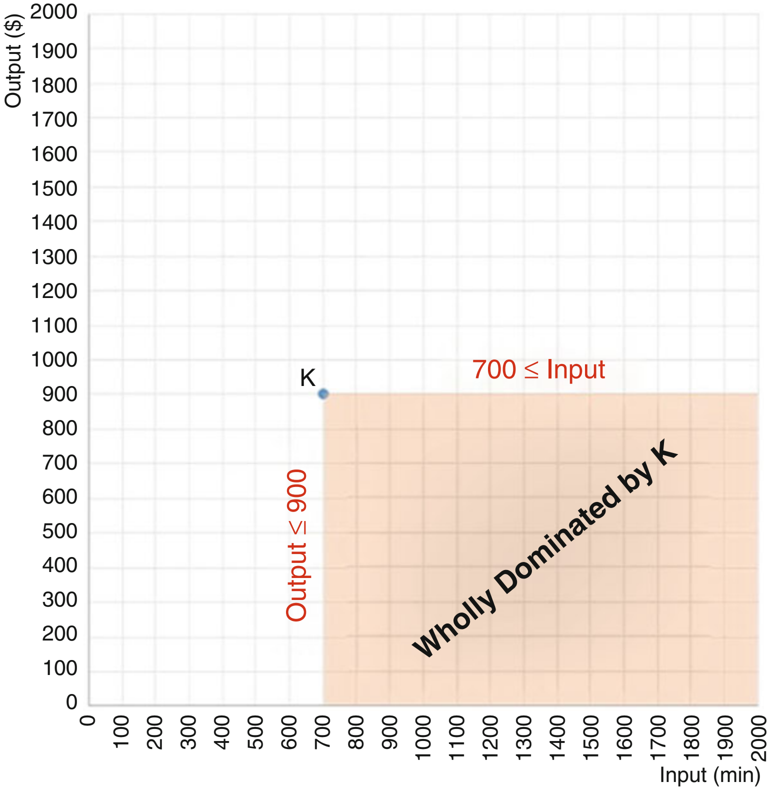


**Fig. 1.11** Region (1)

*Regions (5) and (6)*: Similar to the discussion for Regions (3) and (4), the points in Regions (5) and (6) can be discriminated by comparing the rates of increasing input and output factors (Type 3). Indeed, the points in Regions (5) and (6) have greater values of input and output factors via K (Fig. 1.17).

These points satisfy the inequalities in Eq. 1.5, that is, $700 < x$ & $900 < y$. Therefore, the ratio of the rate of output factor over the rate of input factor for a point in Regions (5) is less than 1, and in Region (6) is greater than 1.

For instance, O (1800, 1600) via K has used 1100 extra values of input factor to earn 700 extra values of output factor. Therefore, the rate of increasing output factor ($\approx 0.78$) is less than the rate of increasing input factor ($\approx$1.57), and K has done the job well in comparison with O (Fig. 1.18).

In contrast, I (800, 1500) has used 100 extra values of input factor and earned 1500 extra values of output factor, that is, the rate of increasing the output factor is almost 0.67 whereas the rate of increasing the input factor is almost 0.14, and I has done the job well in comparison with K (Fig. 1.19).

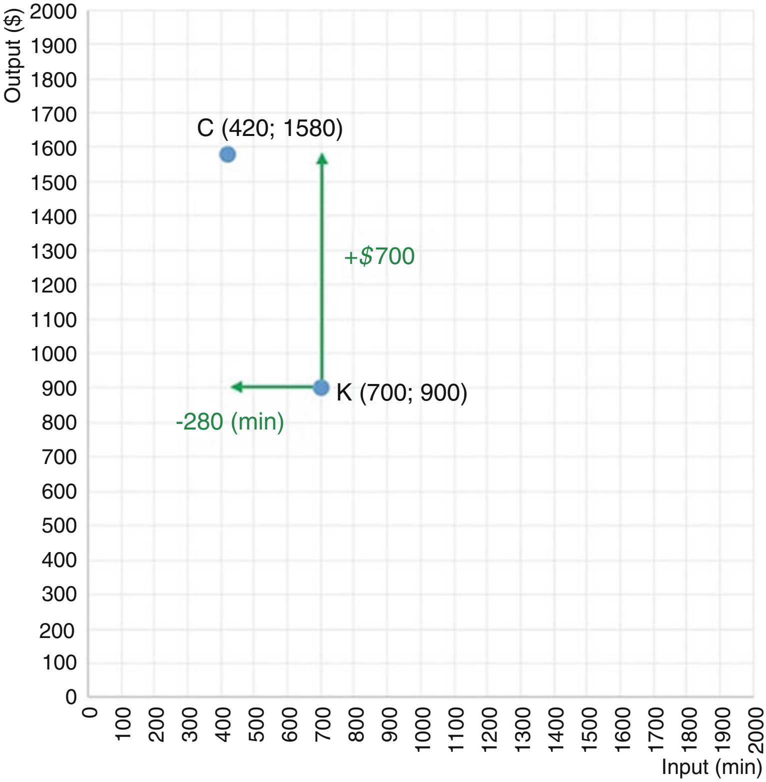


**Fig. 1.12** The performance of K via C

## 1.5 The Rank of the Candidates

As illustrated in the previous section, the wholly dominant concept is not enough to discriminate each two of candidates fairly. For instance, the candidates which are in regions (3)–(6) are not compared by the wholly dominant definition. But, the partially dominant concept can be applied for each candidate and discriminate the candidate with all other candidates according to Types (1)–(4), in order to arrange/ rank them and find who has done the job well in comparison with the other. Indeed, a candidate who partially dominates all other candidates has done the job well in comparison with all other candidates, and should have a greater score than other candidates, and hence the candidate should get the highest rank. Likewise, a candidate who is partially dominated by all the other candidates should get the lowest rank.

For instance, candidate A (400, 1600) gets the best rank among the other candidates, because A partially dominates candidates B–Q, as Fig. 1.20 depicts.

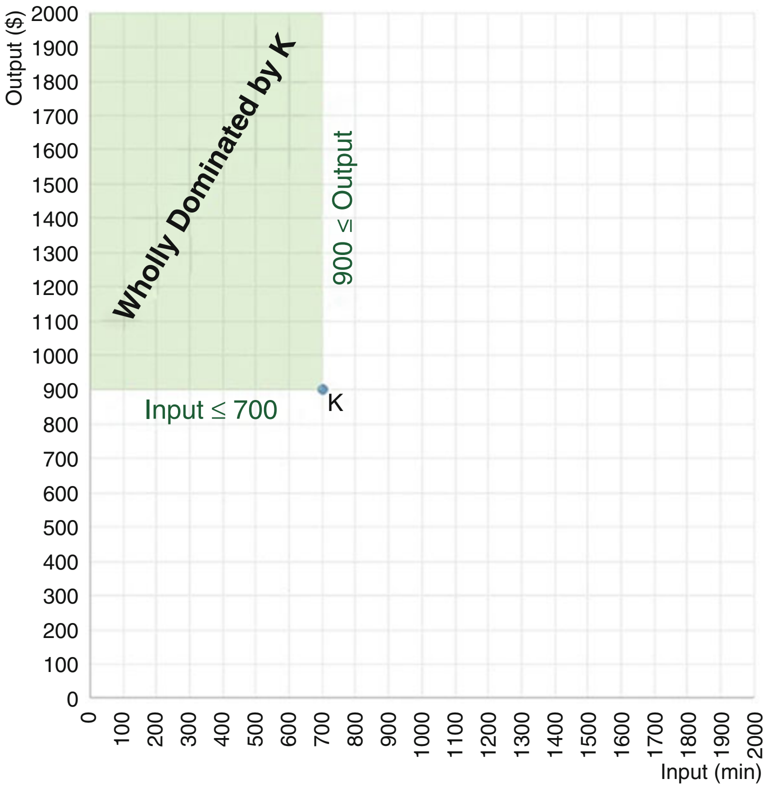


**Fig. 1.13** Region (2)

Candidate A wholly dominates the candidates in Fig. 1.21, and partially dominates the candidates in Fig. 1.22.

As can be seen, the candidates who are wholly dominated by A, are also partially dominated by A, but there might be some candidates such as B and E which are partially dominated by A, but are not wholly dominated by A.

The points $(x, y)$ which are wholly dominated by A(400, 1600) are those points which satisfy the two following inequalities

$$400 \leq x \quad \& \quad y \leq 1600. \tag{1.7}$$

When at least one of the inequalities in Eq. 1.7 is not satisfied, the point is not wholly dominated by A.

For instance, the value of input factor of N satisfies $400 \leq x$ (because, $400 \leq 1800$), but the value of output factor of N does not satisfy $y \leq 1600$ ($1800 \leq 1600$). Therefore N is not wholly dominated by A, but as can be seen in Fig. 1.22, N is partially dominated by A, and A has done the job well in comparison with N.

As Fig. 1.23 depicts, none of the candidates A, B, E, F, J, M and N are wholly dominated by each other. This means, if the areas which are wholly dominated by

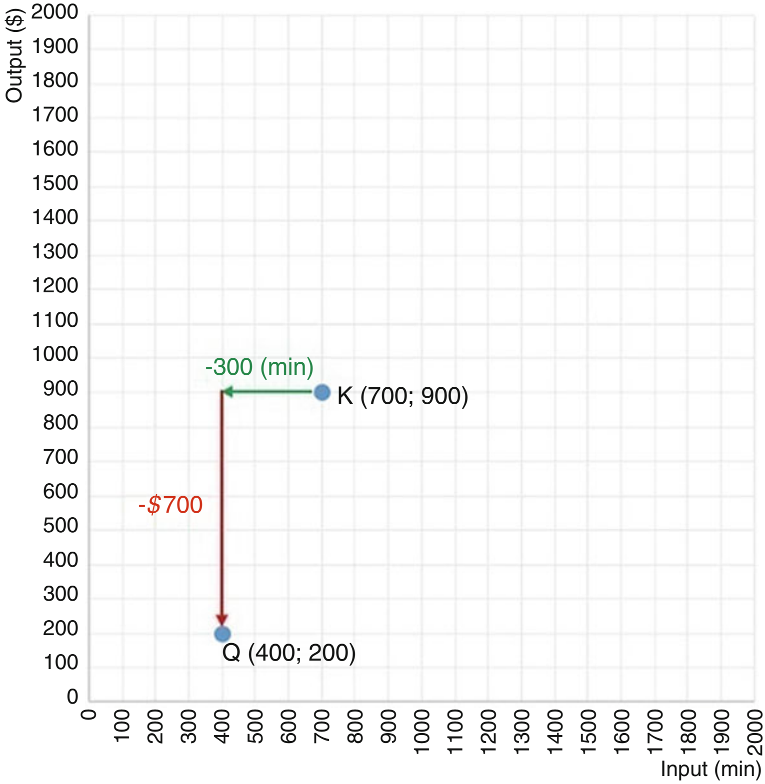


**Fig. 1.14** The performance of K via Q

candidates A, B, E, F, J, M and N are depicted, none of them belong to the other wholly dominated area. However, A partially dominates B, E, F, J, M and N, and B partially dominates E, F, J, M and N, and so on, which illustrates A has done the job well in comparison with B, E, F, J, M and N, and B has done a better job than E, F, J, M and N, and so on.

For example, the shaded rectangle, which depicts the area dominated by M in Fig. 1.23, is made by the inequalities in Eq. 1.8.

$$200 \leq x \quad \& \quad y \leq 200. \tag{1.8}$$

None of A(400, 1600), B(380, 1500), E(500, 1620), F(220, 700), J(1300, 1780) or N(1800, 1800) satisfy Eq. 1.8. Therefore, none of A, B, E, F, J and N are wholly dominated by M. Moreover, the area dominated by B, E, F, J and N are respectively made by the following Eqs. 1.9, 1.10, 1.11, 1.12, and 1.13:

$$380 \leq x \quad \& \quad y \leq 1500. \tag{1.9}$$

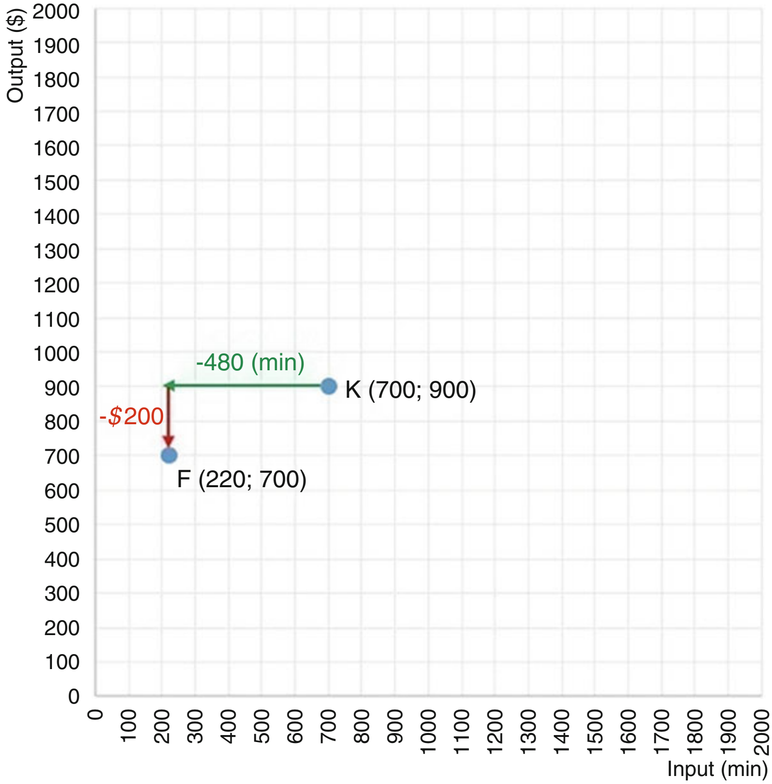


**Fig. 1.15** The performance of K via F

$$500 \leq x \quad \& \quad y \leq 1620. \tag{1.10}$$

$$220 \leq x \quad \& \quad y \leq 700. \tag{1.11}$$

$$1300 \leq x \quad \& \quad y \leq 1780. \tag{1.12}$$

$$1800 \leq x \quad \& \quad y \leq 1800. \tag{1.13}$$

As can be seen, at least one of the coordinates of A, B, E, F, J and M does not satisfy Eq. 1.9, which means none of A, B, E, F, J and M are wholly dominated by N, and so on.

On the other hand, the points which are wholly dominated by A in Fig. 1.21, that is, C, D, G, H, I, K, L, O, P and Q, may be wholly dominated with other points as well. For instance, P is wholly dominated by all points A–L, but P should be compared with the most eligible candidate, A, in order to get a fair score. This

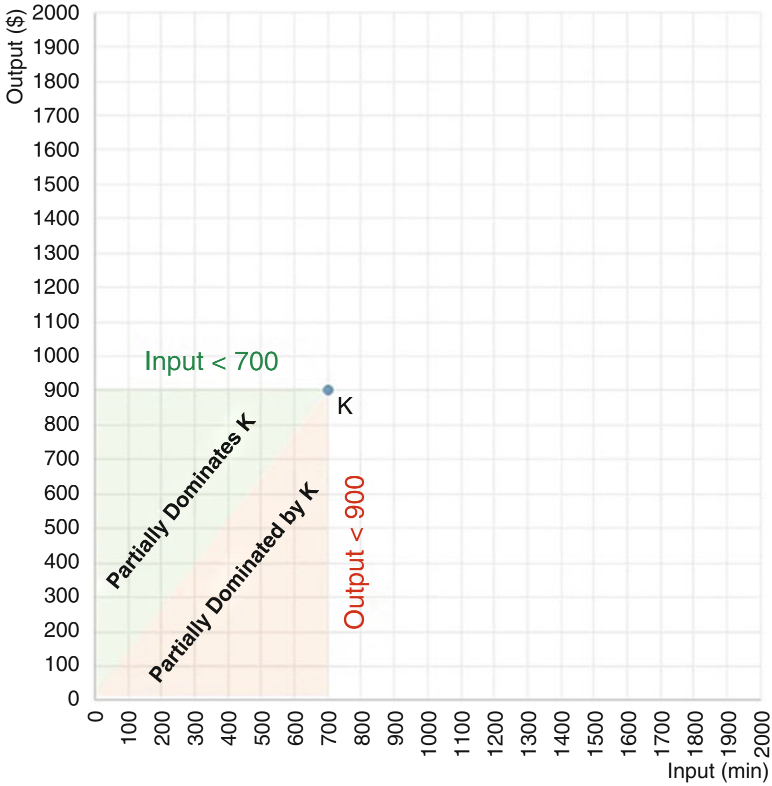


**Fig. 1.16** Regions (3) and (4)

phenomenon provides a very important issue to benchmark the factors of P which will be discussed in the next chapters.

By the same illustration, M only dominates Q, partially dominates O and P and has the same performance as N. In fact, the rate of increasing input factor from 200 min by M to 1800 min by N is the same as the rate of increasing output factor from $200 by M to $1800 by N. Indeed, both M and N partially dominate each other, and therefore, they have the same score for their performances in this task, as Table 1.2 represents.

C is wholly dominated by A, and is partially dominated by B. But, C partially dominates E, F, J, M, N, O and Q, and dominates D, G, H, I, K, L and P. Therefore, the score of C is less than A and B, and greater than D–Q, which means, A and B have done the job well in comparison with C and, C has done the job well in comparison with all other candidates D–Q.

H is wholly dominated by A, B, C, D and E, and is partially dominated by F and G. However, H partially dominates I, J, M, N, O and Q, and wholly dominates K, L and P. Hence, the score of H's performance is less than A–G, and greater than I–Q, which means H has done the job well in comparison with I–Q.

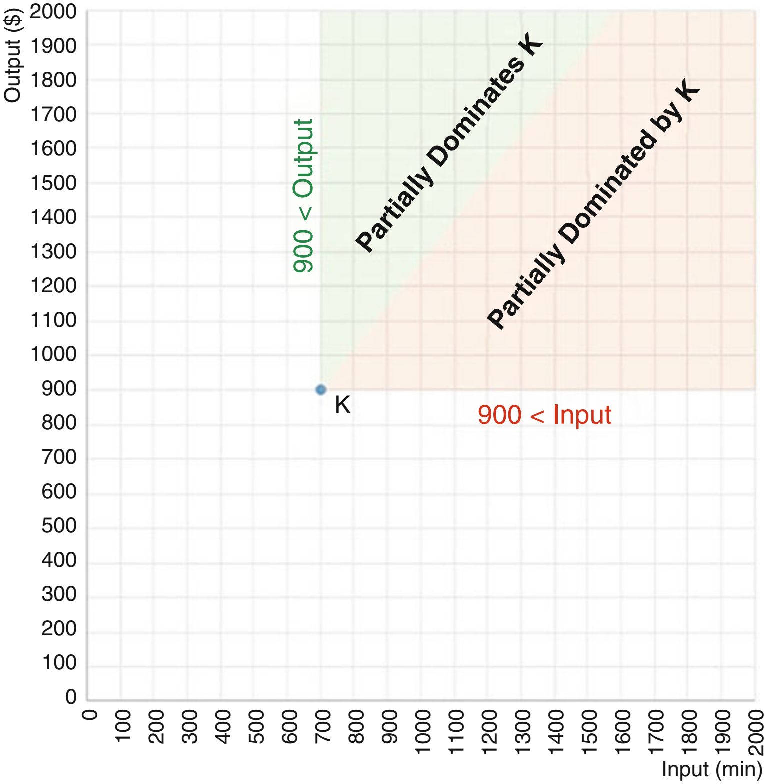


**Fig. 1.17** The performance of K via I

Moreover, a candidate who is wholly dominated by another such as A, may partially dominate a candidate that is not wholly dominated by A. For example, C is wholly dominated by A, which means, A has done the job well in comparison with C. The candidate C partially dominates the candidates F, J, M and N, which means C has done the job well in comparison with the candidates F, J, M and N. As can be seen, the candidates F, J, M and N are partially dominated by A, but are not wholly dominated by A. Therefore, C is wholly dominated by A, but C partially dominates the candidates F, J, M and N who are partially dominated by A (and are not wholly dominated by A). This phenomenon also provides a very important issue to discrimination between candidates which will be discussed in the next chapters.

Table 1.3 illustrates the relationship between each two candidates with three symbols “++”, “+” and “–”, which means, “wholly dominated”, “partially dominated” and “not partially dominated”, respectively.

Every row in Table 1.3 illustrates whether a candidate is wholly/partially dominated by other candidates or not. For instance, the corresponding row for candidate I says that I is wholly dominated by A–E, is partially dominated by F–H, is wholly/partially dominated by itself, and is not wholly/partially dominated by the other candidates J–Q.

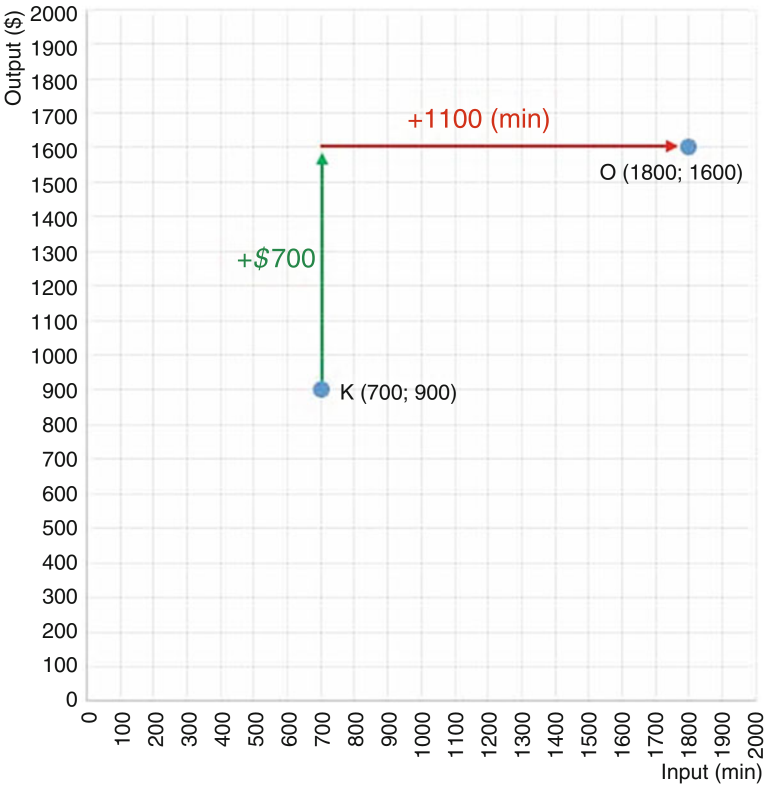


**Fig. 1.18** The performance of K via O

Likewise, every column in Table 1.3 illustrates whether a candidate wholly/partially dominates other candidates or not. For example, the corresponding column for candidate I says that I does not wholly/partially dominate candidates A–H, but, I wholly dominates I, L and P, and partially dominates J, K, M, N, O and Q.

In short, Table 1.3 elucidates the results in Table 1.2, and clarifies the introduced Types 1–4 based upon Eq. 1.2.

## 1.6 The Relative Score

In the gemstone example, candidate A wholly dominates the candidates in Fig. 1.21, partially dominates the candidates in Fig. 1.22, and therefore, has the maximum score among other candidates in Table 1.2. The weak performance of other candidates is identified, when they are compared with A. This phenomenon can be shown by measuring the ratio of the score of each candidate to the score of A, as Table 1.4 illustrates.

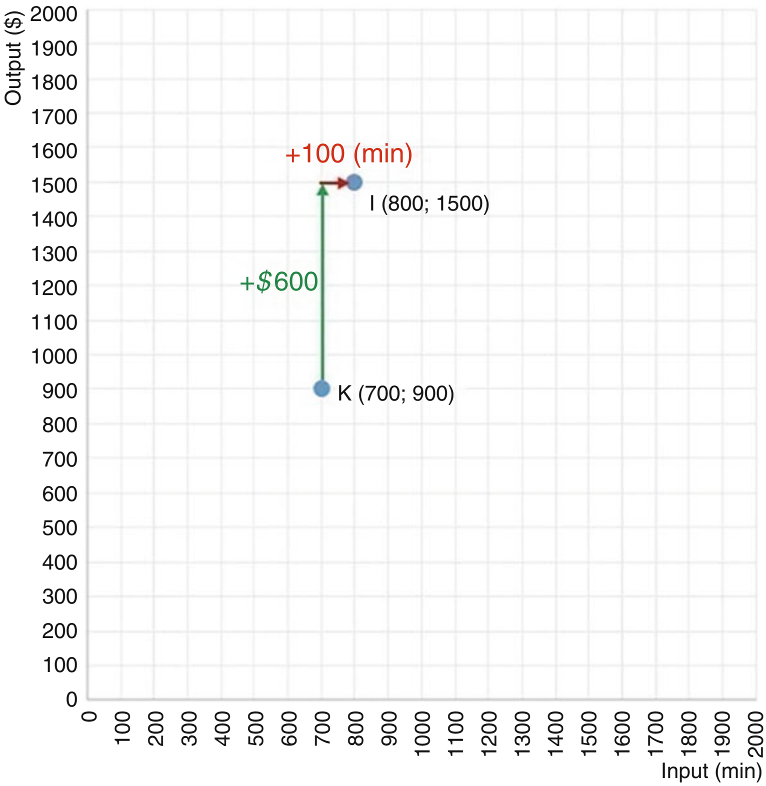


**Fig. 1.19** The performance of K via O

The scores in Table 1.4 are relatively meaningful. For instance, the relative score of C is 0.94, because the score of C (=3.95) is compared with the best available score, that is the score of A (= 4), by measuring the ratio of the score of C to the score of A, that is, 3.95/4 (= 0.94).

Note that, A has done the job well in comparison with all other candidates in this task. If there were another group of candidates or another approach to compare the candidates and so on, A might not get the best score. For instance, if there was a candidate who used 200 min to earn $1800, A could not get the highest score, because A (400, 1600) is wholly dominated by (200, 1800). Therefore, the relative score illustrates the score of each candidate in comparison with the best observed score.

As can be seen, the relative scores of C and D (which are wholly dominated by A) are greater than the relative scores of E, F, J, M and N (which are not wholly dominated by A, but are partially dominated by A). This is due to the fact that, although, C and D are wholly dominated by A, they partially dominate E, F, J, M and N (Table 1.3), and therefore, they have greater relative scores than E, F, J, M and N, which means, they have done the job well in comparison with E, F, J, M and N.

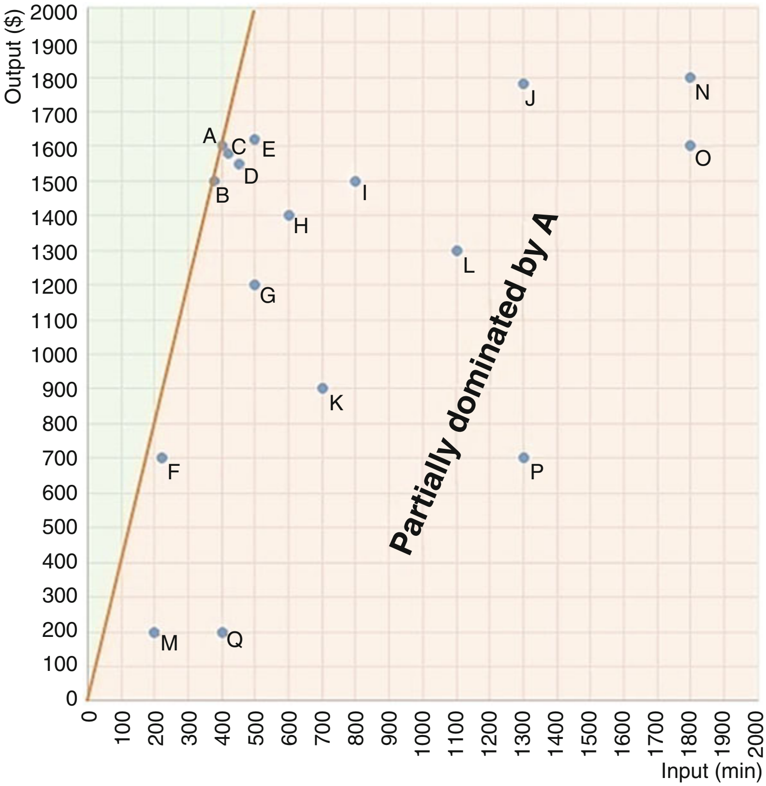


**Fig. 1.20** A partially dominates B–Q

In short, the relative scores in Table 1.4 display the results of Table 1.3 appropriately as well as providing the scores between 0 and 1.

## 1.7 The Unity Scale and the Unit Invariant Property

In Figs. 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.10, 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.19, 1.20, 1.21, 1.22, and 1.23, the horizontal and the vertical axes are scaled 100:100 from 0 to 2000 while the units of measurement in these figures are in minutes and \$, respectively. Every candidate in the gemstone example has two factors which their values can be considered as the units of measurement to rescale the axes. For instance, M has two factors that the values for both factors are 200 in minute unit and 200 in \$ unit, respectively, and the company can suppose these values as units of measurement.

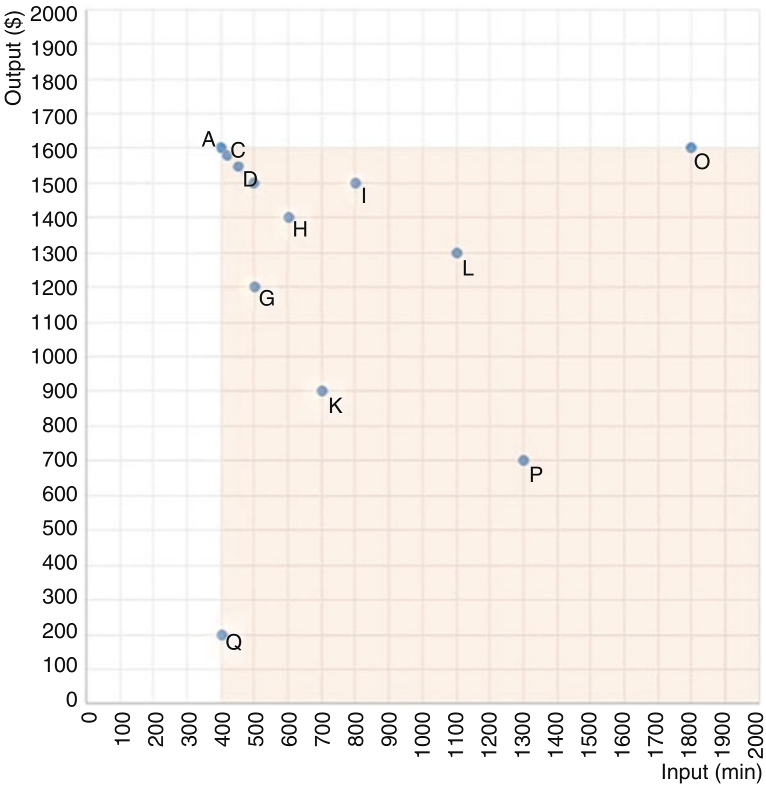


**Fig. 1.21** Candidates who are wholly dominated by A

For example, suppose that the coordinates of M are considered as the units of measurement. Thus, the value of input factor of A, 400 in minute unit, over the value of input factor of M, 200 in minute unit, is equal to 2 in minute/minute unit, and the value of output factor of A, 1600 in $ unit, over the output factor of M, 200 in $ unit, is equal to 8 in $/$ unit. After rescaling the factors of A by the coordinates of M, the units of measurement for the factors of A are dimensionless, because minute/minute = 1 and $/$ = 1. The unit of measurement in this case is called *the unity scale,* and is often used in this book.

**Definition 1.6** A factor has the unity scale, if it is dimensionless or unit-less. The unity scale is shown by (1).

The coordinates of A–Q can also be divided over 200 in the unity scale. In this case the units of measurement are not changed. Table 1.5 represents the data for each candidate in 200:200.

Furthermore, as Fig. 1.24 depicts, the locations of A–Q in Fig. 1.2 in comparison with each other are equivalent with the locations of A–Q in Fig. 1.24.

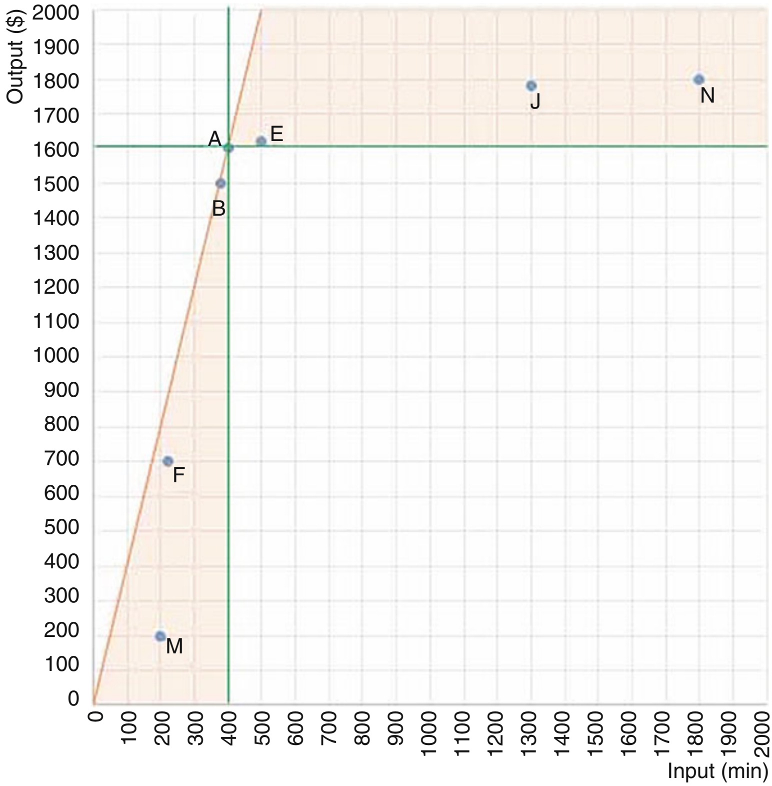


**Fig. 1.22** Candidates who are not wholly dominated by A

The scores and the relative scores of data in Table 1.5 by using Eq. 1.2, are also shown in Table 1.6, which are the same as the results in Tables 1.2 and 1.4.

As Fig. 1.25 depicts, by rescaling the axes differently, the locations of A–Q are still equivalent with Fig. 1.2. Although, the value of Eq. 1.2 is changed while the value of input factor of each point is multiplied to a positive value such as $\alpha$ and the value of output factor is multiplied to a positive value such as $\beta$, which $\alpha \neq \beta$, but the relative scores of A–Q are not changed. In other words, if the points $(\alpha x, \beta y)$'s are replaced by $(x, y)$'s, the value of Eq. 1.2 is measured by $\beta y/\alpha x$'s or $(\beta/\alpha)$'s. This means, every score in Table 1.2 is multiplied by $(\beta/\alpha)$. However, in calculating the relative score, which is the score of a point over the best observed score, the multiplier $(\beta/\alpha)$ is simplified and the relative score is not changed.

Indeed, the measured relative scores in the previous section are independent of the units of measurement. This phenomenon is read as *unit invariant property*. For example, the score of M in Fig. 1.25 is measured by 2/200 (= 0.01 in 100$/min. unit) and the score of A is measured by 16/400 (= 0.04 in 100$/min. unit). Hence, the relative score of M is 0.25 in the unity scale.

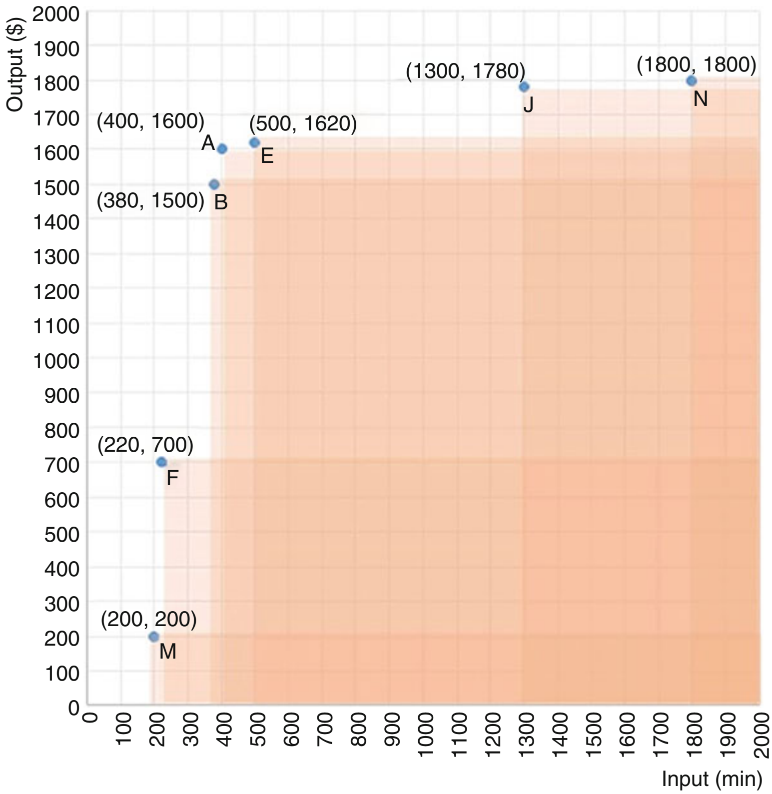


**Fig. 1.23** The wholly dominated area by A, B, E, F, J, M and N

Table 1.7 illustrates the replaced data of candidates which have the same relative scores as the relative scores in Tables 1.4.

**Definition 1.7** A measure has the unit invariant property, if the measure is not changed when the input and output factors are multiplied to positive values.

## 1.8 Conclusion

This chapter provides a fundamental step to introduce the basis of a methodology for making a fair decision concerning the performance of a candidate (a factory, a doctor and so on among a set of homogenous candidates, factories, doctors and so on, respectively). An example of one input factor and one output factor is presented to evaluate the performance of 17 candidates for a task in a gemstone company. The

**Table 1.3** Wholly dominated and partially dominated candidates

| Candidate | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A | ++ | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |
| B | + | ++ | – | – | – | – | – | – | – | – | – | – | – | – | – | – | – |
| C | ++ | + | ++ | – | – | – | – | – | – | – | – | – | – | – | – | – | – |
| D | ++ | + | ++ | ++ | – | – | – | – | – | – | – | – | – | – | – | – | – |
| E | + | + | + | + | ++ | – | – | – | – | – | – | – | – | – | – | – | – |
| F | + | + | + | + | + | ++ | – | – | – | – | – | – | – | – | – | – | – |
| G | ++ | ++ | ++ | ++ | ++ | + | ++ | – | – | – | – | – | – | – | – | – | – |
| H | ++ | ++ | ++ | ++ | ++ | + | + | ++ | – | – | – | – | – | – | – | – | – |
| I | ++ | ++ | ++ | ++ | ++ | + | + | + | ++ | – | – | – | – | – | – | – | – |
| J | + | + | + | + | + | + | + | + | + | ++ | – | – | – | – | – | – | – |
| K | ++ | ++ | ++ | ++ | ++ | + | ++ | + | + | + | ++ | – | – | – | – | – | – |
| L | ++ | ++ | ++ | ++ | ++ | + | + | ++ | ++ | + | + | ++ | – | – | – | – | – |
| M | + | + | + | + | + | + | + | + | + | + | + | + | ++ | + | – | – | – |
| N | + | + | + | + | + | + | + | + | + | + | + | + | + | ++ | – | – | – |
| O | ++ | + | + | + | ++ | + | + | + | + | ++ | + | + | + | ++ | ++ | – | – |
| P | ++ | ++ | ++ | ++ | ++ | ++ | ++ | ++ | ++ | ++ | ++ | ++ | + | + | + | ++ | – |
| Q | ++ | ++ | + | + | + | ++ | + | + | + | + | + | + | ++ | + | + | + | ++ |

**Table 1.4** The relative scores of candidates

| Candidate | Relative Score (1) | Candidate | Relative Score (1) |
|---|---|---|---|
| A | 1.000 | J | 0.342 |
| B | 0.987 | K | 0.321 |
| C | 0.940 | L | 0.295 |
| D | 0.861 | M | 0.250 |
| E | 0.810 | N | 0.250 |
| F | 0.795 | O | 0.222 |
| G | 0.600 | P | 0.135 |
| H | 0.583 | Q | 0.125 |
| I | 0.469 | | |

**Table 1.5** Rescaled Data in 200:200

| Candidate | Time (200 × min) | Earned (200 × $) | Candidate | Time (200 × min) | Earned (200 × $) |
|---|---|---|---|---|---|
| A | 2.00 | 8.00 | J | 6.50 | 8.90 |
| B | 1.90 | 7.50 | K | 3.50 | 4.50 |
| C | 2.10 | 7.90 | L | 5.50 | 6.50 |
| D | 2.25 | 7.75 | M | 1.00 | 1.00 |
| E | 2.50 | 8.10 | N | 9.00 | 9.00 |
| F | 1.10 | 3.50 | O | 9.00 | 8.00 |
| G | 2.50 | 6.00 | P | 6.50 | 3.50 |
| H | 3.00 | 7.00 | Q | 2.00 | 1.00 |
| I | 4.00 | 7.50 | | | |

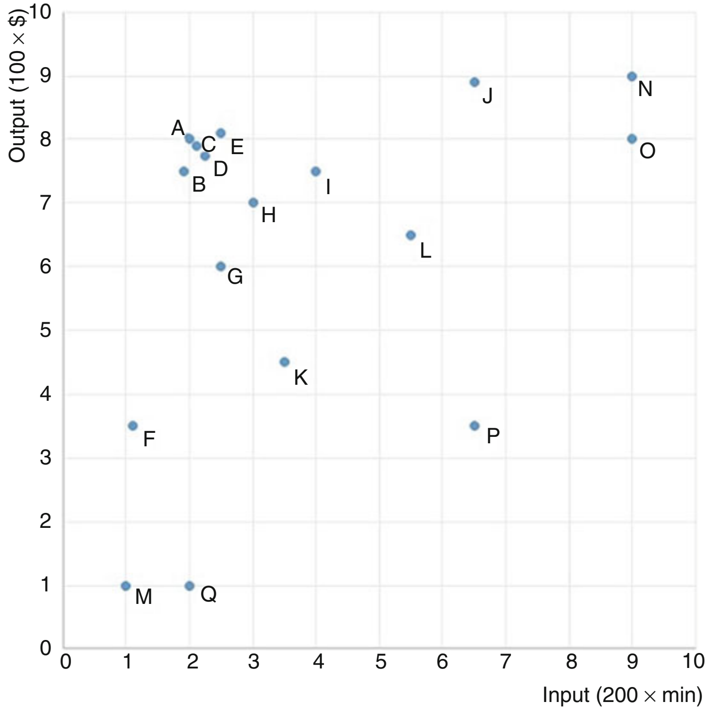


**Fig. 1.24** Locations of A–Q by rescaling 200:200

**Table 1.6** The scores and the relative scores for Data in Table 1.5

| Candidate | Score ($/min) | Relative Score | Candidate | Score ($/min) | Relative Score |
|---|---|---|---|---|---|
| A | 4.00 | 1.000 | J | 1.37 | 0.342 |
| B | 3.95 | 0.987 | K | 1.29 | 0.321 |
| C | 3.76 | 0.940 | L | 1.18 | 0.295 |
| D | 3.44 | 0.861 | M | 1.00 | 0.250 |
| E | 3.24 | 0.810 | N | 1.00 | 0.250 |
| F | 3.18 | 0.795 | O | 0.89 | 0.222 |
| G | 2.40 | 0.600 | P | 0.54 | 0.135 |
| H | 2.33 | 0.583 | Q | 0.50 | 0.125 |
| I | 1.88 | 0.469 | | | |

example is easily solved, the reasons to select the best candidate are explained, and the fundamental subjects are discussed to measure the relative scores, to know the meaning of the unity scale and the unit invariant property, to show the advantages of introducing the partially dominant and the wholly dominant concepts.

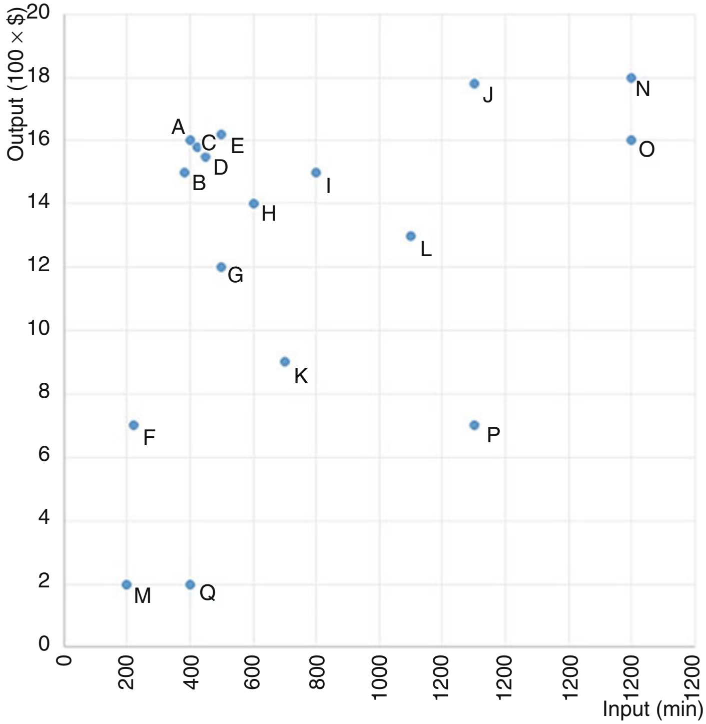


**Fig. 1.25** Locations of A–Q by 1:100

**Table 1.7** Data in (minute, $100)

| Candidate | Time (min) | Earned (100 × $) | Candidate | Time (min) | Earned ($) |
|---|---|---|---|---|---|
| A | 400 | 16.0 | J | 1300 | 17.8 |
| B | 380 | 15.0 | K | 700 | 9.0 |
| C | 420 | 15.8 | L | 1100 | 13.0 |
| D | 450 | 15.5 | M | 200 | 2.0 |
| E | 500 | 16.2 | N | 1800 | 18.0 |
| F | 220 | 7.0 | O | 1800 | 16.0 |
| G | 500 | 12.0 | P | 1300 | 7.0 |
| H | 600 | 14.0 | Q | 400 | 2.0 |
| I | 800 | 15.0 | | | |

**Table 1.8** The property investor example

| Investor | I | II | III | IV | V | VI | VII | IIX | IX |
|---|---|---|---|---|---|---|---|---|---|
| Capital (thousands$) | 12,000 | 18,000 | 15,000 | 10,000 | 13,000 | 17,000 | 22,000 | 11,000 | 10,000 |
| Income (million$) | 23 | 26 | 27 | 17 | 26 | 19 | 27 | 21 | 11 |

## 1.9 Exercises

1.1. A group of 9 property investors review their performances together based upon the capital expenditure and income earned in their city according to the following table. Each investor is numbered with a Latin number and the one who made least capital expenditure while gaining the greatest income earned is the best.

  1.1.1. Graph the location of data in Table 1.8 in the Cartesian coordinate plane.
  1.1.2. Which investors partially dominate the investor IV?
  1.1.3. Which investors is partially dominated by the investor IV?
  1.1.4. Display the area that investor V wholly dominates, and is wholly dominated.
  1.1.5. Measure the scores of each investor.
  1.1.6. Measure the relative scores the investors.
  1.1.7. Display the feasible area by applying the wholly dominant approach for data in Table 1.8.
  1.1.8. Rank the investors in Table 1.8, and find the best investor and the weakest investor.
  1.1.9. Similar to the results in Table 1.3, provide a table to show which investor wholly dominated and partially dominated with other investors.

1.2. What does a relative score mean and why such a score is important in measuring the performance evaluation of a set of homogenous firms?
1.3. What does a unit invariant property mean? Explain with an example.
1.4. What are the differences between the wholly dominant concept and the partially dominant concept?

# Chapter 2
# Possibility and Practicability

## 2.1 Introduction

In this chapter, we continue the discussions in Chap. 1. Step by step, we built the mathematical background of the requirement axioms to measure the performance of homogenous firms. We similarly start with a question from the gemstone company and gradually illustrate the new concepts. The provided concepts are not new, and have been mentioned in the literature of operations research; however, in order to avoid any confusion and pretend judgment based on the literature, only the concepts are re-introduced and the pros and cons of each concept are demonstrated. We also provide a philosophical argument to clarify some of the concepts. At the end of this chapter, readers will have an understanding of the overall concepts used to measure the performance of firms and will be ready to develop the mathematical background for complex situations.

## 2.2 Possibility and Practicability

In the Gemstone example in Chap. 1, the company may look for a relationship between the input and output factors in order to know how much an amount of input factor would possibly yield an amount of output factor.

The above objective may lead the company to conduct its dealers for future business in Sri Lanka. However, there is not a grading system or formula for such an aim and it is almost always impossible to find one, because gathering enough data is not usually possible, reasonable or economically appropriate. A regression approach also focuses on the average values which do not appropriately reflect how much a value of input factor obtain a value of output factor. So, how can the company



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**Table 2.1** The record data for candidates of gemstones

| Candidate | Time (min) | Earned ($) | Candidate | Time (min) | Earned ($) |
|---|---|---|---|---|---|
| A | 400 | 1600 | J | 1300 | 1780 |
| B | 380 | 1500 | K | 700 | 900 |
| C | 420 | 1580 | L | 1100 | 1300 |
| D | 450 | 1550 | M | 200 | 200 |
| E | 500 | 1620 | N | 1800 | 1800 |
| F | 220 | 700 | O | 1800 | 1600 |
| G | 500 | 1200 | P | 1300 | 700 |
| H | 600 | 1400 | Q | 400 | 200 |
| I | 800 | 1500 | | | |

achieve its objective through the available sample in Table 2.1? In other words, how much money can possibly be earned after using an amount of time?

Note that, this new objective is different from the previous objective of the company in Sect. 1.2 to compare the candidates to find the best performers.

### *2.2.1 The Wholly Dominant Approach*

A simple way to achieve the above objective is to use the dominant concept, discussed in Sect. 1.4, which allows the company to find some possible or practical points from the available data which are already feasible and practical.

For instance, as depicted in Fig. 2.1, when the company sees that M used 200 values of the input factor and earned 200 values of the output factor, the company can deduce that it is possible to use a value of the input factor greater than 200 to earn 200 values of the output factor; and by using 200 values of the input factor, it is possible to earn a value of the output factor less than 200.

In other words, by the coordinates of M, the company can deduce that all the points which are wholly dominated by M (which are generated by Eq. 2.1) are practical.

$$200 \leq x \quad \& \quad y \leq 200. \tag{2.1}$$

The above approach can be applied for every practical point A–Q. For instance, Eq. 2.2 generates all the practical points which are wholly dominated by A.

$$400 \leq x \quad \& \quad y \leq 1600. \tag{2.2}$$

Since A wholly dominates C, D, G, H, I, K, L, O, P and Q, there is no need to apply such an approach for these points, because all points which are wholly dominated by these points, are already generated by Eq. 2.2, as Fig. 2.2 illustrates.

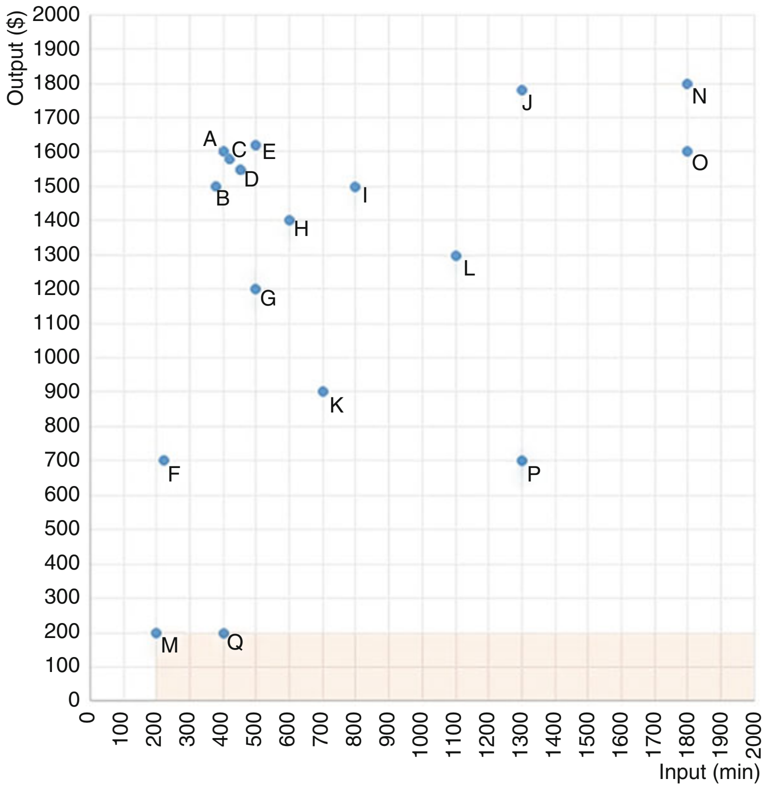


**Fig. 2.1** The wholly dominated area by M

As a result, all the points in the shaded area in Fig. 2.3 are practical and feasible, and can be generated by Eqs. 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, and 2.7.

$$380 \leq x \quad \& \quad y \leq 1500. \tag{2.3}$$

$$500 \leq x \quad \& \quad y \leq 1620. \tag{2.4}$$

$$220 \leq x \quad \& \quad y \leq 700. \tag{2.5}$$

$$1300 \leq x \quad \& \quad y \leq 1780. \tag{2.6}$$

$$1800 \leq x \quad \& \quad y \leq 1800. \tag{2.7}$$

In other words, every point in the shaded area in Fig. 2.3, satisfies one of the Eqs. 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, and 2.7. From this outcome, the following theorem can be proposed.

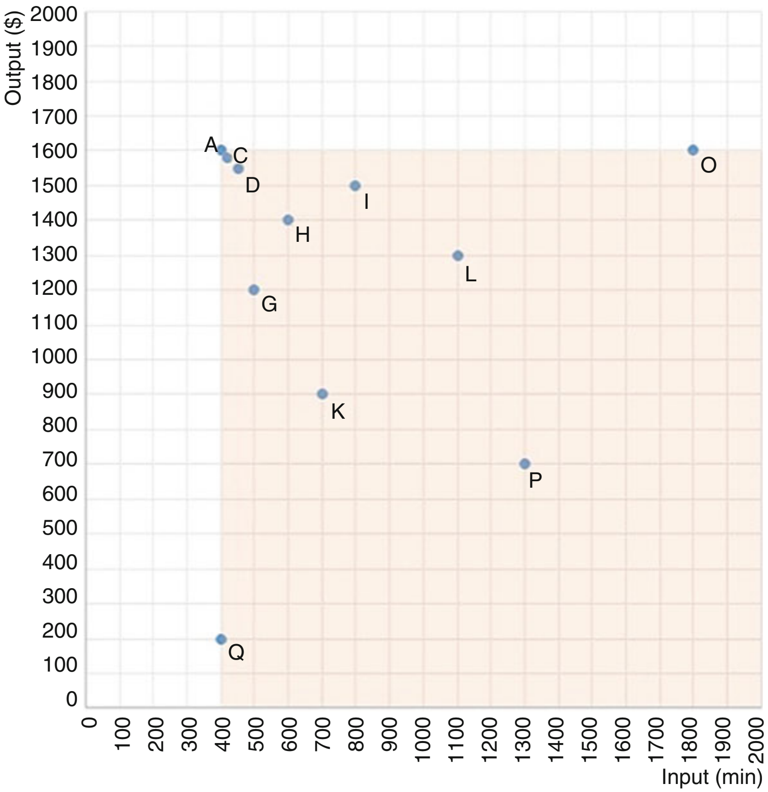


**Fig. 2.2** Candidates who are wholly dominated by A

**Theorem 2.1** Suppose that $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, … and $A_{17}(x_{17}, y_{17})$ denote the observations A–Q, respectively. The practical area from these points, using the wholly dominant approach, is generated by the following set:

$$\bigcup_{i=1}^{17} \{(x, y) : x_i \le x, y \le y_i\} \tag{2.8}$$

*Proof* The proof is leaved as an exercise. □

The Eq. 2.8 can also be rewritten by the following equation:

$$\{(x, y) : x_i \le x, y \le y_i, \text{for some } \ i = 1, 2, \ldots, n\} \tag{2.9}$$

In Eq. 2.9, 'for some $i = 1, 2, \ldots, n$' means 'for at least one $i = 1, 2, \ldots, n$', that is, 'for $i = 1$ (2, 3, …, $n$), for $i = 1$ and 2 (1 and 3; 2 and 3; 1 and 4; and so on), for $i = 1$, 2 and 3 (1, 3 and 4; and so on), or for all $i = 1, 2, \ldots, n$'.

Now, the company can introduce the frontier of the shaded area in Fig. 2.3, as an approximation to find a relationship between the factors, in order to know how much

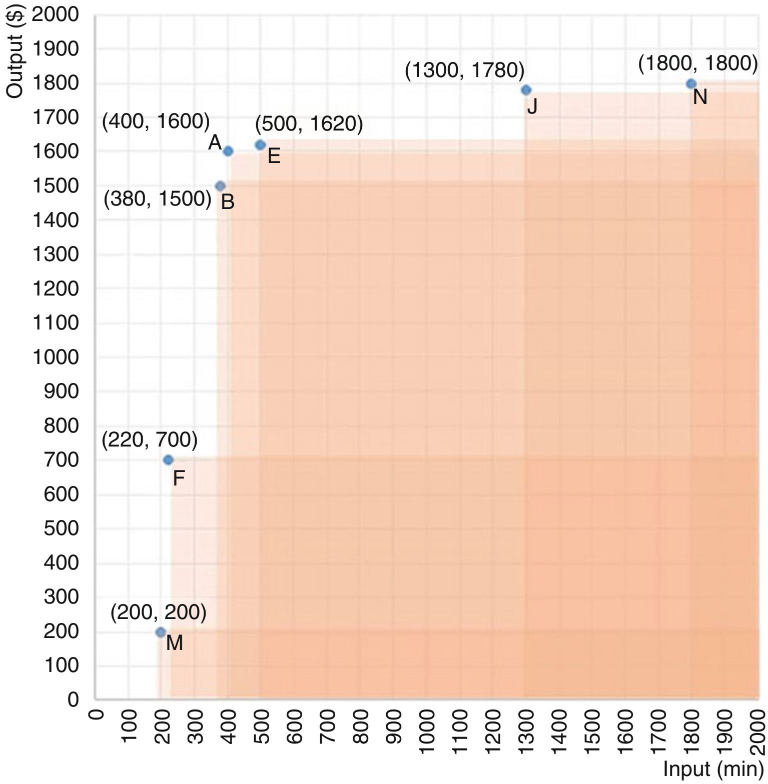


**Fig. 2.3** The wholly dominated area by A, B, E, F, J, M and N

a value of the input factor can practically obtain a maximum value of the output factor, as shown in Fig. 2.4.

Note that, the shaded areas in Fig. 2.3 are the feasible areas which can be generated by the proposed approach (using the wholly dominant approach) and the sample in Table 2.1. The other points which are out of the shaded area in Fig. 2.3 might be feasible or practical as well; however, the company is not able to introduce extra feasible points from the used approach and the data in Table 2.1. Therefore, by such an approach, the company can assume that there are not any feasible points out of the shaded area in Fig. 2.3, which means, the company supposes that if an arbitrary candidate uses 200 min to buy a gemstone, and then sells it without re-cutting/re-polishing with the same situations and conditions in this task, the candidate can *at most* earn $200.

In other words, the company assumes that even if A, B or another candidate C–Q uses the same amount of time, similar to the time that M has used, none of them are able to earn more than $200.

From the above illustration by using the wholly dominant approach, the company can introduce the points on the frontier in Fig. 2.4 as the points which have' *done the*

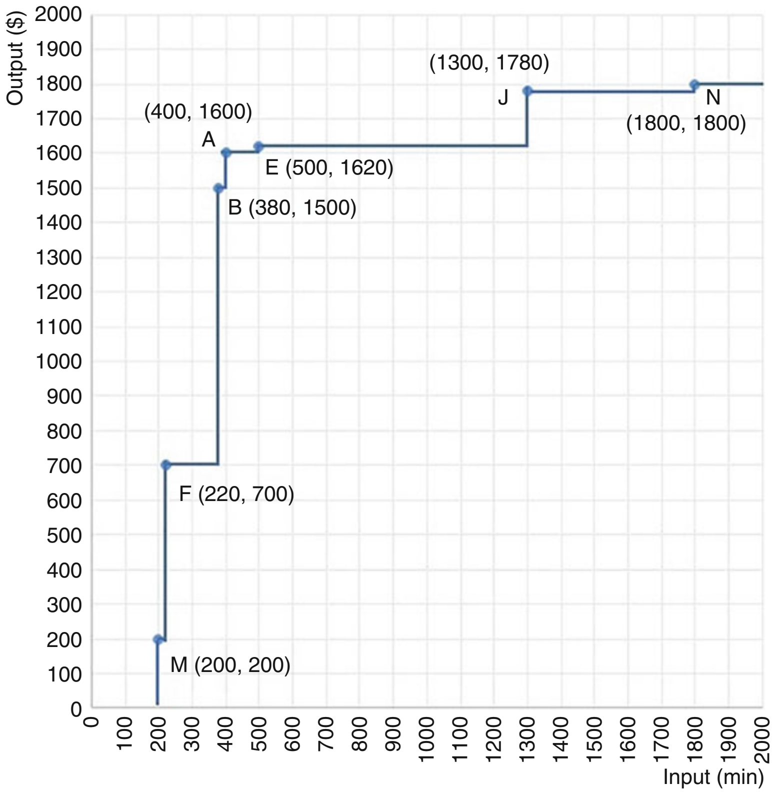


**Fig. 2.4** The frontier of the area in Fig. 1.23

*job right'*. The company can say the candidates A, B, E, F, J, M and N have done the job right, because it is impossible to earn more than what these candidates have earned with their corresponding amount of time (using the wholly dominant concept).

In contrast, candidate C has not done the job right, because C has used 20 min more than A, but earned $20 less than A. Here the word 'right' means 'correct for a particular situation or thing' or 'correct in the company's opinion or judgment'.

**Definition 2.1** A candidate has done the job right, who used an amount of time and earned the maximum amount of money according to that amount of time.

#### 2.2.1.1 Differences Between Doing the Job Well and Right

Are there any relations between the findings in Sect. 2.2.1 and the outcomes in Table 1.4 (which shows the relative scores of candidate)? Do these findings support

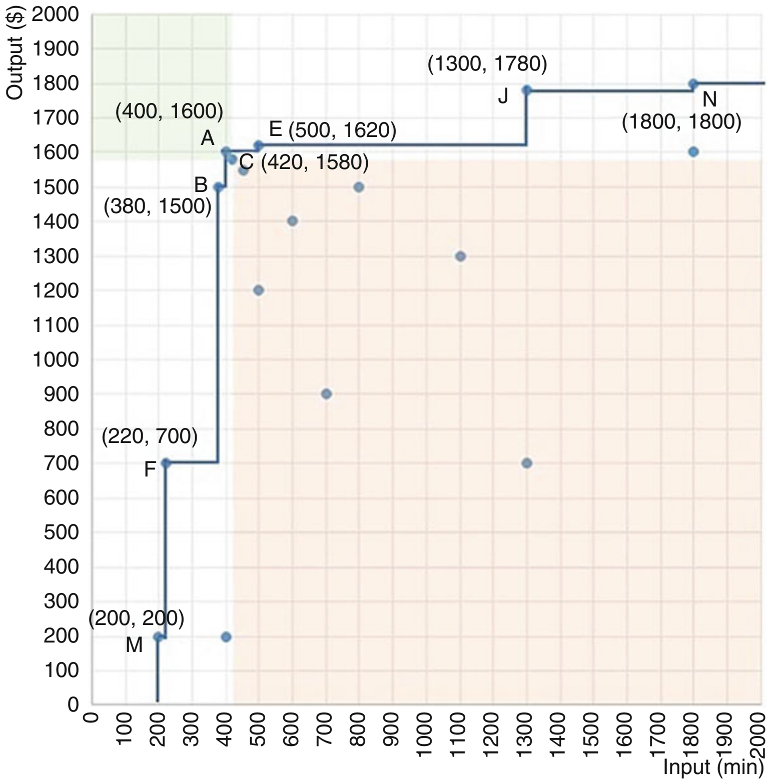


**Fig. 2.5** The area dominates C, or is dominated by C

the outcomes in Table 1.2 (which shows the scores of candidate)? Is the meaning of 'doing the job well' the same as the meaning of "doing the job right"?

As can be seen, candidate C with the relative score 0.95 in Table 1.4 is known as the candidate who has not done the job right, whereas candidate M with the relative score 0.25 has done the job right. This phenomenon at least shows that the meaning of doing the job right does not satisfy the meaning of doing the job well. In other words, C has not done the job right in comparison with A, but C has done the job well in comparison with M (E, F, J and N), (see Sects. 1.2, 1.3, 1.4, 1.5, and 1.6).

From Fig. 2.5, by applying the wholly dominant concept, C is only comparable with A, and none of the other candidates wholly dominate C (or even partially dominate C, except B which partially dominates C). As a result, the great performance of C is not measured by using the concept of doing the job right. This means, introducing the frontier in Fig. 2.4, as the extreme practical location from the available sample in Table 2.1, does not represent the discrimination between the candidates completely. Therefore, the concept of doing the job right by the wholly dominant approach does not logically support the measured scores in Table 1.4 which discriminate the performance of each candidate appropriately.

The company can debate that if really using 200 min provides only $200, why M should not use 20 min more to earn $700 as F has done (Fig. 1.3). In fact, F not only has done the job right (Fig. 2.4), but at the same time has done the job well in comparison with M, as discussed in Fig. 1.3. Or the company can even say, what are the advantages of doing the job right for M, whereas C has done the job well in comparison with M and can get higher rank?

Note that, the concept of doing the job right is good to find the possible waste of the used time and the earned money, but it is neither enough to measure the waste completely nor provides a ranking or benchmarking tool. For instance, the frontier in Fig. 2.4 shows that, Q has not done the job right and it could improve its performance (in comparison with one of A, B, E, F, J, M and N which have done the job right); however, it can mislead Q, for instance, toward M which has had the weakest performance among A–L. As a result, a candidate, who has not done the job right, might have done the job well in comparison with some of the other candidates who have done the job right.

On the other hand, the meaning of the relative score is missed as well as the excellent performance of A, if only the frontier in Fig. 2.4 is estimated, and the candidates are classified into two different groups such as 'those have done the job right' and 'those have not done the job right'. Indeed, the points on the frontier might not have the same performances, and need to be discriminated to provide an appropriate relative score to estimate the performance of each candidate, thus, it is necessary to classify the candidates into two groups of 'those have done the job well' and 'those have not done the job well'.

In short, assuming the frontier in Fig. 2.4 by the wholly dominant approach is useful to achieve the objective of the company (to display how much a value of the input factor can at most obtain a value of the output factor), however, the points on the frontier have different relative scores, and require to be discriminated before any judgment which can be made about candidates. Of course, the points on the frontier can be estimated as those points which have done the job right, however, they might not have done the job well, and those which have not done the job right should not blindly be compared with the points on the frontier in Fig. 2.4.

### *2.2.2 The Convexity Approach*

Another way to achieve the objective of the company (to know how much an amount of input factor would possibly yield an amount of output factor) is to use the *convexity approach*, that is, when two points such as F(220,700) and B(380,1500) are practical, therefore, the points on the line-segment BF are also practical.

Figure 2.6 depicts the practical points generated by the convexity approach while B and F are observed. In other words, the convexity approach says that

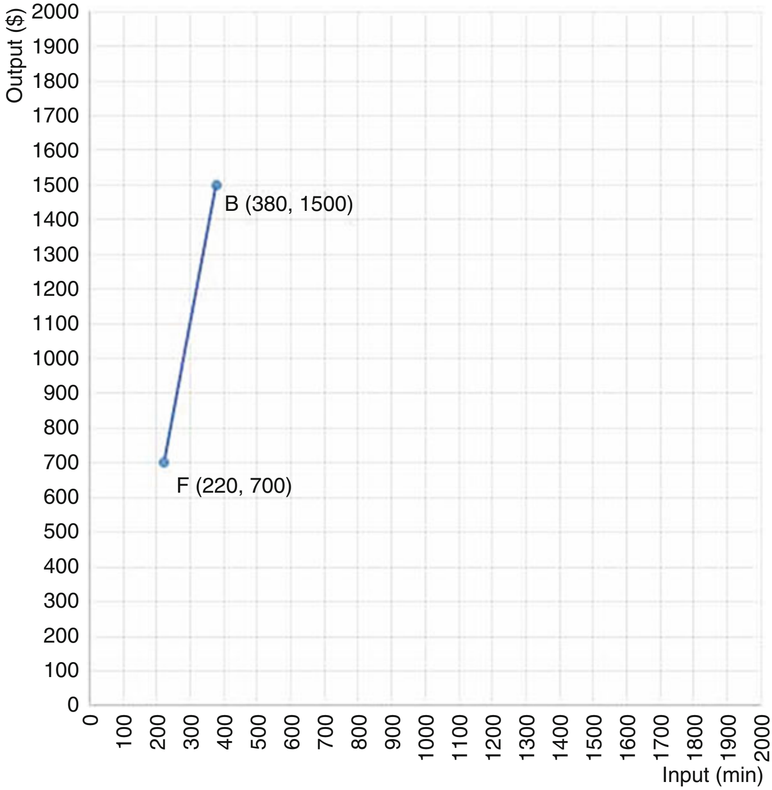


**Fig. 2.6** The feasible points on the line-segment BF

when two points such as F(220,700) and B(380,1500) are practical, a linear combination of these two points, that is,

$$\lambda(220, 700) + (1 - \lambda)(380, 1500) \text{ where } 0 \leq \lambda \leq 1, \tag{2.10}$$

is also practical, for a λ in the set of real numbers. If $\lambda = 0$, point B is generated and if $\lambda = 1$, point F is generated. Considering a λ between 0 and 1 yields to find another point between B and F. Indeed,

$$\begin{aligned}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} 220 \\ 700 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 380 \\ 1500 \end{bmatrix} &= \begin{bmatrix} 220\lambda \\ 700\lambda \end{bmatrix} + \begin{bmatrix} 380(1-\lambda) \\ 1500(1-\lambda) \end{bmatrix} \\ &= \begin{bmatrix} 220\lambda + 380(1-\lambda) \\ 700\lambda + 1500(1-\lambda) \end{bmatrix} \\ &= \begin{bmatrix} 380 - (380-220)\lambda \\ 1500 - (1500-700)\lambda \end{bmatrix}.\end{aligned}$$

As can be seen, (380–220) is the distance value between the values of input factors of B and F. Multiplying this value to a λ for $0 \leq \lambda \leq 1$, yields measuring a proportion of the distance, and subtracting from 380 yields a value of the input factor of the line-segment BF. For instance, if $\lambda = 1$, the whole distance value (= 160), is subtracted from 380, and hence the value of input factor of F (= 220) is found; if $\lambda = 0.5$, half of the distance value (= 80), is subtracted from 380, and hence the middle of the values of input factor (= 300) is found; and so on. The same illustration can also be demonstrated for the output factor coordinate.

On the other hand, the above equations yield the formula of the line which passes the points B and F, by the following steps:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 380 - (380 - 220)\lambda \\ 1500 - (1500 - 700)\lambda \end{bmatrix}$$

$$\Rightarrow \begin{cases} x = 380 - (380 - 220)\lambda \\ y = 1500 - (1500 - 700)\lambda \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{x - 380}{-(380 - 220)} = \lambda \\ \dfrac{y - 1500}{-(1500 - 700)} = \lambda \end{cases}$$

$$\Rightarrow \frac{y - 1500}{-(1500 - 700)} = \frac{x - 380}{-(380 - 220)}$$

$$\Rightarrow y - 1500 = \frac{1500 - 700}{380 - 220}(x - 380).$$

The formula of the line-segment BF, the convexity approach, can also be written as follows:

$$\lambda_B(x_B, y_B) + \lambda_F(x_F, y_F), \text{where}\, \lambda_B + \lambda_F = 1, \text{for } 0 \leq \lambda_B, \lambda_F \leq 1. \tag{2.11}$$

In Eq. 2.11, $x_B$ and $x_F$ refer to the values of input factors of B and F, respectively, and $y_B$ and $y_F$ refer to the values of output factors of B and F, respectively. Moreover, $\lambda_B = 1 - \lambda_F$ shows that both Eqs. 2.10 and 2.11 are equivalent.

Similarly, the convexity approach can be applied for every set of two observations in Table 2.1. For instance, Eqs. 2.12 and 2.13 represent the lines-segment between B and K and K and F, respectively.

$$\lambda_B(x_B, y_B) + \lambda_K(x_K, y_K), \text{where } \lambda_B + \lambda_K = 1, \text{for } 0 \leq \lambda_B, \lambda_K \leq 1. \tag{2.12}$$

$$\lambda_K(x_K, y_K) + \lambda_F(x_F, y_F), \text{where } \lambda_K + \lambda_F = 1, \text{for } 0 \leq \lambda_K, \lambda_F \leq 1. \tag{2.13}$$

The convexity approach can also be extended for more than two points. For instance, the convexity approach can be simultaneously applied for the three points B, K and F as Eq. 2.14 represents.

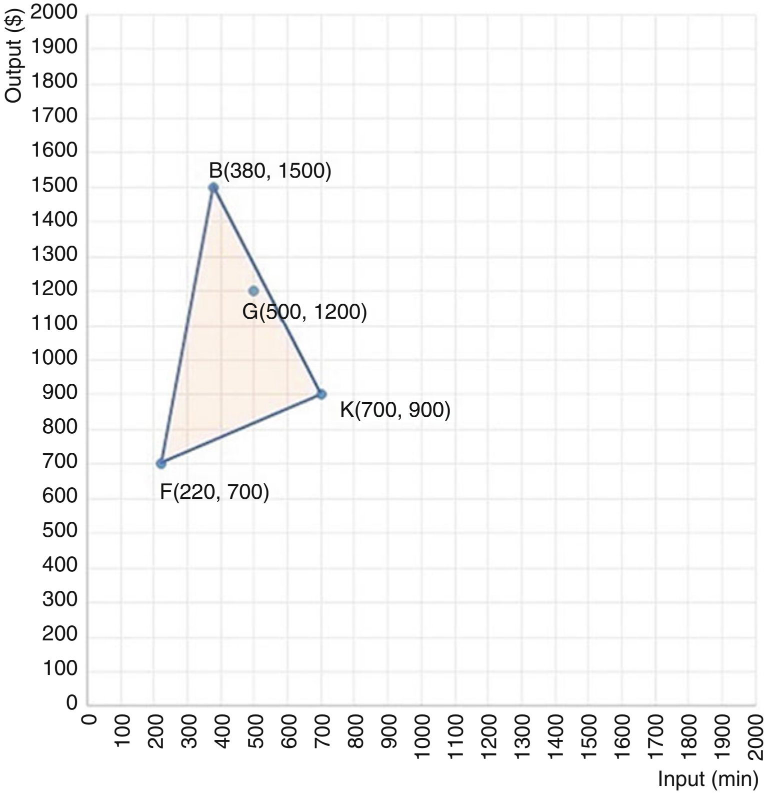


**Fig. 2.7** The three-sided polygon BKF

$$\begin{aligned} &\lambda_B(x_B, y_B) + \lambda_K(x_K, y_K) + \lambda_F(x_F, y_F), \\ &\text{where } \lambda_B + \lambda_K + \lambda_F = 1, \text{for } 0 \le \lambda_B, \lambda_K, \lambda_F \le 1. \end{aligned} \tag{2.14}$$

If it is supposed that, $\lambda_K = 0$, Eq. 2.14 is the same as Eq. 2.11, if $\lambda_F = 0$, Eqs. 2.12 and 2.14 are the same, and if $\lambda_B = 0$, Eq. 2.13 is yielded from Eq. 2.14. Therefore, not only all the generated points by Eqs. 2.11, 2.12, and 2.13 are generated by Eq. 2.14, but also all points in the triangle BKF, are also generated, as the three-sided polygon is depicted in Fig. 2.7.

For instance, G is in the triangle BKF, which means, there are at least some $\lambda_B$, $\lambda_K$, $\lambda_F$, between 0 and 1 where $\lambda_B + \lambda_K + \lambda_F = 1$ and the following equation is satisfied:

$$\lambda_B(x_B, y_B) + \lambda_K(x_K, y_K) + \lambda_F(x_F, y_F) = (x_G, y_G).$$

The above equation yields the following equations:

$$\begin{bmatrix} x_G \\ y_G \end{bmatrix} = \begin{bmatrix} \lambda_B x_B + \lambda_K x_K + \lambda_F x_F \\ \lambda_B y_B + \lambda_K y_K + \lambda_F y_F \end{bmatrix},$$
$$\Leftrightarrow \begin{bmatrix} 500 \\ 1200 \end{bmatrix} = \begin{bmatrix} 380\lambda_B + 700\lambda_K + 220\lambda_F \\ 1500\lambda_B + 900\lambda_K + 700\lambda_F \end{bmatrix}.$$

Thus, the following system should be solved.

$$\begin{cases} 380\lambda_B + 700\lambda_K + 220\lambda_F = 500, \\ 1500\lambda_B + 900\lambda_K + 700\lambda_F = 1200, \\ \lambda_B + \lambda_K + \lambda_F = 1, \\ \lambda_B \geq 0, \\ \lambda_K \geq 0, \\ \lambda_F \geq 0. \end{cases}$$

The unique answer for $(\lambda_B, \lambda_K, \lambda_F)$ in the above system is

$$\left(\frac{95}{182}, \frac{75}{182}, \frac{6}{91}\right).$$

Moreover, none of the points which are out of the triangle BKF, can be generated by Eq. 2.14, as Lemma 2.1 illustrates.

**Lemma 2.1** *Equation 2.14 generates the area and circumstance of the three-sided polygon BKF in Fig. 2.7.*

*Proof* The Proof is leaved as an exercise. □

Lemma 2.1 can be extended for four-sided polygon, five-sided polygon, six-sided polygon and so on, as Theorem 2.2 represents.

**Theorem 2.2** *The area and circumference of an* n*-sided polygon, $A_1A_2...A_n$, where $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, ..., and $A_n(x_n, y_n)$, are generated by Eq. 2.15,*

$$\begin{aligned} &\lambda_1(x_1, y_1) + \lambda_2(x_2, y_2) + \ldots + \lambda_n(x_n, y_n), \\ &\text{where}\, \lambda_1 + \lambda_2 + \ldots + \lambda_n = 1, \text{for } 0 \leq \lambda_1, \lambda_2, \ldots, \lambda_n \leq 1. \end{aligned} \tag{2.15}$$

*Proof* The proof is leaved as an exercise. □

Equation 2.16 can be merged as follows:

$$\begin{aligned} &\textstyle\sum_{i=1}^{n} \lambda_i(x_i, y_i), \cdots \text{or} \cdots (\sum_{i=1}^{n} x_i\lambda_i,\ \sum_{i=1}^{n} y_i\lambda_i) \\ &\text{where } \textstyle\sum_{i=1}^{n} \lambda_i = 1 \text{ and } 0 \leq \lambda_i \leq 1 \text{ for } i = 1, 2, \ldots, n. \end{aligned} \tag{2.16}$$

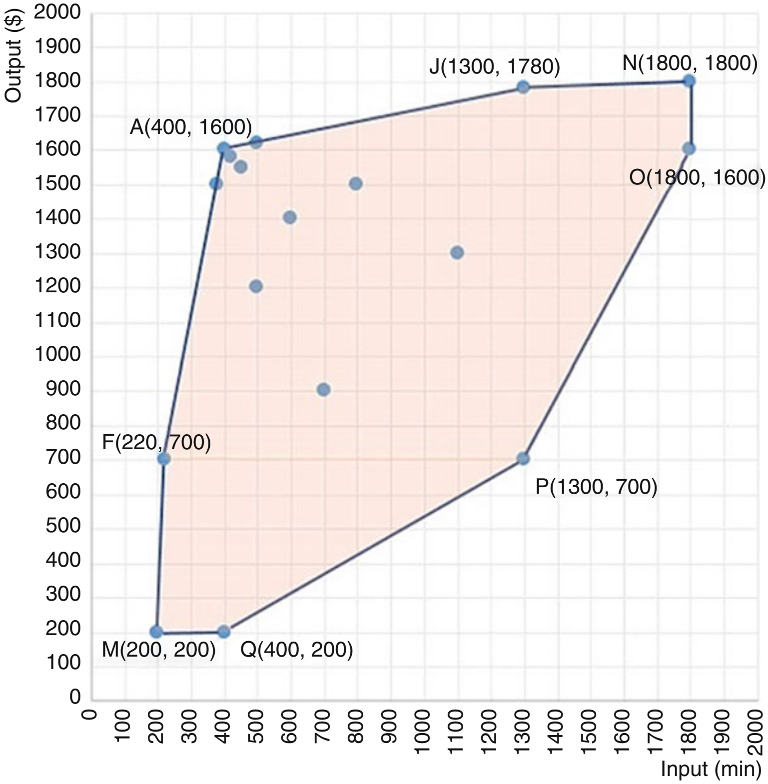


**Fig. 2.8** The octagon AJNOPQMF

Theorem 2.2 suggests the practical area from the observations A–Q by using the convexity approach, as an octagon (eight-sided polygon) depicted in Fig. 2.8. The company can introduce the circumference of the octagon in Fig. 2.8 to display how much an amount of input factor can obtain at most (at least) an amount of output factor from the practical observation in Table 2.1.

The circumference of the eight-sided polygon can be divided into the two following parts, MFAJN and MQPON. The practical points on MFAJN can be illustrated as the points which have done the job right, however, similar to the Sect. 2.2.1.1, these points are not representing the set of the points which have done the job well. Moreover, the points on MQPON can be introduced as the points which have done the job worst, because of wasting a huge amount of the input factor to earn a lesser amount of the output factor. Note that, the points M and N are simultaneously known as those who are doing the job right and worst, while the convexity approach is applied. In order to remove this contradiction, M and N can be classified to the part MFAJN. However, there is still a need to discriminate the points on the circumference of the octagon whether they have done the job well.

In short, the objective of the company (to display how much a value of the input factor can at most obtain a value of the output factor) can be achieved by the convexity approach, however, the outcomes should not be wrongly interpreted to provide a relative score for each practical point, as discussed in Sect. 2.2.1.1.

### 2.2.3 The Radiate Approach

The objective of the company may also be achieved by utilizing *the radiate approach*. A radiate approach means that, if $(x, y)$ is practical, then $\lambda\,(x, y)$ is also practical, for every positive real number $\lambda$. Figure 2.9 depicts the points which are generated by applying the radiate approach for the practical points in Table 2.1. As can be seen in Fig. 2.9, the radiate approach assumes that increasing/decreasing the value of input factor has the same effect to increase/decrease the value of output factor. In other words, if the time is become doubled, the earned money should have

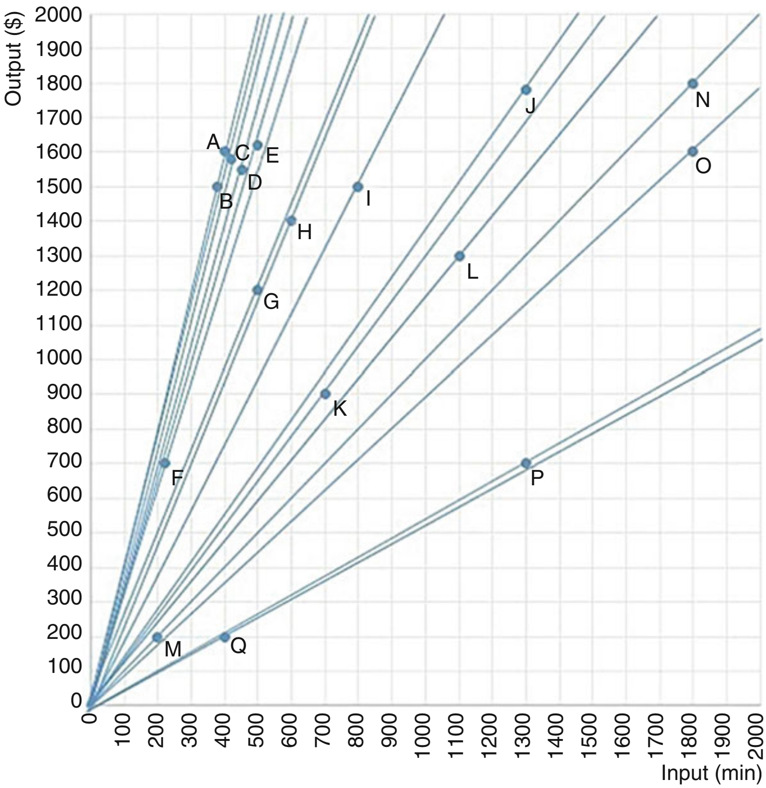


**Fig. 2.9** The radiate approach

also become twice, which means, the rate of increasing (decreasing) the input factor is the same as the rate of increasing (decreasing) the output factor. Therefore, the score of an observed point by Eq. 1.1, that is, Earned($)/Time(minute), is the same as the scores of the points which are generated by the radiate approach from that observed point. For instance, suppose that the radiate approach is applied for A. The following set represents the generated points:

$$\{(x, y): \ x = x_A\lambda \ \& \ y = y_A\lambda, \ \lambda > 0\}. \tag{2.17}$$

which is equivalence with the following set:

$$\{(x_A\lambda, y_A\lambda): \ \lambda > 0\}. \tag{2.18}$$

By using Eq. 1.2, that is, output/input, the score of each point in the above set is measured by

$$\frac{y_A\lambda}{x_A\lambda},$$

which is the same as the score of A's performance, that is, $y_A/x_A$. The following theorem mathematically represents the generated points in Fig. 2.9.

**Theorem 2.3** Suppose that $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, …, and $A_{17}(x_{17}, y_{17})$ denote the observations A–Q, respectively. The following set displays all the practical points which are generated by the radiate approach, as depicted in Fig. 2.9.

$$\bigcup_{i=1}^{17} \{(x_i\lambda, y_i\lambda): \lambda > 0\}. \tag{2.19}$$

*Proof* The proof is leaved as an exercise. □

Equation 2.19 can also be rewritten as follows:

$$\{(x, y): x = x_i\lambda, \ \ y = y_i\lambda, \ \ \lambda > 0, \text{for some } i = 1, 2, \ldots, 17\}. \tag{2.20}$$

If the domain of the multiplier, λ, is restricted to $0 < \lambda < 1$ (or λ > 1), *the inner* (or *outer*) *radiate approach* can be defined, as Figs. 2.10 and 2.11 represent.

The following set generates the inner radiate approach,

$$\{(x_i\lambda, y_i\lambda): 0 < \lambda \leq 1, \text{for some } i = 1, 2, \ldots, n\}. \tag{2.21}$$

and the Eq. 2.22 generates the outer radiate approach,

$$\{(x_i\lambda, y_i\lambda): \ \ \lambda \geq 1, \text{for some } i = 1, 2, \ldots, n\}. \tag{2.22}$$

The radiate approach may rarely be applicable; however, it allows generating some feasible points with the same relative scores corresponding to the scores of the observed points.

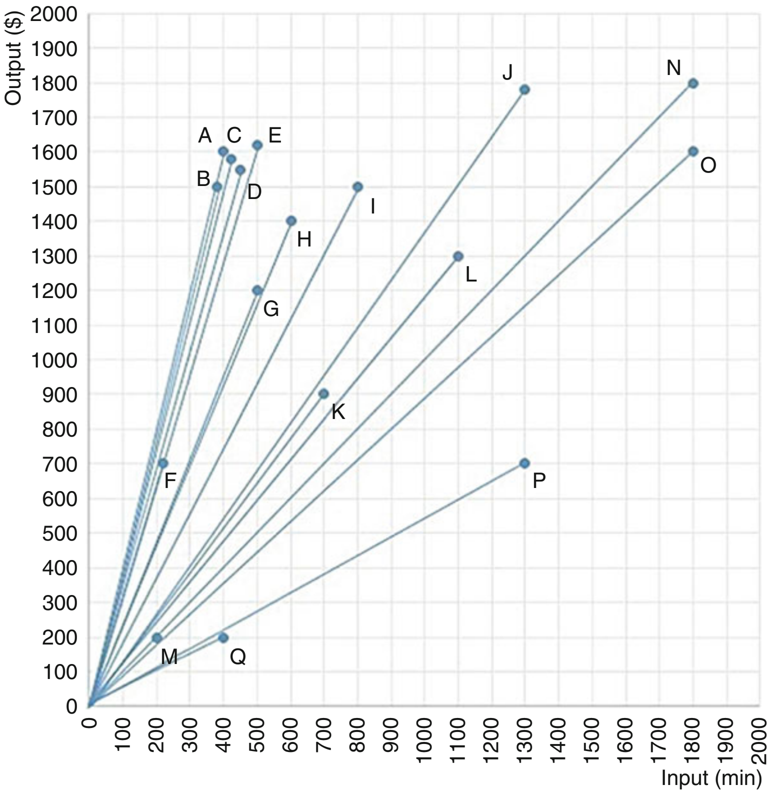


**Fig. 2.10** The inner radiate approach

On the other hand, Eq. 2.18 is the set of all the points on the line which passes origin and A, and can be considered to define how much a value of the input factor can at most obtain a value of the output factor, because these points have the same relative scores as A. Since A partially dominates B–Q, every point of the set in Eq. 2.18 also partially dominates B–Q. Therefore, the points in Eq. 2.18 are the points which have done the job right and at the same time have done the job well. Therefore, it seems that if a candidate which partially dominates all other candidates is known, the radiate approach can introduce other practical points which satisfy both concepts of 'doing the job right' and 'doing the job well'. *Nonetheless, it is not correct as it is illustrated in the next chapters*.

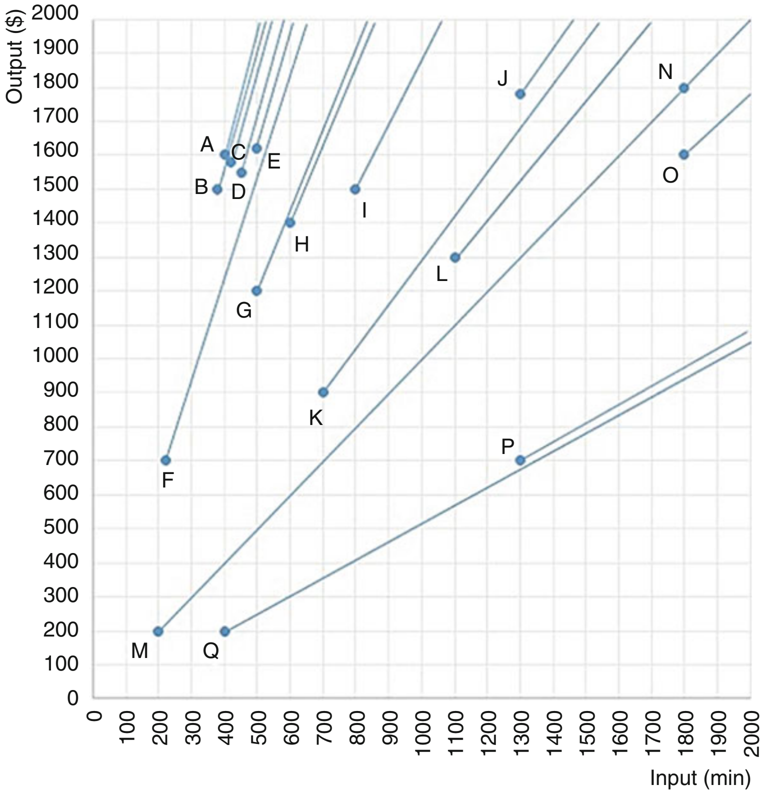


**Fig. 2.11** The outer radiate approach

### *2.2.4 The Mixed Approaches*

The company can combine the previous approaches in Sects. 2.2.1, 2.2.2, and 2.2.3 to generate some practical points from the observations in Table 2.1, and supposes some other frontiers to introduce the points which have done the job right, although, may not have done the job well.

Note that, as it is discussed in the previous Sects. 2.2.1, 2.2.2, and 2.2.3, doing the job right is a necessary condition to find the best performers, and is not enough to measure the relative scores of the candidates. The company should use another methodology to find those who have done the job well (which are among those have done the job right and those which partially dominates all the others), and after that, should estimate the relative scores. Nonetheless, this section only focuses to find how much a value of the input factor can at most obtain a value of the output factor, regardless of the concept of doing the job well and finding the relative scores.

The following sections illustrate the combination of each two or all approaches in Sects. 2.2.1, 2.2.2, and 2.2.3.

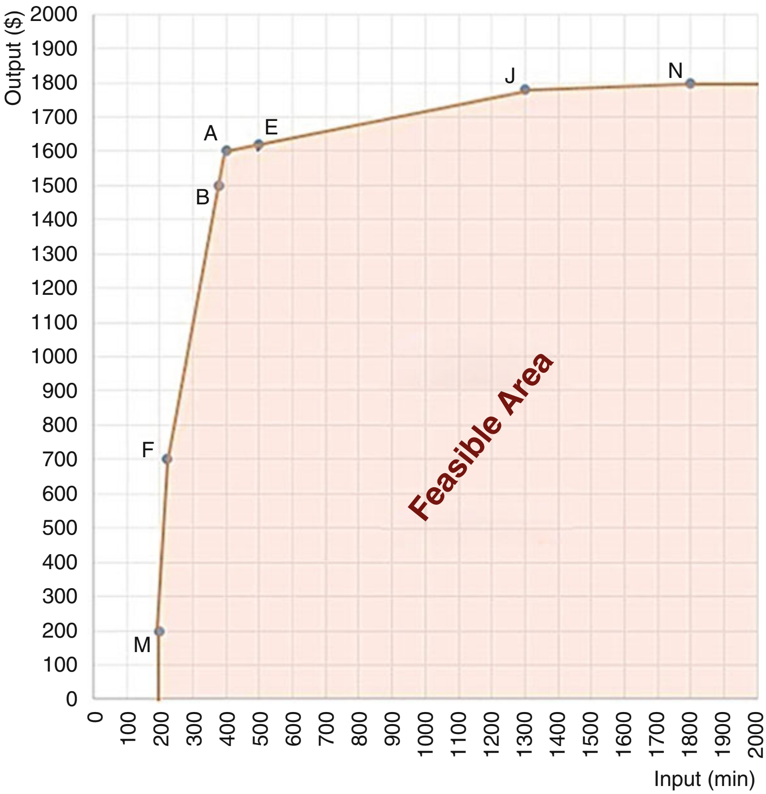


**Fig. 2.12** The wholly dominant and the convexity approaches

#### 2.2.4.1 The Wholly Dominant and the Convexity Approaches

There are two different ways to generate the practical points by combination of the wholly dominant and the convexity approaches. The first way is to apply the wholly dominant approach and then apply the convexity approach, whereas the second way is to apply the convexity approach and then apply the wholly dominant approach.

If the company first uses the wholly dominant approach, the points in Fig. 2.3 are generated and then, by applying the convexity approach, the shaded area in Fig. 2.12 is generated. By applying the convexity approach, the points in Fig. 2.8 are generated and then the wholly dominant approach results the shaded area in Fig. 2.12 as well. As a result, there are no differences between the outcomes of both ways graphically. Is the mathematical modeling in both ways to generate the shaded area in Fig. 2.12 the same, too?

The answer to the above question is negative. In order to display the differences, first the meanings of *linear equation* and *non-linear equation* are illustrated. An equation is called a linear equation, if it is an algebraic polynomial of degree 1 as follows:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n, \tag{2.23}$$

where, $x_1, x_2, \ldots, x_n$ are the *n variables* or unknowns with the exponent one, and $a_1$, $a_2, \ldots, a_n$ represent numbers which are called the *coefficients* ($n \in \mathbb{N}$).

An equation which is not linear is called a non-linear equation. For instance, if a variable with an exponent other than one appears in an equation, the equation is non-linear. Or if two variables are multiplied together, even if the degree of each variable is one, the degree of the equation is not one, and therefore, the equation is non-linear. For instance, the following equations are linear equations,

$$x, \quad 3x + y, \quad 5x + 2y + 7z - t,$$

however, the following equations are non-linear,

$$x^2, xy, \sqrt{x}, 1/x, x/(y+z).$$

Now, suppose that $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, …, and $A_{17}(x_{17}, y_{17})$ denote the observations A–Q, respectively. For the first way, the company should use Eq. 1.14 and then use Eq. 2.15. The combination of such an approach is non-linear. Because there are two variables $x_i'$ and $y_i'$ corresponding to each observations, in the generated point by the first two inequalities of the wholly dominant approach, that is, $x_i \le x_i'$, $y_i' \le y_i$. Applying the convexity approach for each two sets of variables, $x_i'$ and $y_i'$, for $i = 1, 2, \ldots, 17$, yields multiplying $\lambda_i x_i'$ and $\lambda_i y_i'$ with the exponent of $1 + 1 = 2$. Thus, the first way is a non-linear approach, as Eq. 2.24 represents.

$$\left\{ \begin{array}{l} (x'', y'') : \quad x'' = \sum_{i=1}^{17} x_i'\lambda_i, x_i \le x_i', y'' = \sum_{i=1}^{17} y_i'\lambda_i, \\ y_i' \le y_i, \sum_{i=1}^{n} \lambda_i = 1, \ \ 0 \le \lambda_i \le 1, \text{for } i = 1, 2, \ldots, 17. \end{array} \right\}. \tag{2.24}$$

For the second way, the company should use Eq. 2.15 and then use Eq. 2.8. Equation 2.25 represents the combination of both approaches by the second way.

$$\left\{ \begin{array}{l} (x', y') : \quad \sum_{i=1}^{17} x_i\lambda_i \le x', \ \ y' \le \sum_{i=1}^{17} y_i\lambda_i, \\ \sum_{i=1}^{n} \lambda_i = 1, \quad 0 \le \lambda_i \le 1, \ \ \text{for } \ i = 1, 2, \ldots, 17. \end{array} \right\}. \tag{2.25}$$

Since the convexity approach applies for the observations with known data, therefore, the equations, $\sum_{i=1}^{n} x_i\lambda_i$ and $\sum_{i=1}^{n} y_i\lambda_i$, are two polynomials of degree 1. Now, applying the wholly dominant approach yields two inequalities $\sum_{i=1}^{n} x_i\lambda_i \le x'$ and $y' \le \sum_{i=1}^{n} y_i\lambda_i$, which are both linear equations. Thus, the second approach is a linear approach and the following theorem is proposed.

**Theorem 2.4** *Suppose that $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, …, and $A_{17}(x_{17}, y_{17})$ denote the observations A–Q, respectively. Equations 2.24 and 2.25 yield the shaded area in Fig. 2.12 by a non-linear approach and a linear approach, respectively.*

*Proof* The proof is leaved as an exercise. □

#### 2.2.4.2 The Wholly Dominant and the Radiate Approaches

Similar to the previous section, the area generated by applying the wholly dominant approach in the first step and then applying the radiate approach in the second step, yields a non-linear approach, while applying the radiate approach in the first step and then applying the wholly dominant approach in the second step, yields a linear approach. However, the generated areas by both approaches are the same graphically, as Fig. 2.13 illustrates, (see Figs. 2.9 and 2.10).

Figure 2.13 represents all the points which are wholly dominated by A (See Fig. 1.20), and can be generated by Eqs. 2.9 and 2.10 according to the following theorem:

**Theorem 2.5** *Suppose that* $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, …, *and* $A_{17}(x_{17}, y_{17})$ *denote the observations A–Q, respectively. Equations 2.26 and 2.27 generate the area which is depicted in Fig. 2.13, non-linearly and linearly, respectively.*

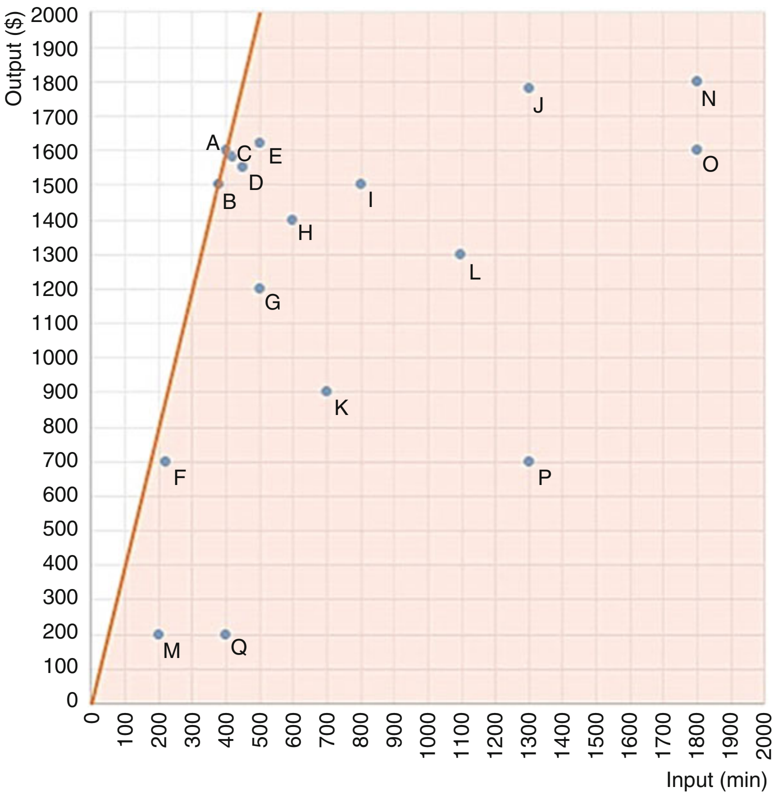


**Fig. 2.13** The wholly dominant and the radiate approaches

$$\left\{ \begin{array}{c} (x'', y''): \quad x_i \leq x'_i, \;\; y'_i \leq y_i, \;\; x'' = x'_i \lambda, \;\; y'' = y'_i \lambda, \\ \lambda > 0, \;\; \text{for some} \;\; i = 1, 2, \ldots, 17. \end{array} \right\}. \tag{2.26}$$

$$\{(x, y): \;\; x_i \lambda \leq x, \;\; y \leq y_i \lambda, \;\; \lambda > 0, \;\; \text{for some } i = 1, 2, .., 17.\} \tag{2.27}$$

*Proof* The proof is leaved as an exercise. □

In the same manner, the combination of the inner (outer) radiate approach and the wholly dominant approach can be introduced. Figures 2.14 and 2.15 depict the generated areas by these two combinations, respectively.

Equations 2.28 and 2.29 linearly generate the shaded areas in Figs. 2.14 and 2.15, respectively.

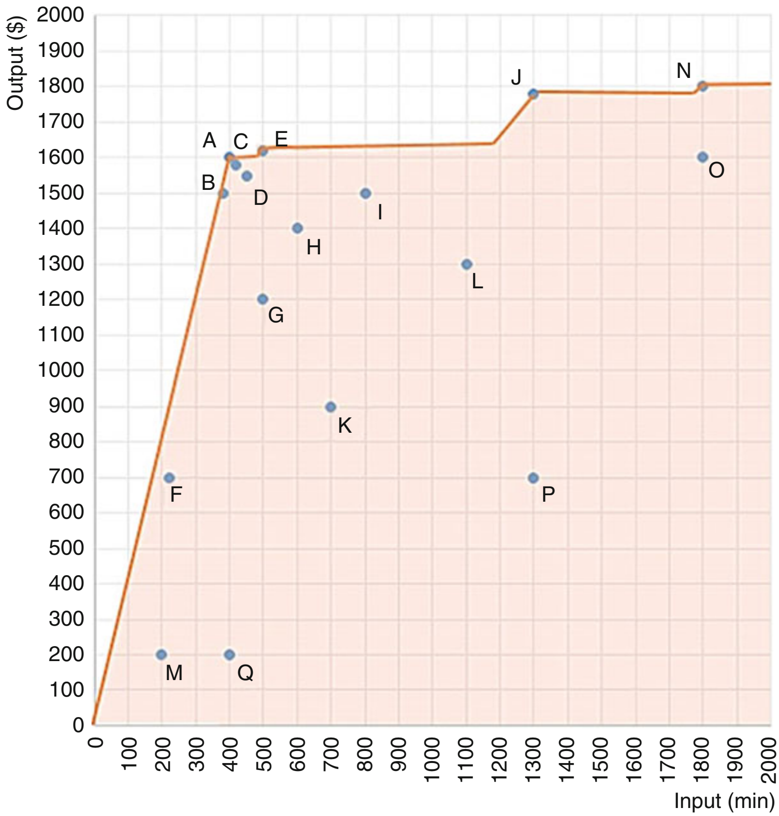


**Fig. 2.14** The wholly dominant and the inner radiate approaches

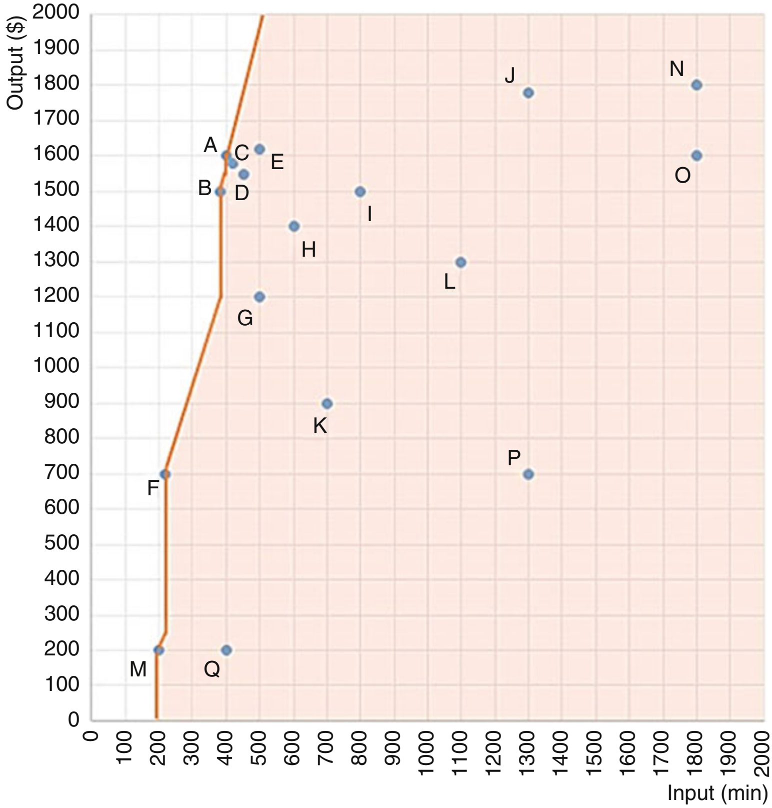


**Fig. 2.15** The wholly dominant and the outer radiate approaches

$$\{(x, y) : \quad x_i\lambda \le x, \quad y \le y_i\lambda, \quad 0 < \lambda \le 1, \text{for some } i = 1, 2, .., 17.\} \tag{2.28}$$

$$\{(x, y) : \quad x_i\lambda \le x, \quad y \le y_i\lambda, \quad \lambda \ge 1, \text{for some } i = 1, 2, .., 17.\} \tag{2.29}$$

The frontier of the shaded areas in Figs. 2.13, 2.14, and 2.15 can be considered to measure how much an amount of input factor can, at most, obtain an amount of output factor. Moreover, it is supposed that the points out of the shaded area are infeasible, that is, impossible to achieve according to the selected approaches.

#### 2.2.4.3 The Convexity and the Radiate Approaches

If it is assumed that the radiate approach is applied in the first step and then in the second step, the convexity approach is applied for the points which are generated in the first step, the combination of these two approaches are non-linear. Moreover, if it

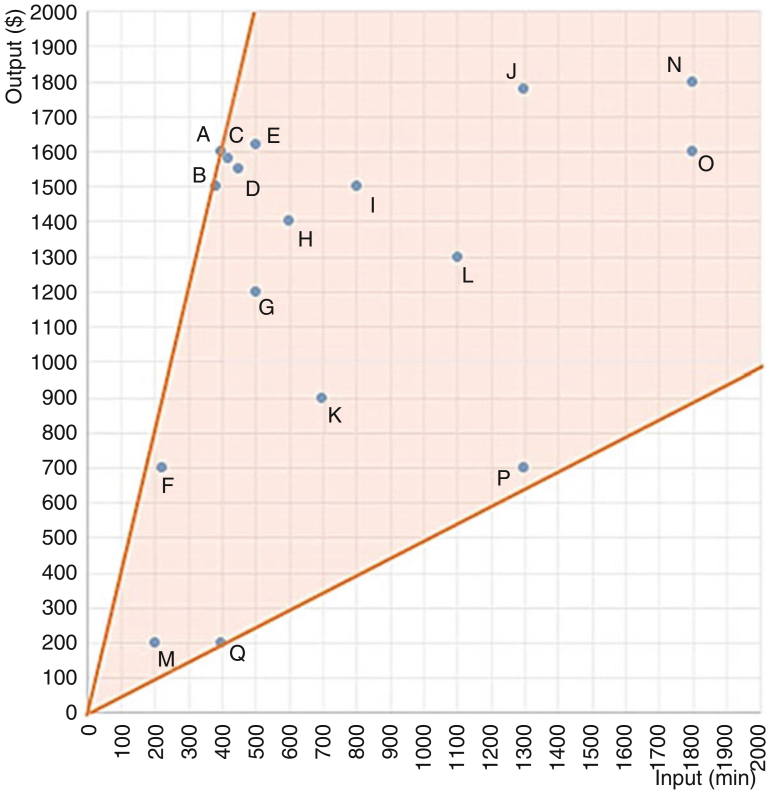


**Fig. 2.16** The convexity and the radiate approaches

is also assumed that the convexity approach by Eq. 2.16 is applied and then the radiate approach is applied, the combination is still non-linear.

However, the convexity approach can be extended by the increasing the domains of the multipliers $\lambda_i$, for $i = 1, 2, \ldots, n$, in Eq. 2.16. In other words, if it is supposed that $\lambda_i \geq 0$, for $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} \lambda_i = 1$ is removed in Eq. 2.16, both the convexity and the radiate approaches are simultaneously applied and the mathematical formula is also linear (See Theorem 2.6). For instance, if it is supposed that $\lambda_1 = \lambda$ and $\lambda_i = 0$ for $i = 2, \ldots, 17$, the point $\left(\sum_{i=1}^{17} x_i\lambda_i, \sum_{i=1}^{17} y_i\lambda_i\right)$ is the same as $(\lambda x_1, \lambda y_1)$ which displays Eq. 2.18. Every line in Fig. 2.9 can be illustrated by the pair $\left(\sum_{i=1}^{17} x_i\lambda_i, \sum_{i=1}^{17} y_i\lambda_i\right)$.

**Theorem 2.6** *Suppose that* $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, …, and $A_{17}(x_{17}, y_{17})$ denote the observations A–Q, respectively. Equation 2.30 generates the shaded area in Fig. 2.16, linearly.

$$\{(x', y') : \quad x' = \sum_{i=1}^{17} x_i \lambda_i, \ \ y' = \sum_{i=1}^{17} y_i \lambda_i, \ \ \Lambda \in \mathbb{R}_{+}^{17}\}. \tag{2.30}$$

where $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{17})$. and $\lambda_i$'s are non-negative real numbers for $i = 1, 2, \ldots,$ 17, which at least one of them is positive.

*Proof* The proof is leaved as an exercise. □

From Fig. 2.16, the company can introduce two frontiers. One frontier passes the point A and another frontier passes the point Q. The points on the first frontier can be introduced as the points which have done the job right, and the points on the second frontier can be introduced as the points which have done the job worst.

This phenomenon yields that every point in the shaded area can be written with a linear combination of the A and Q coordinates, which means, every point between the two lines in the shaded area in Fig. 2.16, is a combination of two points in which a point has done the job right and another point has done the job worst. For instance, the following equation system illustrates how the point K(700, 900) is generated by the points A and Q.

$$\begin{bmatrix} 700 \\ 900 \end{bmatrix} = \begin{bmatrix} 400\lambda_A + 400\lambda_Q \\ 1600 y_A + 200\lambda_Q \end{bmatrix} \Leftrightarrow \begin{cases} 400\lambda_A + 400\lambda_Q = 700 \\ 1600 y_A + 200\lambda_Q = 900 \end{cases}$$

$$\Leftrightarrow \lambda_A = \frac{11}{28} \quad and \quad \lambda_Q = \frac{19}{14}.$$

By the same illustration, the combination of the convexity and the inner/outer approaches can be proposed by Eqs. 2.31 and 2.32, respectively, which result Figs. 2.17 and 2.18, respectively.

$$\{(x', y') : \ x' = \sum_{i=1}^{17} x_i \lambda_i, \ \ y' = \sum_{i=1}^{17} y_i \lambda_i, \ \ 0 \le \lambda_i \le 1, \ \ \text{for} \ \ i = 1, 2, \ldots, 17\}. \tag{2.31}$$

$$\{(x', y') : \ x' = \sum_{i=1}^{17} x_i \lambda_i, \ \ y_i = \sum_{i=1}^{17} y_i \lambda_i, \ \ \lambda_i \ge 1, \ \ \text{for} \ \ i = 1, 2, \ldots, 17\}. \tag{2.32}$$

Figure 2.17 depicts the combinations of the convexity approach and the inner radiate approach, according to Eq. 2.21, and Fig. 2.18 depicts the combination of the convexity approach and the outer radiate approach, according to Eq. 2.32.

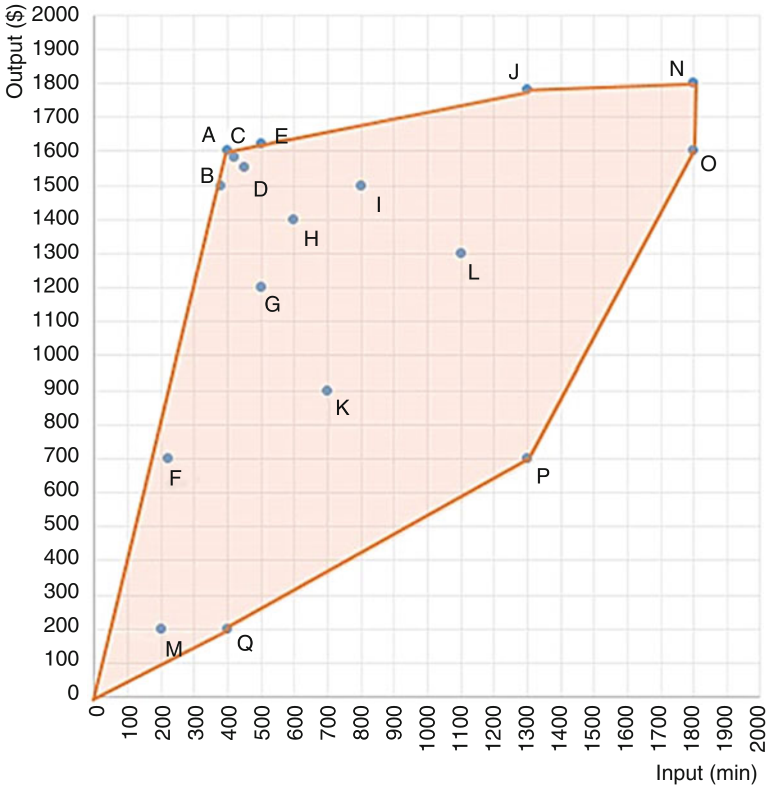


**Fig. 2.17** The convexity and the inner radiate approaches

#### 2.2.4.4 The Wholly Dominant, the Convexity and the Radiate Approaches

Similar to the previous sections, the combination of these three approaches can be considered as Theorem 2.7 illustrates, and the generated area by Eq. 2.33 is the same as the shaded area in Fig. 2.13.

**Theorem 2.7** *Suppose that* $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, …, *and* $A_{17}(x_{17}, y_{17})$ *denote the observations A–Q, respectively. Equation 2.33 generates the shaded area in Fig. 2.13, linearly.*

$$\{(x', y') : \sum_{i=1}^{17} x_i\lambda_i \le x', \ y' \le \sum_{i=1}^{17} y_i\lambda_i, \ \Lambda \in \mathbb{R}_+^{17}\}. \tag{2.33}$$

where $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{17})$ and $\lambda_i$'s are non-negative real numbers for $i = 1, 2, \ldots, 17$, which at least one of them is positive.

*Proof* The proof is leaved as an exercise. □

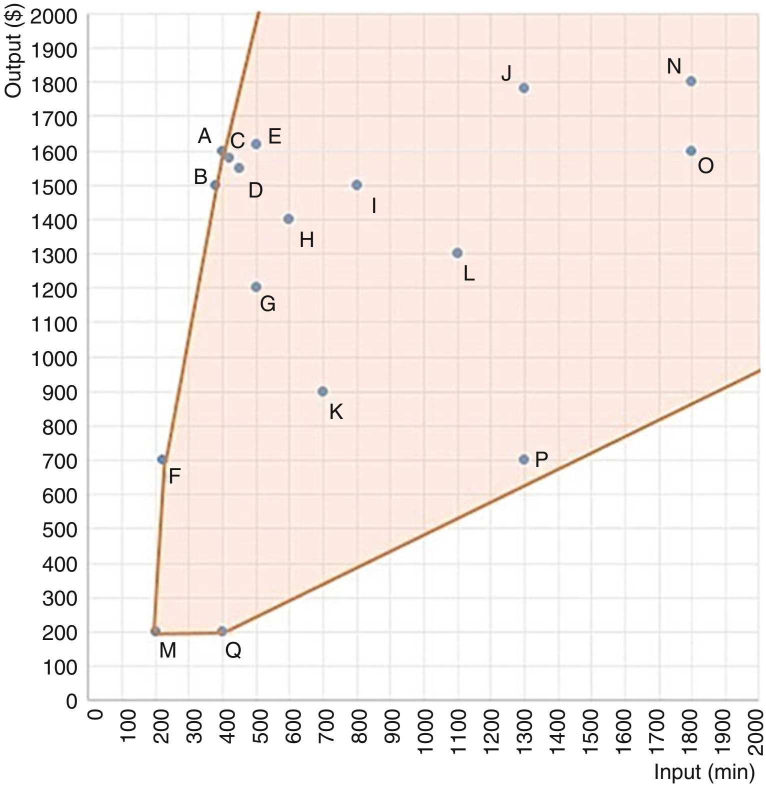


**Fig. 2.18** The convexity and the outer radiate approaches

Note that, in this task with one input factor and one output factor, the combination of the radiate and the wholly dominant approaches and the combination of the convexity, the radiate and the wholly dominant approaches result in the same shaded area as shown in Fig. 2.13. Is it deduced that these two combinations are the same? (See the exercise 2.7.)

By the same illustration, the combination of the convexity, the inner (outer) radiate and the wholly dominant approaches can also be introduced (see the exercise 2.8).

### *2.2.5 The Other Approaches*

The approaches in the previous sections are linear approaches (or, at most, combination of two linear approaches). The company can use a multitude other relations to introduce a practical point from an observed point. For instance, Fig. 2.19 depicts the natural logarithm function $y = \ln x$ on interval $(0, 50]$ to interval $[-1, 5]$.

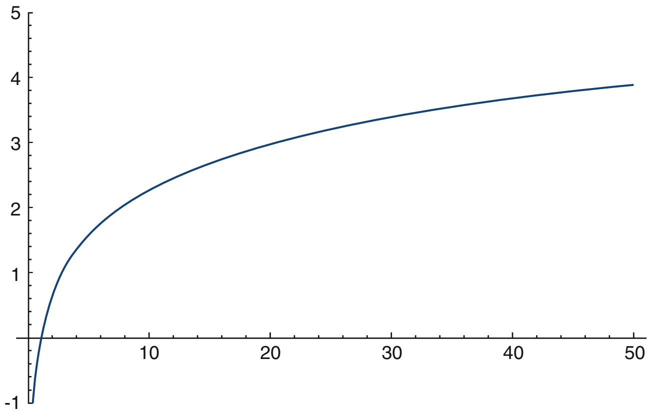


**Fig. 2.19** The natural logarithm function

The company can assume that if a point with positive coordinates, $(x, y)$, is observed, then every point $(t, \ln(t + 1 - x) + y)$ is also practical, for some real value $t$. Figure 2.20 illustrates the practical points from the observations A–Q which are generated by the following equation.

$$\bigcup_{i=1}^{17} \{(t, \ln(t + 1 - x_i) + y_i) : t > x_i - 1, \ln(t + 1 - x_i) \geq -y_i\}. \tag{2.34}$$

## 2.3 Homogeneity and the Relative Score

The comparisons between each two candidates in the gemstone example certainly depend upon the selected input and output factors, their dimensions, units and the worth of each factor in comparison with another factor. For instance, the used time for each candidate should have the same measurement, dimension, unit and worth. In other words, if the used time for one candidate is in minute unit and for another is in hour unit, measuring the relative scores of these two candidates are not appropriate unless both the used times are transferred to one of minute unit or hour unit. In addition, the company cannot determine the worth (price) of the used time for a candidate different with the worth (price) of the used time for another candidate. One minute unit to measure the score of A should have the same meaning with 1 min unit to measure the score of B, and vice versa.

On the other hand, in the gemstone example, the candidate has the same situation. If the company does not give the same amount of money to each candidate, or each candidate was in different state or country, or the worth of time is more than the worth of money for some candidate, and the company displays the same data in

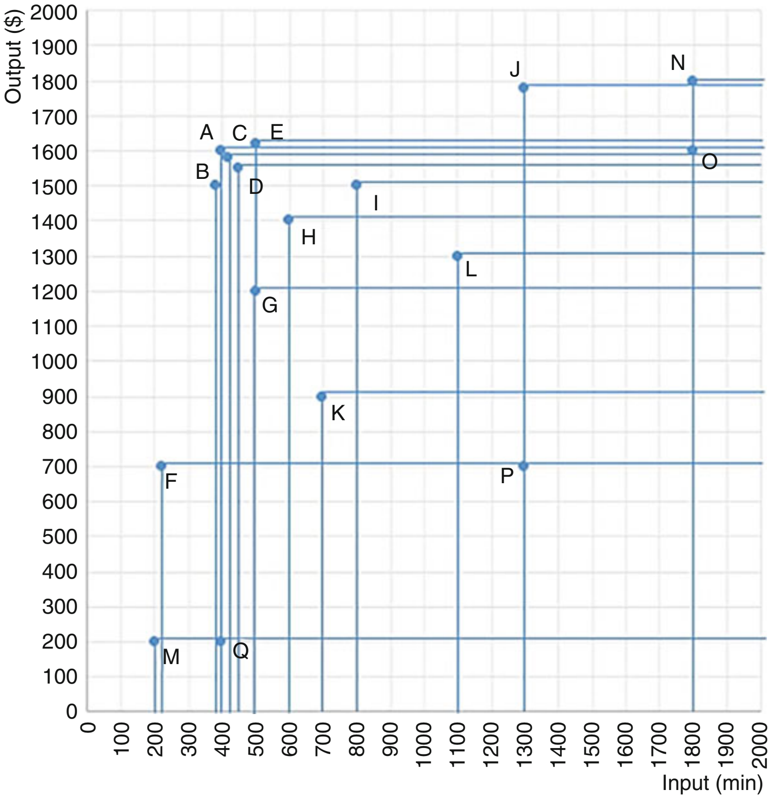


**Fig. 2.20** The natural logarithm approach

Table 2.1, the relative scores in Table 1.4 do not represent the fair discriminations between candidates A–Q. In this situation, for instance, it is not clear whether candidate A had a great performance due to have a higher or a lesser amount of money in comparison with others. Therefore, one of the most important conditions to compare two candidates in gemstone example is that the situations for each candidate should be homogenous. The requirement to define the word 'homogenous' is introduced in the next chapter. Indeed, one way to decrease the concerns about the homogeneity of candidates is to increase the effected factors. The next chapters extend the methodology while the number of input and output factors increases. The following discussions mathematically illustrate the requirement to compare two different candidates regarding Eq. 1.2, that is, output/input, while there are one input and one output factors.

### 2.3.1 A Philosophical Discussion

Suppose that there are two candidates, labeled by, $A_1$ and $A_2$ which $A_i$ has two factors, for $i = 1, 2$. One factor is an input factor with a positive real value, labeled by $x_i$, and another factor is an output factor with a positive real value, labeled by $y_i$, for $i = 1, 2$. The score of $A_i$ is calculated by $S(A_i) = y_i/x_i$, for $i = 1, 2$, according to Eq. 1.2, that is, output/input.

If the units of measurement for $x_i$ and $y_i$ are in \$ unit, then the unit of measurement for the scores of $A_i$ is calculated by \$/\$, for $i = 1, 2$. Note that, \$/\$ is equal to 1 (unitless or dimensionless) as well as *m*/*m*, £/£, and so on, which in this article is called unity scale and denoted by 1, as described in Sect. 1.7. Here, the symbols \$, £ and *m* indicate US dollar, UK pound and meter, respectively.

If the unit for $x_i$ is \$ and the unit for $y_i$ is *m*, the unit for the scores of $A_i$ is *m*/\$, and called 'meter per dollars', for $i = 1, 2$. In both of these situations, the comparison between the scores of $A_1$ and $A_2$ are valid and the maximum score can be identified. Thus, the relative score for $A_i$, described in Sect. 1.6, can be calculated by the following equation, for $i = 1, 2$.

$$\mathrm{ReS}(A_i) = \frac{y_i/x_i}{\max\{y_1/x_1, y_2/x_2\}}. \tag{2.35}$$

However, if the units for $x_1$ and $x_2$ are £ and \$, respectively, and the units for both $y_1$ and $y_2$ are \$, the units for the scores of $A_1$ and $A_2$ are "\$/£" and "1", respectively. This means that the comparison between the scores of $A_1$ and $A_2$ in Eq. 2.35 is not logical, except when the units of measurement are converted to the same unit.

As a result, the first assumption for comparing the scores of $A_1$ and $A_2$ is that the corresponded units of measurement for the input and output factors of $A_1$ and $A_2$ should be identical. This means, if the unit of $x_1$ is £ and the unit of $y_1$ is *m*, the unit of $x_2$ and $y_2$ should be £ and *m*, respectively, which yields $x_1/x_2$ and $y_1/y_2$have the unity scale.

On the other hand, the equality $x_1 = x_2$ is valid if the worth (price/weight) of one unit of $x_1$ is the same as the worth (price/weight) of one unit of $x_2$. For instance, suppose that $x_1 = 4\,(m^2)$ and $x_2 = 4\,(m^2)$, where the worth (price/weight) of $x_1$ is twice the worth (price/weight) of $x_2$, that is, one square meter of $x_1$ is equal to two square meter of $x_2$, and therefore, $x_1 = 2 \times x_2$, and $x_1 \neq x_2$. In this case, the Eq. 2.35 is not also valid, except the worth (price/weight) of $x_1$ and $x_2$ are identical or their worth (price/weight) are multiplied.

Consequently, if it is supposed that the worth (prices/weights) of $x_1$ and $x_2$, labeled by, $u_1$ and $u_2$, respectively, are unknown while $x_1$ and $x_2$ have an identical unit, the relationship between $x_1$ and $x_2$ are ambiguous and finding the minimum and maximum of $x_1$ and $x_2$ are not possible, except $u_1$ and $u_2$ are specified and multiplied to $x_1$ and $x_2$, respectively. Hence, the relation between $x_1$ and $x_2$ completely depend upon $u_1$ and $u_2$. For example, suppose that $x_1 = 4\,(m^2)$ and $x_2 = 8\,(m^2)$, and $u_1$ and $u_2$ are unknown. The equations $x_1 < x_2$ and $u_1x_1 < u_2x_2$ are ambiguous, that is, they can be true (if $u_1 < 2u_2$) or false (if $u_1 \geq 2u_2$). Thus, how can we depict $x_1$ and $x_2$ on an axis, if it is supposed that the worth (price/weight) of $x_1$ and $x_2$ are unknown?

As a result, the second assumption for comparing the scores of $A_1$ and $A_2$ is that the corresponding worth (prices/weights) for the input and output factors of $A_1$ and $A_2$ should be identical. This means, if the worth (price/weight) of $x_1$ is $u_1$ and the worth (price/weight) of $y_1$ is $\nu_1$, the worth (price/weight) of $x_2$ (that is, $u_2$) and the worth (price/weight) of $y_2$ (that is, $\nu_2$) should be equal to $u_1$ and $\nu_1$, respectively, that is $u_1 = u_2$ and $v_1 = v_2$. Otherwise, if $u_1, u_1, \nu_1$ and $\nu_2$ are unknown the locations of $A_1$ and $A_2$ in the Cartesian coordinate plane are also unknown.

In short, for each two sets of DMUs, $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$, the relation between $x_1$ and $x_2$, and the relation between $y_1$ and $y_2$, should be available, and let us to write the following inequalities, for instance, $x_1 < x_2$ and $y_1 > y_2$, and let us to depict the locations of $A_1$ and $A_2$ in the Cartesian coordinate plane, that is, both $x_1$ and $x_2$ (or $y_1$ and $y_2$) should have an identical unit and an identical worth (prices or weights). Note that, the relation between $x_i$ and $y_i$ can be unavailable, because Eq. 2.35 has the unity scale under the above assumptions, and yields that the comparison between the scores of $A_1$ and $A_2$ become valid.

## 2.4 The Preplanned Purpose/Goal

In the gemstone example in Sect. 1.2, the aim is to find the candidates who performed in the best possible manner with least used time and most earned money. This concept is called as ‘doing the job well’, and the ratio of the earned money to the used time is introduced to calculate which one of the candidates have done the job well. In addition, a feasible area is generated from the observations in Sect. 2.2, regarding to an introduced approach, and the frontier of that area is supposed as the maximum possible amount of money from a given amount of time. This concept is called ‘doing the job right’ which is a necessary condition for the concept of ‘doing the job well’.

On the other hand, it is always possible and common for a company to have a goal or a purpose to hire a candidate who is at least eligible for some specified requirements. In such situation, the concept of doing the job well is not enough to select the best candidates.

For example, suppose that the company has aimed that each candidate should at least have earned \$1500, and the results in Table 2.1 is given. From the table, only the nine candidates A, B, C, D, E, I, J, N and O are eligible, and have accomplished this purpose. This concept, to find the candidates who are successful in performing the desired result, is called ‘*doing the well job*’, as the following definition illustrates.

**Definition 2.2** A candidate, who has done the well job, should satisfy all the purposes/goals of the company.

**Table 2.2** The candidates who have done the well job

| Candidate | Score | Relative score | Candidate | Score | Relative score |
|---|---|---|---|---|---|
| A | 1.07 | 0.89 | J | 1.19 | 0.99 |
| B | 1.00 | 0.83 | K | 0.60 | 0.50 |
| C | 1.05 | 0.88 | L | 0.87 | 0.72 |
| D | 1.03 | 0.86 | M | 0.13 | 0.11 |
| E | 1.08 | 0.90 | N | 1.20 | 1.00 |
| F | 0.47 | 0.39 | O | 1.07 | 0.89 |
| G | 0.80 | 0.67 | P | 0.47 | 0.39 |
| H | 0.93 | 0.78 | Q | 0.13 | 0.11 |
| I | 1.00 | 0.83 | | | |

As Eq. 2.36 illustrates, the ratio of the earned money to the desired outcome can be introduced as a measure to find which one of the candidates have done the well job. Similarly, while the desired results are unknown the concept of *doing the right job* can be introduced, from different approaches.

$$\frac{\text{Outcome}}{\text{Desired outcome}} \tag{2.36}$$

When the value of Eq. 2.36 for a candidate is greater than or equal to 1, the candidate has done the well job. As the numerator in Eq. 2.36 is increases the value of Eq. 2.36 is also increased. The unit of measurement in Eq. 2.36 is the unity scale. In order to have a score between 0 and 1, the relative score for each candidate can be measured by the score of that candidate over the maximum score. Table 2.2 illustrates the results of Eq. 2.36 and the relative scores for each candidate. The bolded scores represent the candidates who have done the well job, regarding to at least earning $1500.

Now, for each candidate, there are two different relative scores, corresponding to the concepts of doing the job well and doing the well job. For instance, the company can depict the results in the Cartesian coordinate plane, as Fig. 2.21 illustrates.

The horizontal axis in Fig. 2.21 represents the relative scores in Table 1.4, corresponding to the concept of doing the job well, and the vertical axis displays the relative scores in Table 2.2, corresponding to the concept of doing the well job. There are four different area in Fig. 2.21, colored green, yellow and red. The green area illustrates the points which their relative scores by the concept of doing the job well is greater than 0.70 and their relative scores by the concept of doing the well job is also greater than 0.70. This area can be introduced as the area that represents the candidates who have done *the perfect job* according to the aim of the company and the available information, according to following definition. For example, candidates A–E have done the perfect job.

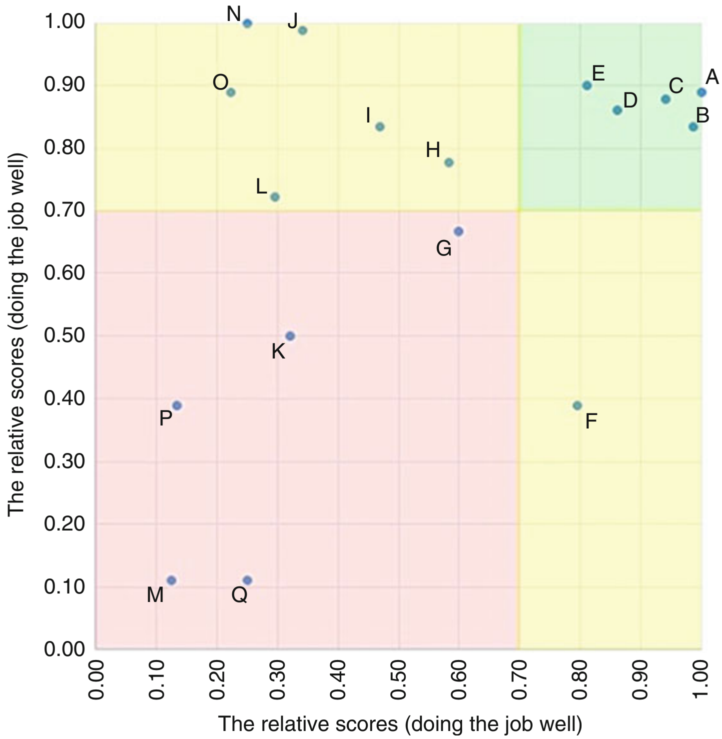


**Fig. 2.21** The candidates who have done the perfect job

**Definition 2.3** A candidate, who has done the perfect job in comparison with another candidate, should simultaneously have done the job well and the well job in comparison with the other candidate.

The yellow area represents that the points have satisfied only one of the concepts of doing the job well and doing the well job. For example, F has done the job well, and has not done the well job, whereas the candidates H, I, J, L, N and O have done the well job, and have not done the job well. The yellow area can be introduced as the area that represents the candidates who have done *the semi-perfect job*, according to the aim of the company and the available information.

The red area displays that the points have neither done the job well nor done the well job. This area can be introduced as the area that represents the candidates who have done *the imperfect job*, according to the aim of the company and the available information. Thus, the candidates G, K, M, P and Q have done the imperfect job.

The value 0.70 to partition each region in Fig. 2.21 can be changed according to the purpose of the company. While the point with the coordinates (0.70, 0.70) in Fig. 2.21 is changed, different candidates can be introduced as those have done the perfect job, the semi-perfect job or the imperfect job. The candidates in each area in

Fig. 2.21 can also be ranked according to the purpose of the company to weight the relative scores in each axis. The points which lie between two regions can also be considered as the points of the weaker region.

On the other hand, the desired outcome value may not be available either from an unclear view of the company or lack of information. The desired outcome can also be depended upon the number of dealers that the company needs to hire, and so on. In addition, both the input and output factors may have desired values as well. For example, suppose that the company has aimed that each candidate should at most use 600 min and at least earn \$1500, and the results in Table 2.1 is obtained. The points which satisfy this purpose should dominate the point (600, 1600). Thus, only candidates A–E would be eligible, and have done the well job. In order to discriminate A–E according to the concept of doing the well job, the company can find the value of Eq. 2.36 for the output factor and the inverse value of Eq. 2.36 for the input factor. After that, the company should weight each factor, and can introduce the sum of the calculated values as a measure to discriminate A–E, according to the concept of doing the well job. Of course, different set of weights for the factors yields different outcomes, and there is a need to introduce the weights by expert judgment or some statistical confidence intervals. This situation is discussed in the next chapters.

## 2.5 Conclusion

This chapter provides important concepts and axioms to introduce a methodology for making a fair decision. The gemstone example, with one input factor and one output factor, is reviewed to introduce several new concepts. Meanwhile, the required concepts for future applications are illustrated step by step, and every concept is illustrated by the gemstone example. Mathematical background and techniques as well as logical interpretations of each technique are clearly elucidated in order to assist readers to learn the calculations and methodologies, especially while the number of input and output factors increases in the next chapters. The major subjects which are discussed in this chapter are to describe and understand their possible meanings as well as the dominant, convexity and radiate approaches, and to reveal the importance of the homogeneity condition. In addition, a valuable discussion is provided to discriminate the meaning of doing the job well and doing the job right. It is illustrated that the relative scores are supported by the meaning of doing the job well, not doing the job right. In addition, the concepts of doing the well (right) job and doing the perfect job are also required to discriminate between each candidate. Without the concept of doing the job well, doing the well job, and doing the perfect job, the discrimination is ambiguous and incomplate. These concepts demonstrate how to measure the performance of factories and organizations where there are multiple input factors and multiple output factors.

## 2.6 Exercises

2.1. Using Exercise 1.1, answer the following questions:

  2.1.1. Display the feasible area by applying the convexity approach for data in Table 1.8.
  2.1.2. Display the feasible area by applying the outer radiate approach for data in Table 1.8.
  2.1.3. Display the feasible area by applying the wholly dominant and the inner radiate approaches for data in Table 1.8.
  2.1.4. Display the feasible area by applying the wholly dominant and the outer radiate approaches for data in Table 1.8.
  2.1.5. Write the mathematical equation to generate the feasible area in Exercise 2.1.1, linearly.
  2.1.6. Write the mathematical equation to generate the feasible area in Exercise 2.1.3, linearly.
  2.1.7. Which investors have done the jobs right by the wholly dominant approach?
  2.1.8. Which investors have done the jobs right by the convexity approach?
  2.1.9. Assume that the purpose of this task is to earn at least 23 million dollars. Which investors have done the well job? Which investors have done the right job?
  2.1.10. Display the area that the investors have done the useful job, while the investor should at least obtain the third quartile of the relative scores based upon both the concepts of doing the job well and doing the well job.

2.2. Can one use the following equation instead of Eq. 1.2 to find the best candidate in the gemstone example?

$$\text{Output} - \text{Input} \tag{2.37}$$

What are the advantages, disadvantages, shortcomings or requirements for such an aim?

2.3. Can Eq. 2.37 be used for Exercise 1.1? What are the pros and cons for such an aim?
2.4. Prove Theorems 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, and 2.7 and Lemma 2.1.
2.5. What are the differences between the concepts of doing the job well and doing the job right?
2.6. What are the differences between the concepts of partially dominant and wholly dominant?

2.7. Give a counterexample to prove that the combination of the radiate and the wholly dominant approaches and the combination of the convexity, the radiate and the wholly dominant approaches do not result the same feasible area.
2.8. Depict the feasible area by the combination of the convexity, the inner (outer) radiate and the wholly dominant approaches for the gemstone example.

# Chapter 3
# The Petroleum Example

## 3.1 Introduction

In this chapter, the introduced concepts in Chaps. 1 and 2 are adjusted for an example of two input factors. When there is only one input factor, it is enough to know that the input factor of each firm has the same unit of measurement, regardless of what exactly that unit is. But, when there are two input factors, in order to have a linear combination of input factors, not only should the previous condition be true for each factor, but also the unit of measurement for each input factor should be available. Detailed discussions are provided in this chapter to deal with the situations when the units of measurement are unknown. In other words, in most real-life applications the unit of measurement or relationships between two factors are unknown. At the end of this chapter, readers learn how to measure the performance of firms in which each firm has two input factors, and how the outcome can be different when the introduced weights or prices for the two input factors are changed.

## 3.2 Petroleum Example

The Environmental Protection Agency (EPA) plans to encourage its subdivisions to consume lesser amounts of diesel fuel and gasoline. An environmentalist fairly divides the subdivisions into 18 homogenous divisions.

Table 3.1 illustrates the expenditures of diesel fuel and gasoline consumption for the 18 divisions, labeled A-R, after a year of starting the plan. Both diesel fuel and gasoline amounts are represented in 10,000 US dollars units; and the cost of each gallon of diesel fuel or gasoline is the same from one division to another. How can the environmentalist introduce the best divisions which consumed lesser amounts of



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**Table 3.1** The costs of consuming Diesel fuel and Gasoline

| Division | Diesel fuel ($10,000) | Gasoline ($10,000) |
|---|---|---|
| A | 12,247.448710 | 24,494.897430 |
| B | 53,888.774340 | 20,820.662810 |
| C | 34,292.856400 | 34,292.856400 |
| D | 7348.469228 | 46,540.305110 |
| E | 19,595.917940 | 31,843.366660 |
| F | 58,787.753830 | 19,595.917940 |
| G | 14,696.938460 | 36,742.346140 |
| H | 39,191.835880 | 41,641.325630 |
| I | 19,595.917940 | 22,045.407690 |
| J | 17,146.428200 | 41,641.325630 |
| K | 12,247.693660 | 24,495.142380 |
| L | 61,237.243570 | 24,494.897430 |
| M | 9797.958971 | 31,843.366660 |
| N | 68,585.712800 | 29,393.876910 |
| O | 44,090.815370 | 26,944.387170 |
| P | 8573.214100 | 44,090.815370 |
| Q | 12,222.953820 | 24,568.382120 |
| R | 36,742.346140 | 20,973.755920 |

both diesel fuel and gasoline as well as arranging the divisions based on the lesser amounts of both petroleum products?

Similar to Chap. 1, a division which consumed the least amount of both diesel fuel and gasoline can be introduced as the division which has "done the job well".

A simple way to answer the question is to calculate the sum of the diesel fuel and the gasoline amounts for each division, and then find the minimum value of the summations to identify the least amounts of diesel fuel and gasoline. Since both the petroleum products amounts are in ten thousand dollars units, they can easily be added together, as Eq. 3.1 illustrates.

$$\text{Diesel fuel } (\$10,000) + \text{Gasoline } (\$10,000) \tag{3.1}$$

The third column in Table 3.2 illustrates the results of Eq. 3.1 for A-R and the fourth column displays the ranks of divisions. A division with a lesser value of Eq. 3.1 gets a higher rank.

Similar to Sect. 1.5, a score can be provided for each division. The score should be decreased (increased) while the value of Eq. 3.1 increased (decreased).

Therefore, the score for each division can be calculated by Eq. 3.2. The results of Eq. 3.2 are also represented in the fifth column of Table 3.2. If the scores are sorted descending, the results are the same as the results in the fourth column of Table 3.2.

$$\frac{1}{\text{Diesel fuel } (\$) + \text{Gasoline } (\$)} \tag{3.2}$$

**Table 3.2** The relative scores of divisions A-R

| Division | N | Sum ($10,000) | Rank | Score | Relative score |
|---|---|---|---|---|---|
| A | 1 | 36,742.346140 | 1 | 0.0000272166 | 1.000000 |
| B | 2 | 74,709.437150 | 14 | 0.0000133852 | 0.491803 |
| C | 3 | 68,585.712800 | 12 | 0.0000145803 | 0.535714 |
| D | 4 | 53,888.774338 | 9 | 0.0000185567 | 0.681818 |
| E | 5 | 51,439.284600 | 6 | 0.0000194404 | 0.714286 |
| F | 6 | 78,383.671770 | 15 | 0.0000127578 | 0.468750 |
| G | 7 | 51,439.284600 | 6 | 0.0000194404 | 0.714286 |
| H | 8 | 80,833.161510 | 16 | 0.0000123712 | 0.454545 |
| I | 9 | 41,641.325630 | 4 | 0.0000240146 | 0.882353 |
| J | 10 | 58,787.753830 | 11 | 0.0000170103 | 0.625000 |
| K | 11 | 36,742.836040 | 2 | 0.0000272162 | 0.999987 |
| L | 12 | 85,732.141000 | 17 | 0.0000116642 | 0.428571 |
| M | 13 | 41,641.325631 | 5 | 0.0000240146 | 0.882353 |
| N | 14 | 97,979.589710 | 18 | 0.0000102062 | 0.375000 |
| O | 15 | 71,035.202540 | 13 | 0.0000140775 | 0.517241 |
| P | 16 | 52,664.029470 | 8 | 0.0000189883 | 0.697674 |
| Q | 17 | 36,791.335940 | 3 | 0.0000271803 | 0.998668 |
| R | 18 | 57,716.102060 | 10 | 0.0000173262 | 0.636605 |

In order to have a relative score between 0 and 1, similar to Sect. 1.7, the maximum score of divisions should be found; and then by calculating the score of each division over the maximum score, the relative score can be measured.

For instance, suppose that the divisions A-R are numbered from 1–18 and the diesel fuel and the gasoline factors are shown with, $x_1$ and $x_2$, respectively. Equation 3.3 illustrates the formula to measure the relative score of the division $l$, for $l = 1$, 2, . . ., 18, where $x_{i1}$ and $x_{i2}$ represents the diesel fuel and the gasoline amounts for division $i$, respectively ($i = 1, 2, \ldots, 18$).

$$\frac{\frac{1}{x_{l1}+x_{l2}}}{\max\left\{\frac{1}{x_{i1}+x_{i2}} : i = 1, 2, \ldots, 18\right\}}, \quad \text{for } l = 1, 2, \ldots, 18. \tag{3.3}$$

As can be seen, A has the best relative score among divisions followed by K, Q, I and M and so on, while divisions N, H, L and F have the worst relative scores, respectively. These results interpret that divisions A, K, Q, I and M have done the job well in comparison with other divisions, respectively, and divisions N, H, L and F have not done the job well in comparison with other divisions, respectively.

The sixth column of Table 3.2 illustrates the relative score for divisions A-R. Equation 3.3 can also be simplified as Eq. 3.4

$$\frac{\min\{x_{i1} + x_{i2} : i = 1, 2, \ldots, 18\}}{x_{l1} + x_{l2}}, \quad \text{for } l = 1, 2, \ldots, 18. \tag{3.4}$$

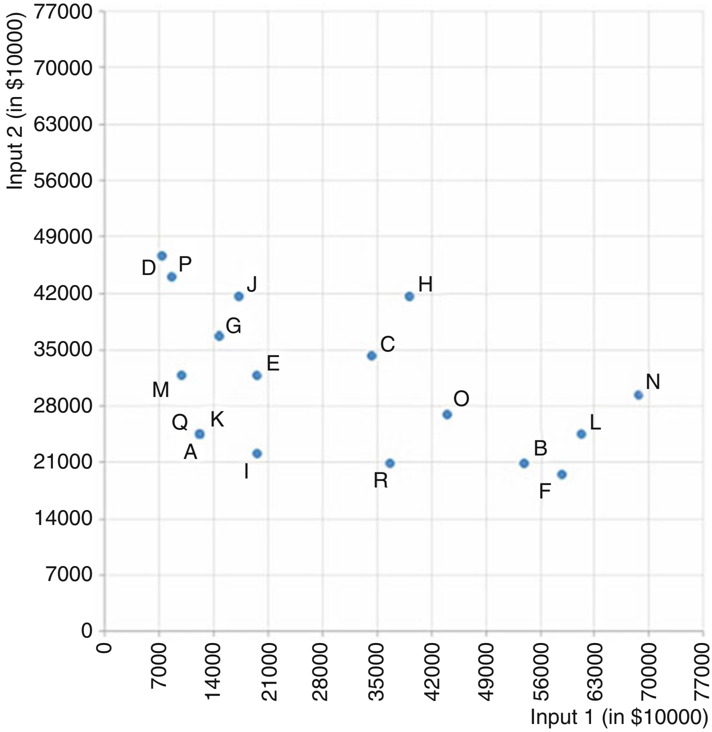


**Fig. 3.1** The divisions A-R

The environmentalist can suggest the combination of the diesel fuel and the gasoline amounts by A as an example to other divisions and suggest that they should use lesser amounts of these two petroleum products. If the relative scores are sorted descending, the results are the same as the results in the fourth column of Table 3.2.

Note that, if the environmentalist assigned different worth (prices or weights) between the amounts of diesel fuel and gasoline, that is, the wroth (prices or weights) of diesel fuel is not the same as the worth (prices or weights) of gasoline, Eqs. 3.1, 3.2, 3.3, and 3.4 should not be used. This important discussion is elucidated in Sect. 3.5.

## 3.3 The Geometry Interpretation

Figure 3.1 depicts the location of each observation in Table 3.1 while the unit of each factor is measured in $10,000 units.

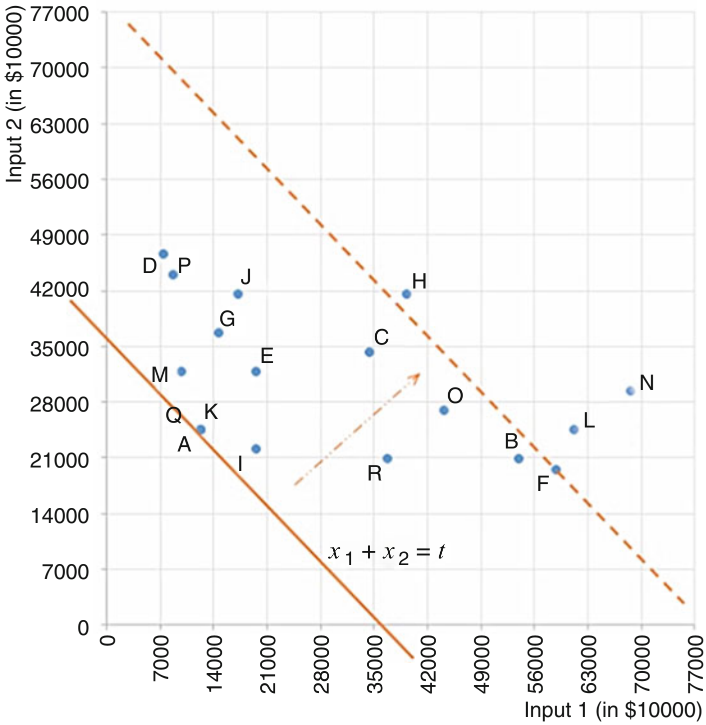


**Fig. 3.2** The line with the slope − 1 which passes A

Since the lesser amounts of the diesel fuel and the gasoline factors have value, they are input factors (see Sect. 1.3).

In Fig. 3.1, the horizontal axis describes the amounts of diesel fuel in $10,000 units and the vertical axis describes the amounts of gasoline in $10,000 units.

Equation 3.1 can be written as $x_1 + x_2$, while $x_1$ displays the first input factor (diesel fuel) and $x_2$ displays the second input factor (gasoline). Now, the formula, $x_1 + x_2 = t$, represents a line with the slope $-1$, as shown in Fig. 3.2, which can be moved to find the best combinations of the two input factors.

If $t = 0$, the line passes the origin. Thus, by moving the line through the observations, the first connection of the line with the observations represents the minimum value of $t$, that is, $367,423,461.40, which displays division A. If moving the line through the observations is continued, the last connection of the line with the observations represents the maximum value of $t$, which displays division N. These outcomes are the same as the results in Table 3.2.

## 3.4 The Practical Points

Similar to Chap. 1, the practical points can be proposed by an introduced approach such as the wholly dominant approach, the convexity approach, the natural logarithm approach and so on.

For instance, Fig. 3.3 depicts the area which is wholly dominated by E and the area which wholly dominates E. As can be seen, divisions A, K, Q, I and M wholly dominate E, whereas E wholly dominates C and H. All the points in the top right of the shaded area in Fig. 3.3 have greater values of the first and the second input factors in comparison with that of E. Every arbitrary point in the shaded area in the top right in Fig. 3.3 with the coordinates $(x_1, x_2)$, satisfies the following inequalities, while $x_1$ displays the first input factor and $x_2$ displays the second input factor.

$$19,595.91794 \leq x_1, \quad 31,843.36666 \leq x_2 \tag{3.5}$$

Figure 3.4 also depicts the area which is partially dominated by E and the area which partially dominates E. Indeed, every point above the line $x_1 + x_2 = 51,439.2846$, is partially dominated by E, and every point lower than that line, partially dominates E. For instance, A, K, Q, I and M partially dominates E and the other

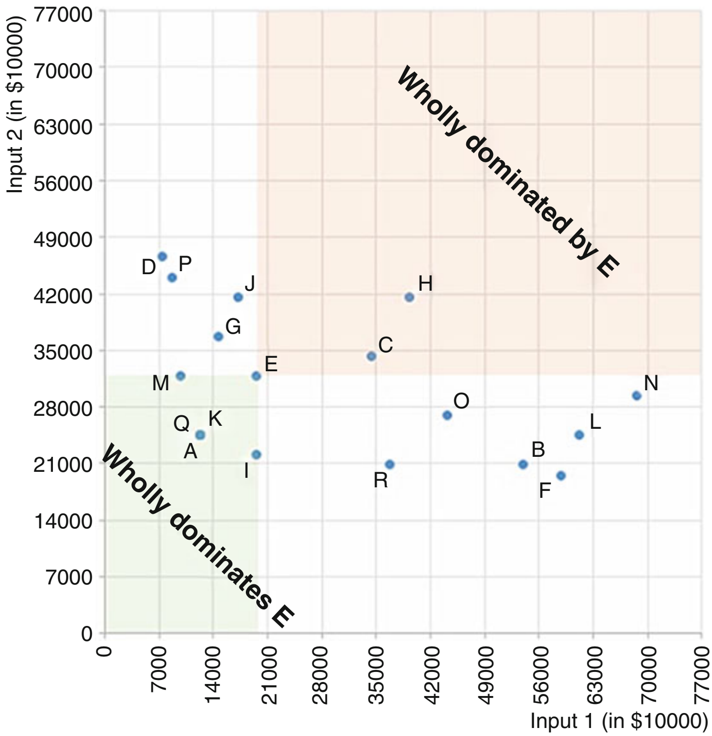


**Fig. 3.3** The area which wholly dominated by E

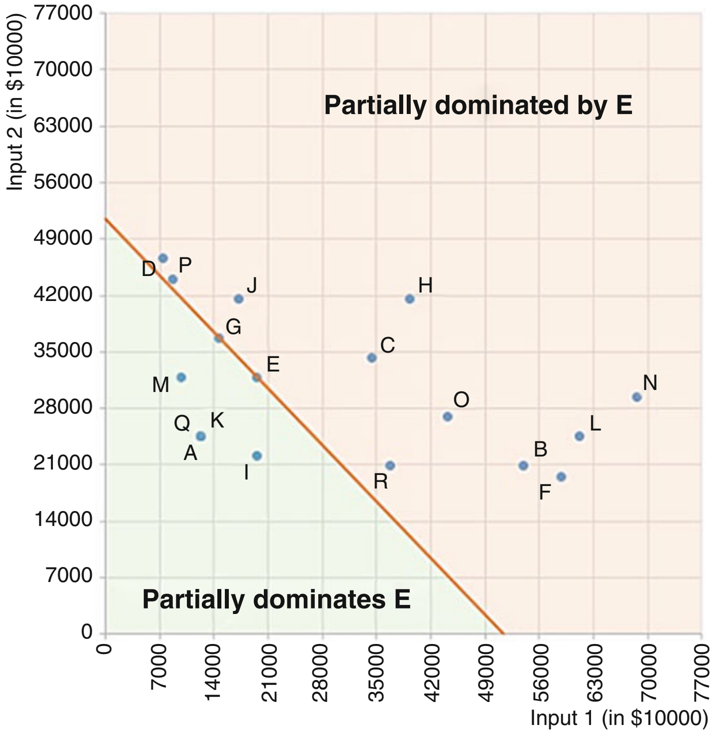


**Fig. 3.4** The area which partially dominated by E

divisions are partially dominated by E. As a result, the relative score of E is less than those of divisions A, K, Q, I and M, and greater than the relative scores of other observations which are partially dominated by E.

From Fig. 3.2, it is clear that division A partially dominates all divisions, and division F partially dominates divisions H, L and N.

Similar to Sect. 2.2.1, by applying the wholly dominant approach for each observation, the shaded area in Fig. 3.5 is generated. The Eq. 3.6 also illustrates the shaded area in Fig. 3.5.

$$\bigcup_{i=1}^{18} \left\{ \left( x_1', x_2' \right) : \ x_{i1} \le x_1', \quad x_{i2} \le x_2' \right\}. \tag{3.6}$$

**Theorem 3.1** *Suppose that $A_i(x_{i1}, x_{i2})$, for $i = 1, 2, \ldots, 18$ denote the observations in Table 3.1. Equation linearly 3.6 yields the shaded area in Fig. 3.5.*

The frontier of the shaded area depicts the border between practical and non-practical points, regarding the wholly dominant approach. In other words, the frontier illustrates that ‘regarding the wholly dominant approach, it is impossible to use lesser amounts of diesel fuel and gasoline’. Therefore, the points on the frontier have done the job right, and of course, it does not represent that they have done the

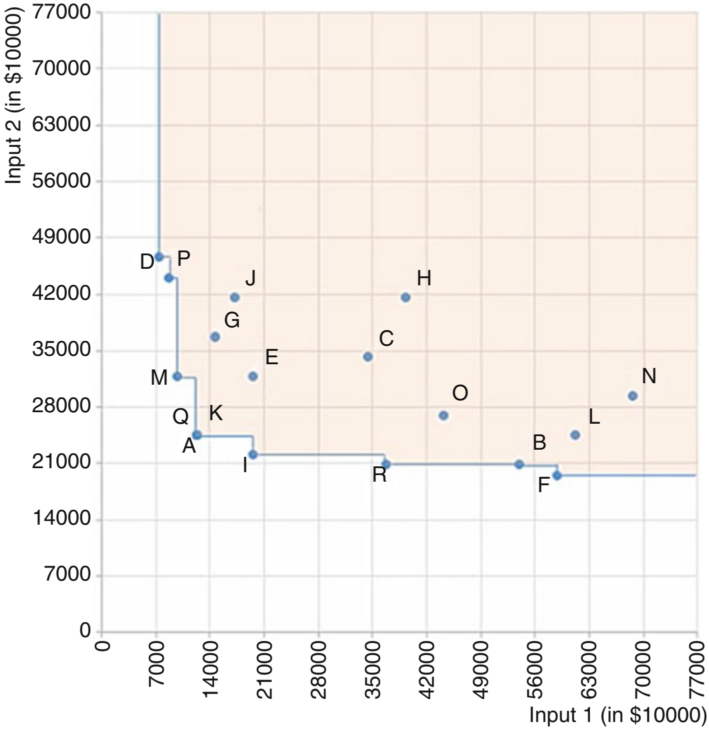


**Fig. 3.5** The wholly dominant approach

job well. Note that, a division which has done the job well in comparison with all other divisions belong to divisions which have done the job right.

Now, suppose that the convexity and the wholly dominant approaches are applied for divisions A-R. The practical points which are generated by these two approaches are the shaded area in Fig. 3.6. As discussed in Sect. 2.2, the mathematical combination of the convexity and the wholly dominant approaches are linear while, at the first step, the convexity approach is applied and after that the wholly dominant approach is applied, as Theorem 3.2 illustrates.

$$\left\{ \begin{array}{ll} (x'_1, x'_2): & \sum_{i=1}^{18} x_{i1}\lambda_i \leq x'_1, \quad \sum_{i=1}^{18} x_{i2}\lambda_i \leq x'_2, \\ & \sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \geq 0, \text{for } i = 1, 2, \ldots, 18. \end{array} \right\} \tag{3.7}$$

**Theorem 3.2** *Suppose that $A_i(x_{i1}, x_{i2})$, for i = 1, 2, …, 18 denote the observations in Table 3.1. Equation 3.7 linearly yields the shaded area in Fig. 3.6.*

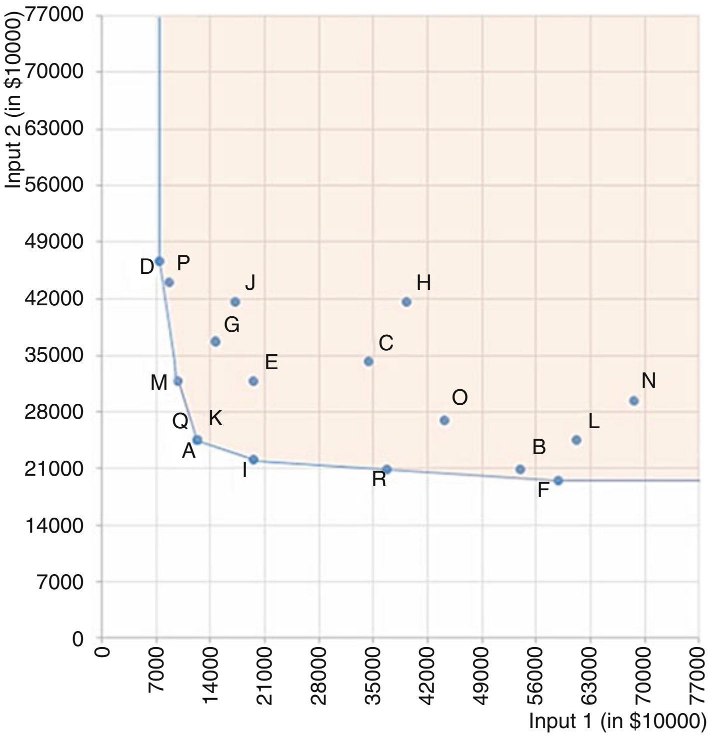


**Fig. 3.6** The convexity and the wholly dominant approaches

## 3.5 The Unit of Measurement

In the petroleum example, if the diesel fuel and the gasoline amounts are not in the same unit, the summation in Eq. 3.1 is not appropriate. For instance, if the diesel fuel was in $10,000 units and the gasoline was in $1,000,000 units, Eq. 3.1 is not valid. This situation is the same as when an environmentalist desires to value the gasoline amounts more than the diesel fuel amounts, or when the worth/weight of the gasoline amounts is more than the worth/weight of the diesel fuel amounts, and so on. Therefore, in these cases, the environmentalist should weigh the amounts to have an appropriate summation in Eq. 3.1, or consider an alternative approach instead of Eq. 3.1.

One way to have the same unit of measurement for each factor is to introduce two weights, $w_1$ and $w_2$, and multiply them to the diesel fuel and the gasoline amounts, respectively. For instance, suppose that the unit of measurement for the diesel fuel is $10,000 and that of the gasoline is $1,000,000, as shown in the third and the fourth columns of Table 3.3. Assume that the divisions A-R are numbered from 1–18 and the amounts of diesel fuel and gasoline are shown with $x_1$ and $x_2$, respectively.

The weights, $w_1$ and $w_2$, can be introduced as $w_1 = 100$ and $w_2 = 1$, in order to have the same units of measurement in \$1,000,000, as the fifth column of Table 3.3 illustrates by adding the diesel fuel and the gasoline amounts. Now, the relative scores, displayed in the sixth column of Table 3.3, can be calculated by the following equation:

$$\frac{\frac{1}{x_{l1}w_1+x_{l2}w_2}}{\max\left\{\frac{1}{x_{i1}w_1+x_{i2}w_2} : i = 1,2,\ldots,18\right\}}, \quad \text{for } l = 1,2,\ldots,18. \tag{3.8}$$

On the other hand, the weights, $w_1$ and $w_2$, can also be considered as $w_1 = 10$ and $w_2 = 0.1$, to have the same unit of measurement in \$1,000,000. Indeed, since the unit of measurement for the first factor, $x_1$, is \$10,000 and that of the second factor, $x_2$, is \$1,000,000, hence, the unit of the ratio of $x_2/x_1$ is 100. As a result, every selection of weights, $w_1$ and $w_2$, while $w_1/w_2$ is 100, yields that the summation of $w_1x_1 + w_2x_2$ becomes appropriate, and in this case, the relative scores of divisions are the same as the sixth column of Table 3.3.

In other words, let's suppose there are two sets of positive weights, ($w_1$, $w_2$), and ($w'_1, w'_2$), where $w_1/w_2 = w'_1/w'_2 = 100$, and assume that an arbitrary division in row $l$ is selected, for $l = 1, 2, \ldots, 18$. The measured relative scores of the division with ($w_1$, $w_2$), and $\left(w'_1, w'_2\right)$ are the same if, and only if, the following equations are satisfied.

$$\frac{\frac{1}{x_{l1}w_1+x_{l2}w_2}}{\max_{1\le i\le 18}\left\{\frac{1}{x_{i1}w_1+x_{i2}w_2}\right\}} = \frac{\frac{1}{x_{l1}w'_1+x_{l2}w'_2}}{\max_{1\le i\le 18}\left\{\frac{1}{x_{i1}w'_1+x_{i2}w'_2}\right\}}$$

$$\Leftrightarrow \frac{w_1\min_{1\le i\le 18}\{x_{i1}\} + w_2\min_{1\le i\le 18}\{x_{i2}\}}{w_1x_{l1} + w_2x_{l2}} = \frac{w'_1\min_{1\le i\le 18}\{x_{i1}\} + w'_2\min_{1\le i\le 18}\{x_{i2}\}}{w'_1x_{l1} + w'_2x_{l2}}$$

$$\Leftrightarrow \frac{(w_1/w_2)\min_{1\le i\le 18}\{x_{i1}\} + \min_{1\le i\le 18}\{x_{i2}\}}{(w_1/w_2)x_{l1} + x_{l2}} = \frac{\left(w'_1/w'_2\right)\min_{1\le i\le 18}\{x_{i1}\} + \min_{1\le i\le 18}\{x_{i2}\}}{\left(w'_1/w'_2\right)x_{l1} + x_{l2}}$$

$$\frac{100\min_{1\le i\le 18}\{x_{i1}\} + \min_{1\le i\le 18}\{x_{i2}\}}{100x_{l1} + x_{l2}} = \frac{100\min_{1\le i\le 18}\{x_{i1}\} + \min_{1\le i\le 18}\{x_{i2}\}}{100x_{l1} + x_{l2}}.$$

Since the last equation is always true, the statement is always true. Therefore, the relative scores of divisions in Table 3.3 are not changed while the ratio of the weights are $w_1/w_2 = 100$, for every set of positi ve numbers $w_1$ and $w_2$.

Moreover, as Fig. 3.7 depicts, the slope of the line $w_1x_1 + w_2x_2 = t$ is $-100$, and the first connection of the line with the observations is D. By moving the line through the observations, the last connection is with division N, which is compatible with the rank in the last column of Table 3.3. From the dotted line

**Table 3.3** The relative scores with different units of measurement

| Division | N | $x_1$($10, 000) | $x_2$($1, 000, 000) | $100x_1 + x_2$($1, 000, 000) | Relative Score | Rank |
|---|---|---|---|---|---|---|
| A | 1 | 12,247.448710 | 24,494.897430 | 1,249,239.768430 | 0.625490 | 5 |
| B | 2 | 53,888.774340 | 20,820.662810 | 5,409,698.096810 | 0.144442 | 15 |
| C | 3 | 34,292.856400 | 34,292.856400 | 3,463,578.496400 | 0.225601 | 11 |
| D | 4 | 7348.469228 | 46,540.305110 | 781,387.227910 | 1.000000 | 1 |
| E | 5 | 19,595.917940 | 31,843.366660 | 1,991,435.160660 | 0.392374 | 10 |
| F | 6 | 58,787.753830 | 19,595.917940 | 5,898,371.300940 | 0.132475 | 16 |
| G | 7 | 14,696.938460 | 36,742.346140 | 1,506,436.192140 | 0.518699 | 7 |
| H | 8 | 39,191.835880 | 41,641.325630 | 3960824.913630 | 0.197279 | 13 |
| I | 9 | 19595.917940 | 22,045.407690 | 1,981,637.201690 | 0.394314 | 9 |
| J | 10 | 17,146.428200 | 41,641.325630 | 1,756,284.145630 | 0.444909 | 8 |
| K | 11 | 12,247.693660 | 24,495.142380 | 1,249,264.508380 | 0.625478 | 6 |
| L | 12 | 61,237.243570 | 24,494.897430 | 6,148,219.254430 | 0.127092 | 17 |
| M | 13 | 9797.958971 | 31,843.366660 | 1,011,639.263760 | 0.772397 | 3 |
| N | 14 | 68,585.712800 | 29,393.876910 | 6,887,965.156910 | 0.113442 | 18 |
| O | 15 | 44,090.815370 | 26,944.387170 | 4,436,025.924170 | 0.176146 | 14 |
| P | 16 | 8573.214100 | 44,090.815370 | 901,412.225370 | 0.866848 | 2 |
| Q | 17 | 12,222.953820 | 24,568.382120 | 1,246,863.764120 | 0.626682 | 4 |
| R | 18 | 36,742.346140 | 20,973.755920 | 3,695,208.369920 | 0.211460 | 12 |

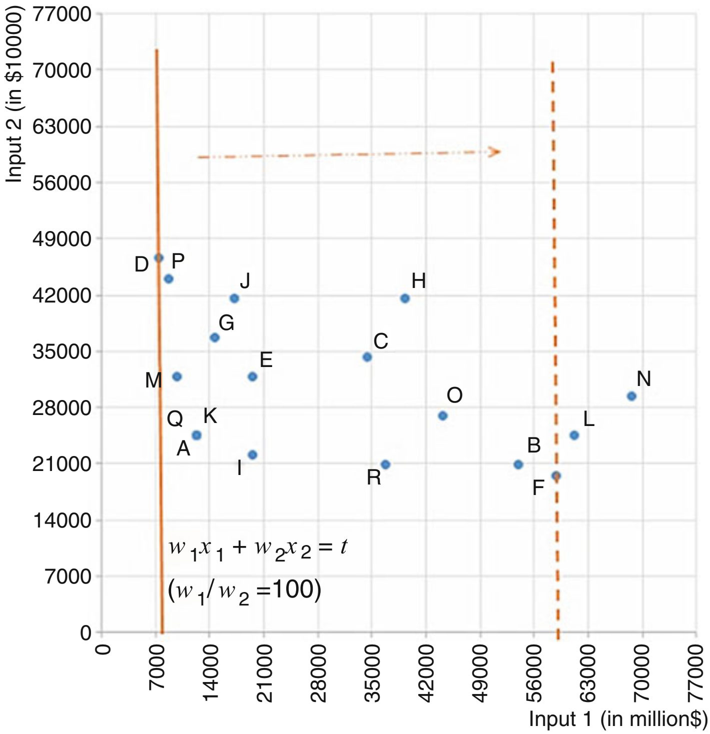


**Fig. 3.7** The line with the slope − 100 which passes D

in Fig. 3.7, division F partially dominates divisions L and N, and is partially dominated by the other divisions.

As can be seen, division D has the highest rank in this example, followed by P, M, Q, A and K, and divisions N, L, F and B have the lowest ranks, respectively. In contrast, if it is supposed that the diesel fuel is in $1,000,000 units and the gasoline is in $10,000 units, the relative scores of divisions and the ranks of them are clearly changed, and division F has the highest rank among the divisions.

As a result, for different relationships between the diesel fuel and the gasoline amounts, one of the observations on the frontiers in Figs. 3.5 and 3.6 can get the highest rank, whereas the frontiers are the same for each situation.

It is transparent that the division which has done the job well in comparison with all other divisions belongs to the divisions which have done the job right, as explained in Chap. 1. However, when the best division is identified, the divisions, which are next to the best division, get higher ranks among other divisions, regardless they are on the frontier or in the proposed practical area.

For instance, in Fig. 3.7, division F has one of the worst combinations of diesel fuel and gasoline with ranking 16th among other divisions, whereas it is on the frontier which is generated by the wholly dominant approach (See Figs. 3.5

and 3.6). In contrast, K has the second rank in Table 3.2, whereas it is not on the frontiers in Figs. 3.5 and 3.6.

Certainly, these results illustrate that the concept of doing the job right is not the same as the concept of doing the job well, and finding the frontier of the practical points with a special approach is not logically yielded to find the relative scores, rank or even suggest divisions to regulate the values of their input factors, as discussed in Chap. 1.

From the other point of view, the relative scores of divisions in Tables 3.2 and 3.3 are measured while the relationship between the diesel fuel and the gasoline amounts are known. In other words, one dollar diesel fuel has the same worth as one dollar gasoline in Table 3.1, whereas one dollar diesel fuel has the same worth as one hundred dollars gasoline in Table 3.3.

These relationships between the two amounts in Tables 3.2 and 3.3 yield that the environmentalists calculate the relative scores by Eq. 3.8, and know that, for instance, divisions A and P have the first and 8th ranks in Table 3.2, respectively, and the corresponding divisions A and P in Table 3.3 have fifth and the second ranks, respectively.

As a result, the relative scores of divisions depend on the relationship between the amounts of diesel fuel and gasoline, which let's select the weights, $w_1$ and $w_2$, appropriately. So, what would happen if the units of measurement or the relationship between the factors are not available?

## 3.6 Unknown Units of Measurement

Suppose the 18 divisions, A-R in Table 3.4, numbered from 1–18, with two factors, the diesel fuel amount, $x_1$, and the gasoline amount, $x_2$, while the units for both factors are unknown, (that is, the units of the diesel fuel and the gasoline amounts are anonymous). How can the best practice be suggested?

As discussed in Sect. 3.3, since the relationship between the factors is unknown, summation of the diesel fuel and the gasoline amounts, (that is, $x_1 + x_2$) is not valid, and there is a need to introduce two weights, $w_1$ and $w_2$, corresponding to the relationship between the factors, as shown in Eq. 3.9.

$$w_1 \times x_1 \text{ (in unknown unit)} + w_2 \times x_2 \text{ (in unknown unit)} \tag{3.9}$$

Different ratio of the weights, that is, $w_1/w_2$, yields different values for Eq. 3.9 and different relative scores for the divisions. So, how can the relative scores of divisions be recommended?

We are certainly able to apply one of the approaches in Sect. 2.2, and find the divisions which can be introduced as those have done the job right, but as illustrated in Sect. 3.5, this way neither discriminates divisions properly nor supports the concept of doing the job well to displays the best division. Misunderstanding this

**Table 3.4** The data with unknown units of measurement

| Division | N | $x_1$(in unknown unit) | $x_2$(in unknown unit) |
|---|---|---|---|
| A | 1 | 12,247.448710 | 24,494.897430 |
| B | 2 | 53,888.774340 | 20,820.662810 |
| C | 3 | 34,292.856400 | 34,292.856400 |
| D | 4 | 7348.469228 | 46,540.305110 |
| E | 5 | 19,595.917940 | 31,843.366660 |
| F | 6 | 58,787.753830 | 19,595.917940 |
| G | 7 | 14,696.938460 | 36,742.346140 |
| H | 8 | 39,191.835880 | 41,641.325630 |
| I | 9 | 19,595.917940 | 22,045.407690 |
| J | 10 | 17,146.428200 | 41,641.325630 |
| K | 11 | 12,247.693660 | 24,495.142380 |
| L | 12 | 61,237.243570 | 24,494.897430 |
| M | 13 | 9797.958971 | 31,843.366660 |
| N | 14 | 68,585.712800 | 29,393.876910 |
| O | 15 | 44,090.815370 | 26,944.387170 |
| P | 16 | 8573.214100 | 44,090.815370 |
| Q | 17 | 12,222.953820 | 24,568.382120 |
| R | 18 | 36,742.346140 | 20,973.755920 |

phenomenon misleads decision makers, managers, economists and researchers to a harmful terminate decision, as the next sections illustrate.

From the lack of information, different approximations can be proposed to measure a relative score for a division; however, each approximation has its own pros and cons and none of the results are pure. The approaches may not be appropriate when simple information from data is added, for example, the desires of the environmentalist on a factor or the available results of a questionnaire to find the relationships between the factors according to expert judgment. Therefore, before using any of the following approaches, the environmentalist should make sure whether there is a logical relationship between the factors, or for which purposes one of the approaches is selected. In addition, the first important assumption of using one of the following approaches is that 'there is no any other information about data in Table 3.4, except desiring the lesser amount of the both factors while the environment is the same for each division in the petroleum example' and 'there is no reasonable way to increase the information about data in Table 3.4'.

### *3.6.1 The Statistical Measurement Approximations*

A simple approach to approximate the relative scores of divisions while the unit of the factors are unknown, is to measure the statistical values of each factor, such as the minimum, maximum, mean, median, standard deviation and standardization values and so on, and after that assuming that the relationship between the factors and the relationship between the measured statistical values are the same.

#### 3.6.1.1 The Minimum Measurement Approximation

Suppose that the minimum values of diesel fuel and gasoline of divisions A-R are measured. Then, the weights, $w_1$ and $w_2$, are given by:

$$w_1 = 1/\min_{1 \le i \le 18}\{x_{i1}\} \,\&\, w_2 = 1/\min_{1 \le i \le 18}\{x_{i2}\}. \tag{3.10}$$

In Eq. 3.10, $x_{i1}$ is a notation for the amount in the row $i$ and the first factor (diesel fuel), and $x_{i2}$ is a notation for the amount in the row $i$ and the second factor (gasoline) in Table 3.4. If there was a zero in data, the minimum value of the diesel fuel and the gasoline amounts can be selected from the positive data, that is, $\min\{x_{i1} : x_{i1} \neq 0\}$ and $\min\{x_{i2} : x_{i2} \neq 0\}$.

Although, the units of the factors are unknown, it is obvious that the minimum of each factor has the same unit of the corresponding factor. Indeed, if the unit for a factor is in \$ unit, \$10,000 unit, £ unit, and so on, the minimum value of that factor is also in \$ unit, \$10,000 unit, £ unit, and so on, respectively. Therefore, the terms $w_1 \times x_{l1}$ and $w_2 \times x_{l2}$ have the unity scale, according to Sect. 1.7, for $l = 1, 2, \ldots,$ 18, and the summation in Eq. 3.9 is valid, as shown in the following equation as well.

$$x_{l1}w_1 + x_{l2}w_2 = \frac{x_{l1}}{\min_{1 \le i \le 18}\{x_{i1}\}} + \frac{x_{l2}}{\min_{1 \le i \le 18}\{x_{i2}\}} \tag{3.11}$$

From Eq. 3.10, the environmentalist assumes that the ratio of $w_1/w_2$, which reflects the relationship between the two factors, is the same as the ratio of the minimum value of the gasoline amounts over the minimum value of the diesel fuel amounts, that is,

$$\frac{w_1}{w_2} = \frac{\min_{1 \le i \le 18}\{x_{i2} : x_{i2} \neq 0\}}{\min_{1 \le i \le 18}\{x_{i1} : x_{i1} \neq 0\}}. \tag{3.12}$$

In other words, the observations are compared by the point with the minimum values of the factors (and this point dominates all the divisions), without assuming that the point with the coordinates of the minimum values are practical or not. Geometrically, the negative value of Eq. 3.12, is the slope of the line $w_1x_1 + w_2x_2 = t$, which let's find the division with the greatest relative score.

Let denote $x'_{l1} = x_{l1}/\min\{x_{i1} : i = 1, 2 \ldots, 18\}$ and $x'_{l2} = x_{l2}/\min\{x_{i2} : i = 1, 2 \ldots, 18\}$, for $l = 1, 2, \ldots, 18$.

The third and the fourth columns of Table 3.5 illustrate the values of $x'_{l1}$ and $x'_{l2}$, for $l = 1, 2, \ldots, 18$, which have the unity scales, and can be added together, that is, the summation $x'_{l1} + x'_{l2}$ is valid. This outcome let us to measure the relative scores by Eq. 3.3. The sixth and the seventh columns of Table 3.5 display the relative scores and the ranks of the divisions.

**Table 3.5** The relative scores with the minimum measurement approximation

| Division | N | $x'_{l2}$ (1) | $x'_{l2}(1)$ | $x'_{l1} + x'_{l2}$ (1) | Relative score | Rank |
|---|---|---|---|---|---|---|
| A | 1 | 1.666667 | 1.250000 | 2.916667 | 1.000000 | 1 |
| B | 2 | 7.333333 | 1.062500 | 8.395833 | 0.347395 | 15 |
| C | 3 | 4.666667 | 1.750000 | 6.416667 | 0.454545 | 12 |
| D | 4 | 1.000000 | 2.375000 | 3.375000 | 0.864198 | 5 |
| E | 5 | 2.666667 | 1.625000 | 4.291667 | 0.679612 | 9 |
| F | 6 | 8.000000 | 1.000000 | 9.000000 | 0.324074 | 16 |
| G | 7 | 2.000000 | 1.875000 | 3.875000 | 0.752688 | 8 |
| H | 8 | 5.333333 | 2.125000 | 7.458333 | 0.391061 | 14 |
| I | 9 | 2.666667 | 1.125000 | 3.791667 | 0.769231 | 7 |
| J | 10 | 2.333333 | 2.125000 | 4.458333 | 0.654206 | 10 |
| K | 11 | 1.666700 | 1.250013 | 2.916712 | 0.999984 | 2 |
| L | 12 | 8.333333 | 1.250000 | 9.583333 | 0.304348 | 17 |
| M | 13 | 1.333333 | 1.625000 | 2.958333 | 0.985915 | 4 |
| N | 14 | 9.333333 | 1.500000 | 10.833333 | 0.269231 | 18 |
| O | 15 | 6.000000 | 1.375000 | 7.375000 | 0.395480 | 13 |
| P | 16 | 1.166667 | 2.250000 | 3.416667 | 0.853659 | 6 |
| Q | 17 | 1.663333 | 1.253750 | 2.917083 | 0.999857 | 3 |
| R | 18 | 5.000000 | 1.070312 | 6.070312 | 0.480480 | 11 |

Figure 3.8 illustrates the locations of the points, A-R, after dividing their coordinates by the corresponding minimum values of the factors, and depicts the feasible area by applying the wholly dominant and the convexity approaches. It is clear that with these two approaches the point with the coordinates (1, 1) is not practical. Nevertheless, in the minimum measurement approximation, the relationship between the factors in Table 3.5 is compared with the point (1, 1).

Since data are divided by the corresponding minimum values of the factors, both axes in Fig. 3.8 are in the unity scale, and it is supposed that one amount of the horizontal axis has the same worth as one amount of the vertical axis.

Note that, the axes in Fig. 3.8 are scaled differently due to having the transparent figure. The horizontal axis is scaled from 0 to 9.9, whereas the vertical axis is scaled from 0 to 3.3, nevertheless, the amount 0.9 of the first input factor has the same worth as the amount 0.9 of the second input factor.

Similar to Sect. 3.3, the first connection of the line, $x'_1 + x'_2 = t$, with the shaded area in Fig. 3.8 is A which identifies that the division A has done the job well in comparison with all other divisions (according to the minimum measurement approximation), and by moving the line through the data, the last connection of the line shows that division N has not done the job well in comparison with all other divisions. These results are the same as the relative scores and the ranks of the divisions shown in the sixth and the seventh columns of Table 3.5. From the dotted line in Fig. 3.8, F partially dominates L and N, and is partially dominated by other divisions.

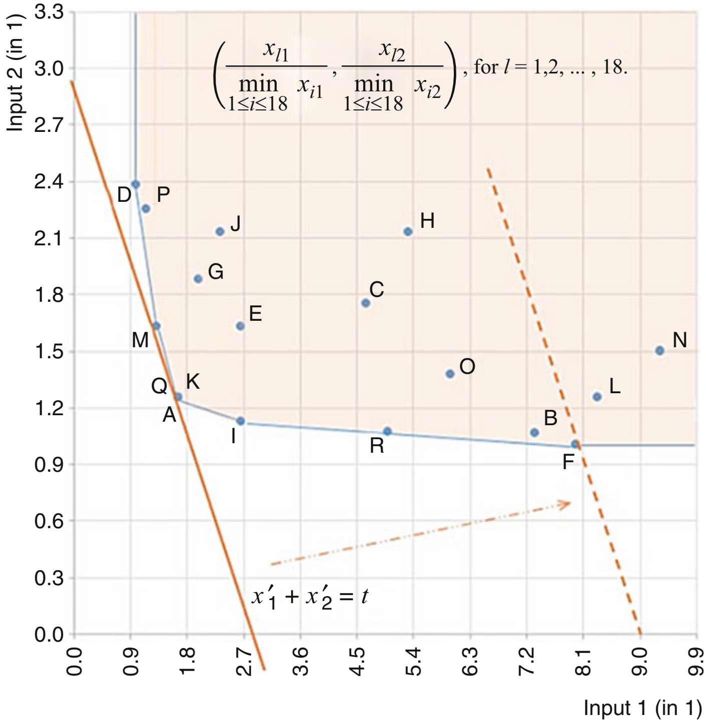


**Fig. 3.8** The minimum measurement approximation

#### 3.6.1.2 The Maximum Measurement Approximation

The discussions for the minimum measurement approximation can also be considered for the maximum measurement approximation by calculating the maximum values of the factors instead of the minimum values. The maximum amounts of the factors also have the same units as the corresponding factors, and thus, an amount of a factor over the maximum amount of the corresponding factor has the unity scale, as shown in Eq. 3.13.

$$x_{l1}w_1 + x_{l2}w_2 = \frac{x_{l1}}{\max_{1\le i\le 18}\{x_{i1}\}} + \frac{x_{l2}}{\max_{1\le i\le 18}\{x_{i2}\}} = x'_{l1} + x'_{l2} \tag{3.13}$$

Furthermore, the environmentalist assumes that the ratio of $w_1/w_2$, which reflects the relationship between the two factors, is the same as the ratio of the maximum value of the gasoline amounts over the maximum value of the diesel fuel amounts. Table 3.6 demonstrates the data with the unity scale by dividing the data in Table 3.4 over the maximum amounts of the corresponding factors.

Figure 3.9 also depicts the locations of divisions after dividing data over the maximum amounts, and displays the feasible area by applying the wholly dominant and the convexity approaches. Both the horizontal and the vertical axes, the first and

**Table 3.6** The relative scores with the maximum measurement approximation

| Division | N | $x'_{l2}$ (1) | $x'_{l2}$ (1) | $x'_{l1} + x'_{l2}$ (1) | Relative score | Rank |
|---|---|---|---|---|---|---|
| A | 1 | 0.178571 | 0.526316 | 0.704887 | 1.000000 | 1 |
| B | 2 | 0.785714 | 0.447368 | 1.233083 | 0.571646 | 13 |
| C | 3 | 0.500000 | 0.736842 | 1.236842 | 0.569909 | 14 |
| D | 4 | 0.107143 | 1.000000 | 1.107143 | 0.636672 | 10 |
| E | 5 | 0.285714 | 0.684211 | 0.969925 | 0.726744 | 6 |
| F | 6 | 0.857143 | 0.421053 | 1.278195 | 0.551471 | 15 |
| G | 7 | 0.214286 | 0.789474 | 1.003759 | 0.702247 | 8 |
| H | 8 | 0.571429 | 0.894737 | 1.466165 | 0.480769 | 17 |
| I | 9 | 0.285714 | 0.473684 | 0.759398 | 0.928218 | 4 |
| J | 10 | 0.250000 | 0.894737 | 1.144737 | 0.615764 | 11 |
| K | 11 | 0.178575 | 0.526321 | 0.704896 | 0.999987 | 2 |
| L | 12 | 0.892857 | 0.526316 | 1.419173 | 0.496689 | 16 |
| M | 13 | 0.142857 | 0.684211 | 0.827068 | 0.852273 | 5 |
| N | 14 | 1.000000 | 0.631579 | 1.631579 | 0.432028 | 18 |
| O | 15 | 0.642857 | 0.578947 | 1.221805 | 0.576923 | 12 |
| P | 16 | 0.125000 | 0.947368 | 1.072368 | 0.657318 | 9 |
| Q | 17 | 0.178214 | 0.527895 | 0.706109 | 0.998270 | 3 |
| R | 18 | 0.535714 | 0.450658 | 0.986372 | 0.714626 | 7 |

the second input factors, are scaled from 0 to 1.1, and one unit on the horizontal axis has the same worth as one unit on the vertical axis. The line $x'_{l1} + x'_{l2} = t$ geometrically displays the best divisions, as well as the area that a division is partially dominated. As can be seen, the points on the frontiers in both Figs. 3.8 and 3.9 are the same. Division A still has the first rank and the rank of D is 10th by the maximum measurement approximation, whereas the rank of D is 5th by the minimum measurement approximation, and so on for other divisions. From the figure, F partially dominates L, N and H.

As Eq. 3.14 illustrates, by applying the maximum measurement approximation, the environmentalist assumes that one unit of the diesel fuel is equal to almost 68% unit of the gasoline amount, whereas, by the minimum approach one unit amount of the diesel fuel is equal to almost 267% unit of the gasoline amount.

This huge difference shows that how an approximation without enough information can be harmful.

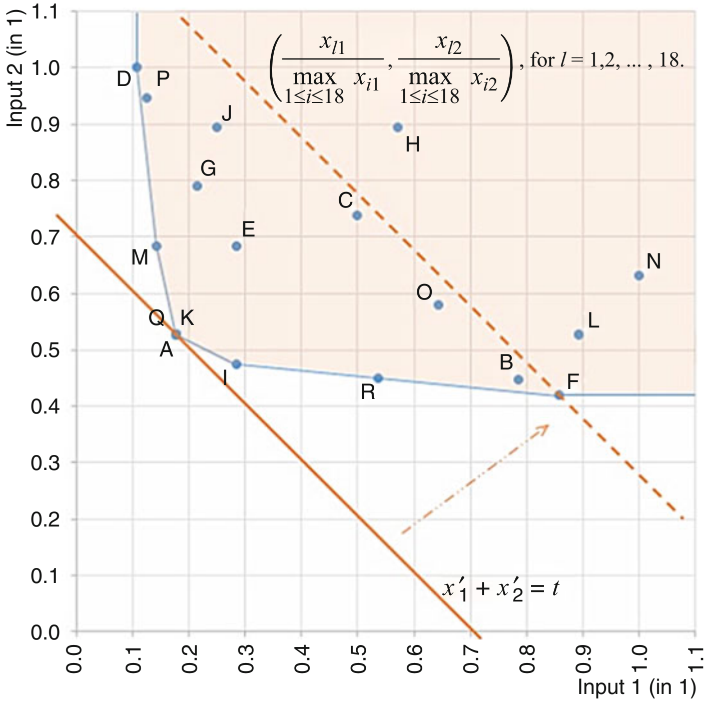


**Fig. 3.9** The maximum measurement approximation

$$\frac{w_1}{w_2} = \frac{\max_{1\le i\le 18}\{x_{i2}\}}{\max_{1\le i\le 18}\{x_{i1}\}} = \frac{46,540.305110}{68,585.712800} = 0.678571429. \tag{3.14}$$

Note that, the point with the minimum values of the input factors wholly dominates all the observations, whereas the point with the maximum values of the input factors is wholly dominated with all the observations. Indeed, the point with the minimum values of the input factors has the best combination of the input factors in comparison with the other observations, while the point with the maximum values of the input factors has the worst combination of the input factors in comparison with the observations.

The point with the minimum values of the input factors may not be practical by using the wholly dominant approach, whereas the point with the maximum values of the input factors is always practical by applying the wholly dominant approach. In contrast, the point with the maximum values of the input factors may not be practical by combination of the convexity and the inner radiate approaches, whereas the point with the minimum values of the factors is always practical by applying the convexity and the inner radiate approaches. Nonetheless, in the extremum measurement approximation is not assumed that the point with extremum values of the input factors is practical or not by applying an approach to introduce the practical points. It is assumed that the relationship between the two input factors is equivalent with the relationship between the observed extremum values, while the unit of measurement is unknown. For instance, in Table 3.3, one hundred amounts of the diesel fuel is equal to one gasoline amount, while the unit of measurement was available, however, it does not mean that a point with the coordinates (100, 1) is practical by the wholly dominant approach.

The relationship between the input factors is independent with the concept of generating the practical points. For a transparent reason, divisions D, M, Q, A, I, R and F lie on the frontiers of the generated area by the convexity and the wholly dominant approaches in Figs. 3.6, 3.7, 3.8 and 3.9, however, the relative scores of these divisions are different in each figure while the relationship between the input factors are changed. Misunderstanding and confusion about these concepts are harmful, and do not provide a fair tool to make decisions properly.

#### 3.6.1.3 The Average Measurement Approximation

The relationship between the diesel fuel and the gasoline amounts can be considered with the mean values of these two input factors as well. While the relationship between the extremum values of the factors are far from the environmentalist expectation, the relationship between the average values of the available input factors can be suggested to calculate Eq. 3.9.

For this aim, the weights, $w_1$ and $w_2$, are introduced with the inverse of the average values of diesel fuel and gasoline of divisions A-R, as Eq. 3.15 represents.

$$w_1 = 1/\underset{1\leq i\leq 18}{\text{average}}\{x_{i1}\} \;\&\; w_2 = 1/\text{average}_{1\leq i\leq 18}\{x_{i2}\}. \tag{3.15}$$

By the average measurement approximation, one unit of diesel fuel is almost equal to one unit of gasoline. In addition, the average values have the same units as the corresponding factors, and therefore as Table 3.7 illustrates, the relative scores of divisions by the average measurement approximation is almost the same as the results in Table 3.2.

Figure 3.9 also depicts the locations of A-R, after dividing their coordinates by the corresponding average values of the factors, and shows the feasible area by applying the wholly dominant and the convexity approaches. Both axes in Fig. 3.10 scaled from 0 to 3.4 with the unity scale, and 0.2 amounts of the horizontal axis have the same worth as 0.2 amounts of the vertical axis. From the figure, division A partially dominates all other divisions, and by moving the line, as shown with the dotted arrow and the dotted line, F partially dominates H, L and N.

**Table 3.7** The relative scores with the average measurement approximation

| Division | N | $x'_{l2}$ (in 1) | $x'_{l2}$ (in 1) | $x'_{l1} + x'_{l2}$ (in 1) | Relative score | Rank |
|---|---|---|---|---|---|---|
| A | 1 | 0.415723 | 0.806840 | 1.222563 | 1.000000 | 1 |
| B | 2 | 1.829183 | 0.685814 | 2.514997 | 0.486109 | 14 |
| C | 3 | 1.164026 | 1.129576 | 2.293601 | 0.533032 | 12 |
| D | 4 | 0.249434 | 1.532996 | 1.782430 | 0.685897 | 9 |
| E | 5 | 0.665157 | 1.048892 | 1.714049 | 0.713260 | 7 |
| F | 6 | 1.995472 | 0.645472 | 2.640944 | 0.462927 | 15 |
| G | 7 | 0.498868 | 1.210260 | 1.709128 | 0.715314 | 6 |
| H | 8 | 1.330315 | 1.371628 | 2.701943 | 0.452476 | 16 |
| I | 9 | 0.665157 | 0.726156 | 1.391313 | 0.878712 | 5 |
| J | 10 | 0.582013 | 1.371628 | 1.953641 | 0.625787 | 11 |
| K | 11 | 0.415732 | 0.806848 | 1.222580 | 0.999987 | 2 |
| L | 12 | 2.078617 | 0.806840 | 2.885457 | 0.423698 | 17 |
| M | 13 | 0.332579 | 1.048892 | 1.381471 | 0.884972 | 4 |
| N | 14 | 2.328051 | 0.968208 | 3.296259 | 0.370894 | 18 |
| O | 15 | 1.496604 | 0.887524 | 2.384128 | 0.512793 | 13 |
| P | 16 | 0.291006 | 1.452312 | 1.743318 | 0.701285 | 8 |
| Q | 17 | 0.414892 | 0.809260 | 1.224152 | 0.998702 | 3 |
| R | 18 | 1.247170 | 0.690857 | 1.938027 | 0.630829 | 10 |

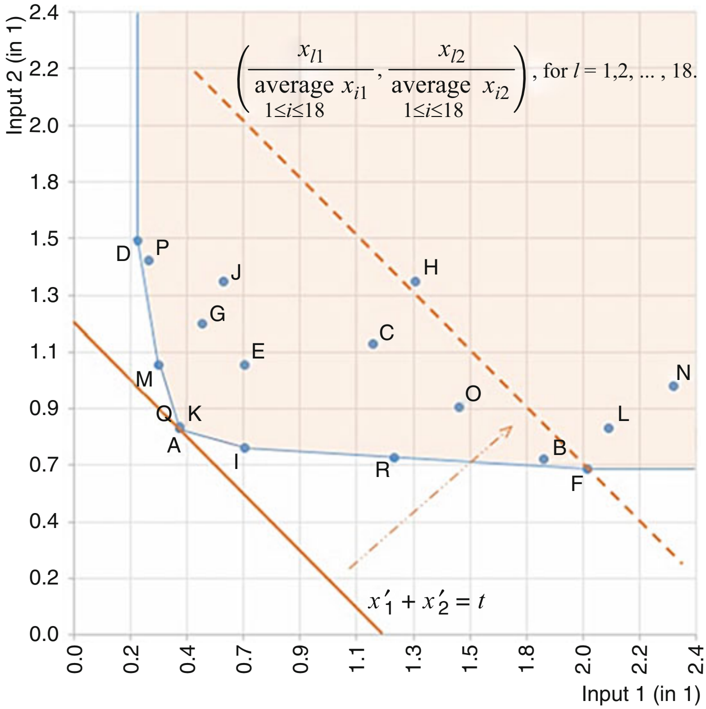


**Fig. 3.10** The average measurement approximation

#### 3.6.1.4 The Standardization Measurement Approximation

The environmentalist can also standardize the data in Table 3.4 to have the unity scale for each factor, and then try to measure the relative scores. Equation 3.16 illustrates how to standardize the data in Table 3.4, for $l = 1, 2, \ldots, 18$. Here 'ave' is the abbreviation for the average (mean) and 'std' is the abbreviation for the standard deviation.

$$(x_{l1}, x_{l2}) \to \left( \frac{x_{l1} - \mathrm{ave}_{1 \le i \le 18}\{x_{i1}\}}{\mathrm{std}_{1 \le i \le 18}\{x_{i1}\}}, \quad \frac{x_{l2} - \mathrm{ave}_{1 \le i \le 18}\{x_{i2}\}}{\mathrm{std}_{1 \le i \le 18}\{x_{i2}\}} \right). \tag{3.16}$$

As can be seen in Eq. 3.16, the relationship between the input factors in this approximation can be introduced as the relationship between the standard deviations of the input factors, that is:

$$w_1 = 1/\mathrm{std}_{1 \le i \le 18}\{x_{i1}\} \;\&\; w_2 = 1/\mathrm{std}_{1 \le i \le 18}\{x_{i2}\}.$$

From Eq. 3.16, at the first step, the point $(x_{l1}, x_{l2})$, is divided by the standard deviation values, $(\mathrm{std}_1, \mathrm{std}_2)$, and, after that, is transferred by the negative ratio of the mean values over the standard deviation values, $(-\mathrm{ave}_1/\mathrm{std}_1, -\mathrm{ave}_2/\mathrm{std}_2)$, for $l = 1, 2, \ldots, 18$. This shift yields the coordinates of some points become as negative, so the data can be shifted again by (+2, +2), (+3, +3) and greater values, in order to have positive data for calculating the relative scores of divisions appropriately. In other words, after dividing the data by the standard deviation values, the new data can be transferred by $(n - \mathrm{ave}_1/\mathrm{std}_1, n - \mathrm{ave}_2/\mathrm{std}_2)$, for $n \in \mathbb{N}$. As $n$ increases, the relative score of each division approaches to 1, but the rank of a division among other divisions is not changed. From the above illustration, the standard deviation measurement approximation is only illustrated in this section.

Similar to the previous sections, the standard deviation of each input factor in Table 3.4 has the same unit as the corresponding input factor. Therefore, dividing each factor over the corresponding standard deviation has the unity scale, as shown in Eq. 3.17. Note that, if the standard deviation is zero for a factor, the factor can be eliminated.

$$x_{l1}w_1 + x_{l2}w_2 = \frac{x_{l1}}{\underset{1 \le i \le 18}{\mathrm{std}}\{x_{i1}\}} + \frac{x_{l2}}{\underset{1 \le i \le 18}{\mathrm{std}}\{x_{i2}\}} = x'_{l1} + x'_{l2} \;\; (\mathrm{in1}) \tag{3.17}$$

As Eq. 3.18 illustrates, by applying the standard deviation measurement approximation, the environmentalist supposes that, one unit of diesel fuel is almost equal to 42% unit of gasoline. Note that, the *biased estimator* of the standard deviation is measured in this approach, that is, the squared deviation from the mean is divided over 17 instead of 18 (the number of divisions).

**Table 3.8** The relative scores with the std. measurement approach

| Division | N | $x'_{l2}$ (in 1) | $x'_{l2}$ (in 1) | $x'_{l1} + x'_{l2}$ (in 1) | Relative score | Rank |
|---|---|---|---|---|---|---|
| A | 1 | 0.597269 | 2.824097 | 3.421366 | 1.000000 | 1 |
| B | 2 | 2.627985 | 2.400482 | 5.028467 | 0.680399 | 9 |
| C | 3 | 1.672354 | 3.953735 | 5.626089 | 0.608125 | 13 |
| D | 4 | 0.358362 | 5.365784 | 5.724145 | 0.597708 | 15 |
| E | 5 | 0.955631 | 3.671326 | 4.626957 | 0.739442 | 7 |
| F | 6 | 2.866892 | 2.259277 | 5.126170 | 0.667431 | 10 |
| G | 7 | 0.716723 | 4.236145 | 4.952868 | 0.690785 | 8 |
| H | 8 | 1.911262 | 4.800964 | 6.712226 | 0.509722 | 17 |
| I | 9 | 0.955631 | 2.541687 | 3.497318 | 0.978283 | 4 |
| J | 10 | 0.836177 | 4.800964 | 5.637141 | 0.606933 | 14 |
| K | 11 | 0.597281 | 2.824125 | 3.421406 | 0.999988 | 2 |
| L | 12 | 2.986346 | 2.824097 | 5.810443 | 0.588830 | 16 |
| M | 13 | 0.477815 | 3.671326 | 4.149141 | 0.824596 | 5 |
| N | 14 | 3.344708 | 3.388916 | 6.733624 | 0.508102 | 18 |
| O | 15 | 2.150169 | 3.106506 | 5.256676 | 0.650861 | 11 |
| P | 16 | 0.418088 | 5.083374 | 5.501463 | 0.621901 | 12 |
| Q | 17 | 0.596075 | 2.832569 | 3.428644 | 0.997877 | 3 |
| R | 18 | 1.791808 | 2.418133 | 4.209941 | 0.812687 | 6 |

$$\frac{w_1}{w_2} = \frac{\text{std}_{1 \le i \le 18}\{x_{i2}\}}{\text{std}_{1 \le i \le 18}\{x_{i1}\}} = \frac{20,505.741139}{8673.533484} = 0.422980736. \tag{3.18}$$

Table 3.8 illustrates the results of the standard deviation measurement approximation. A still has the best relative score among other divisions and the relative score of F is better than that of D.

Figure 3.11 also depicts the standard deviation measurement approximation. The horizontal axis is scaled from 0 to 4.4 and the vertical axis is scaled from 0 to 6.6, in order to sketch a transparent figure. Both axes in Fig. 3.11 are in the unity scale and 1.2 amounts of the horizontal axis is equal to 1.2 amounts of the vertical axis.

As can be seen, the points A, Q, I, M, D and F are on the frontier while the wholly dominant and the convexity approaches are applied, regardless the unit of measurement. In contrast, different measurement approximation yields different results to display the best divisions and to rank divisions reasonably. For example, while D has the highest rank among divisions, F, L and N have the least ranks among divisions, and vice versa. It is impossible to say that both D and F have done the job well at the same time, although, both D and F have done the job right by accepting the convexity and the wholly dominant approaches to generate the practical points. While D has the highest rank among divisions, division P, which is not on the frontier, has also higher rank in comparison with R and F, which are on the frontier.

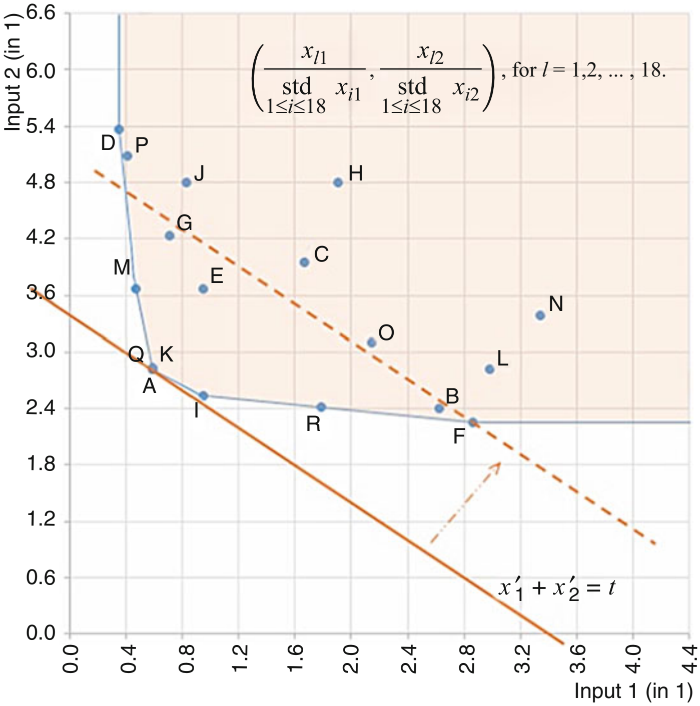


**Fig. 3.11** The standard deviation measurement approximation

While A has the best performance among other divisions, D and F have worse performances than K (which is next to A, and is not on the frontier).

As a result, identifying the divisions which lie on the frontier is not enough to find the best divisions which have done the job well, or to rank divisions and measure the relative scores.

### *3.6.2 The Specified Division Measurement Approximation*

The introduced approaches in Sect. 3.6.1 are simply measured, but a division can always argue that the relationships between the statistical values of the input factors are not meaningful or feasible, or there is no logical reason to compare divisions with these approaches. Every division in the petroleum example has different combinations of the input factors (the diesel fuel and the gasoline amounts) in Table 3.4. These combinations are practical and feasible with no doubts. Therefore, the environmentalist can select the feasible factors of a specified division as the unit of measurement to calculate the relative scores of divisions. For instance, the environmentalist can select the division in row $l$ in Table 3.4 (for $l = 1, 2, \ldots, 18$), and

**Table 3.9** The rank of divisions by specified division measurement

| Division | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A | 1 | 1 | 1 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 1 | 1 | 4 | 1 | 1 |
| B | 15 | 8 | 14 | 15 | 15 | 8 | 15 | 14 | 14 | 15 | 15 | 8 | 15 | 9 | 12 | 15 | 15 | 12 |
| C | 12 | 13 | 12 | 11 | 12 | 13 | 12 | 12 | 12 | 12 | 12 | 13 | 12 | 13 | 14 | 12 | 12 | 15 |
| D | 5 | 16 | 9 | 1 | 7 | 16 | 5 | 9 | 9 | 5 | 7 | 16 | 5 | 15 | 10 | 2 | 5 | 10 |
| E | 9 | 7 | 6 | 10 | 9 | 7 | 9 | 7 | 8 | 9 | 9 | 7 | 9 | 7 | 7 | 10 | 9 | 7 |
| F | 16 | 9 | 15 | 16 | 16 | 9 | 16 | 15 | 16 | 16 | 16 | 10 | 16 | 10 | 15 | 16 | 16 | 14 |
| G | 8 | 10 | 7 | 7 | 8 | 11 | 8 | 6 | 6 | 8 | 8 | 9 | 7 | 8 | 8 | 7 | 8 | 8 |
| H | 14 | 18 | 16 | 13 | 14 | 18 | 14 | 16 | 15 | 14 | 14 | 18 | 13 | 17 | 17 | 13 | 14 | 17 |
| I | 7 | 4 | 4 | 8 | 5 | 2 | 7 | 5 | 5 | 7 | 5 | 4 | 8 | 4 | 4 | 8 | 7 | 4 |
| J | 10 | 15 | 11 | 9 | 10 | 15 | 10 | 10 | 10 | 10 | 10 | 14 | 10 | 14 | 11 | 9 | 10 | 11 |
| **K** | **2** | **2** | **2** | **6** | **2** | **3** | **2** | **2** | **2** | **2** | **2** | **2** | **4** | **2** | **2** | **5** | **2** | **2** |
| L | 17 | 14 | 17 | 17 | 17 | 12 | 17 | 17 | 17 | 17 | 17 | 15 | 17 | 16 | 16 | 17 | 17 | 16 |
| M | 4 | 6 | 5 | 2 | 4 | 6 | 4 | 4 | 4 | 4 | 4 | 6 | 1 | 5 | 5 | 1 | 4 | 5 |
| N | 18 | 17 | 18 | 18 | 18 | 17 | 18 | 18 | 18 | 18 | 18 | 17 | 18 | 18 | 18 | 18 | 18 | 18 |
| O | 13 | 11 | 13 | 14 | 13 | 10 | 13 | 13 | 13 | 13 | 13 | 11 | 14 | 11 | 13 | 14 | 13 | 13 |
| P | 6 | 12 | 8 | 3 | 6 | 14 | 6 | 8 | 7 | 6 | 6 | 12 | 6 | 12 | 9 | 6 | 6 | 9 |
| Q | 3 | 3 | 3 | 4 | 3 | 4 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 3 | 3 | 3 | 3 | 3 |
| R | 11 | 5 | 10 | 12 | 11 | 5 | 11 | 11 | 11 | 11 | 11 | 5 | 11 | 6 | 6 | 11 | 11 | 6 |

introduce the weights, $w_1$ and $w_2$, by the inverse of the diesel fuel and the gasoline amounts in that row, as Eq. 3.19 also illustrates.

$$w_1 = 1/x_{l1} \ \& \ w_2 = 1/x_{l2}. \tag{3.19}$$

Thus, the equation $w_1x_{i1} + w_2x_{i2}$ has the unity scale and the slope of the line $w_1x_{i1} + w_2x_{i2} = 1$ is $x_{l2}/x_{l1}$, for $i = 1, 2, \ldots, 18$. Similar to the previous section, the line can display a division which partially dominates another division. For instance, when $l = 1$, the slope of the line (the relationship between the input factors) is calculated by Eq. 3.20,

$$\frac{w_1}{w_2} = \frac{x_{12}}{x_{11}} = \frac{24,494.897430}{12,247.448710} \approx 2, \tag{3.20}$$

and the relative scores of divisions are measured by Eq. 3.21.

$$\frac{\frac{1}{2x_{i1}+x_{i2}}}{\max\left\{\frac{1}{2x_{i1}+x_{i2}} : i = 1, 2, \ldots, 18\right\}}, \quad \text{for } i = 1, 2, \ldots, 18. \tag{3.21}$$

On the other hand, the same approach can also be applied for $l = 2, 3, \ldots, 18$, because every division has the right to be selected. Table 3.9 illustrates the ranks of divisions by the specified division measurement approximation.

Each row in Table 3.9 represents the rank of a division according to the relationship between the factors of the specified division in each column. For instance, while $l = 11$, the bold data in the twelfth row of Table 3.9 show the ranks of division K by the relationships between the factors of all divisions A-R. Division K has the second rank with the relationships between the factors of all divisions except divisions D, F, M and P. The ranks of K by the relationships between the factors of D, F, M and P, are 6, 3, 4 and 5, respectively.

Furthermore, the highlighted diameter of Table 3.9 shows the rank of each division by the relationship between its own factors. For instance, the rank of B among other divisions is 8 while the relationship between the factors of B is selected, and H has the 16th rank among other divisions, while the factors of H are introduced as the unit of the measurement. Only A, D and M get the first rank among the divisions by the relationships between their factors.

These outcomes by the specified division measurement approximation may be more reasonable than the outcomes by the statistical measurement approximations. But, while the units of the measurement are unknown, making decisions is always rickety and questionable, because the relative scores and the ranks of divisions completely depend on the relationships between the factors or the introduced weights, $w_1$ and $w_2$. For instance, the environmentalist can select two specified divisions or more, and introduce the weights according to their practical data. A table of the weights can also be prepared to display the situations of each division from different selections of the weights. This attempt does not improve the discrimination power of measuring the relative scores of divisions, but provides a transparent table for further predictions and can be used to decrease the risk of selecting the weights.

The environmentalist may also consider the sum (average, median, parametric or non-parametric statistics) of the ranks in Table 3.9 or their corresponding relative scores to identify a division which has done the job well; however, such an approach is harmful.

### *3.6.3 The Classifications of Weights*

In order to increase the accuracy of decision making, the environmentalist needs to provide some more information about the relationships between the factors, for instance, by providing a questionnaire. The relative scores and the ranks of the divisions undoubtedly depend upon the selected weights. While the relationship between the factors is unknown or the unit of measurement is not available, different values of the weights, $w_1$ and $w_2$, may yield different relative scores for the divisions in Table 3.4. Geometrically, as shown in Figs. 3.2, 3.7, 3.8, 3.9, 3.10, and 3.11, while the slope of the line, $w_1x_1 + w_2x_2 = t$, which is $-w_1/w_2$, is changed the first connection of the line with the locations of the divisions in the Cartesian coordinate plan may be changed as well.

For instance, Fig. 3.7 depicts that the line with the slope of $-100$, identifies D as the best division with the minimum values of the diesel fuel and the gasoline amounts, that is, D is the best *performer* among the other divisions while the slope of the line is $-100$. This phenomenon guides us to ask which values of the slope, $-w_1/w_2$, identify division D as the best performer among other divisions? Can a value of the slope, $-w_1/w_2$, be found which both D and M are introduced as the best performers? What about D and R? Is there a value for the slope, $-w_1/w_2$, that introduces the divisions on the frontiers in Figs. 3.8, 3.9, 3.10, and 3.11, (which are D, M, Q, A, I, R and F), as the best performers?

In order to answer the above questions, let's first try to find a slope for the line, $w_1x_1 + w_2x_2 = t$, which shows both D and M as the best performers among the other divisions. For such an aim, the line $w_1x_1 + w_2x_2 = 0$, which passes the origin $(0, 0)$, should be transferred through the locations of the observations, and should simultaneously pass both D and M in the first connection with the locations of the observation. This means, the slope of the line, $w_1x_1 + w_2x_2 = t$, should be the same as the slope of the line which passes D and M. The slope of the line which passes both D$(x_{D1}, x_{D2})$ and M $(x_{M1}, x_{M2})$, can easily be measured by the following equation:

$$\frac{x_{D2} - x_{M2}}{x_{D1} - x_{M1}} = \frac{46,540.3051 - 31,843.3667}{7348.4692 - 9797.9590} = -6. \tag{3.22}$$

Therefore, by moving the line, $6x_1 + x_2 = 0$, through the locations of the observations, both D and M are known as the best performers among the other divisions. As can be seen, finding the slopes of the lines which pass the locations of each two divisions can guide us to estimate the effects of changing the weights and the relationships between the amounts of diesel fuel and gasoline.

On the other hand, as it is illustrated in Sect. 2.2.2, the convexity approach represents the lines which pass the locations of each two divisions. Therefore, suppose that the data in Table 3.4 are given and the relationships between the two input factors are unknown. Figure 3.12 illustrates the shaded area which is generated by applying the convexity approach, as Eq. 3.23 represents.

$$\left\{\left(\sum_{i=1}^{18} x_{i1}\lambda_i, \sum_{i=1}^{18} x_{i2}\lambda_i\right) : \sum_{i=1}^{18} \lambda_i = 1,\ \lambda_i \geq 0,\ \text{for}\, i = 1, 2, \ldots, 18\right\}. \tag{3.23}$$

From Fig. 3.12, it is impossible to find a slope for the line, $w_1x_1 + w_2x_2 = t$, to identify G as the best performer. This is the same for all the points in the shaded area as well. Moreover, transferring the line, $w_1x_1 + w_2x_2 = 0$, through the *heptagon* in Fig. 3.12, in order to find the first connection, is equivalent to find the minimum values of the equation '$w_1x_1 + w_2x_2$' according to the shaded heptagon. Therefore, H and N are never identified as the best performers. Only D, M, Q, A, I, R and F, which

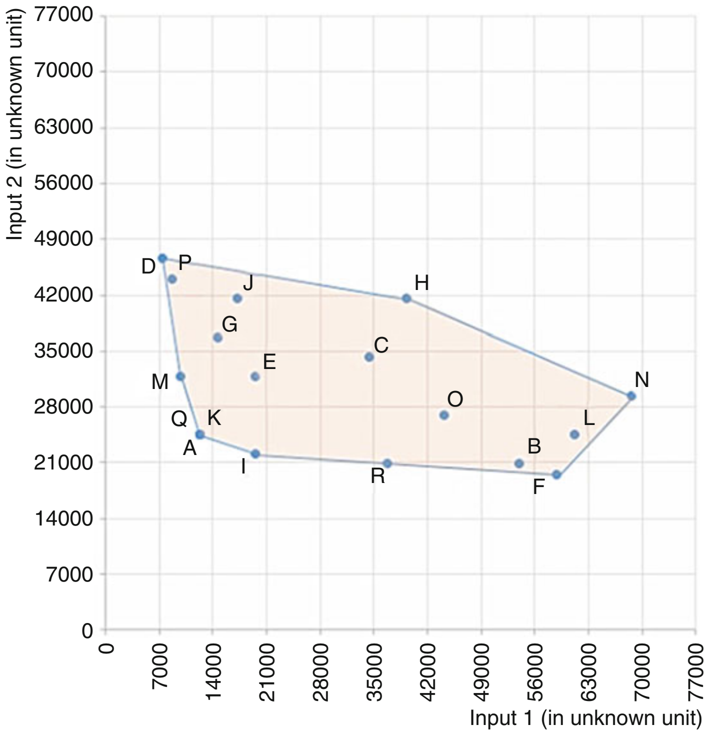


**Fig. 3.12** Applying the convexity approach for data in Table 3.4

are on the frontiers in Figs. 3.5, 3.6, 3.8, 3.9, 3.10, and 3.11, can have a chance to be the best performer among other divisions. Nonetheless, as explained in the previous sections, it is impossible that all of the divisions D, M, Q, A, I, R and F are simultaneously introduced as the best performer.

Figure 3.13 represents the four lines and their equations which pass the heptagon sides from D to F. As can be seen, the line, $6x_1 + x_2 = t_1 \cong 90,631.1205$, passes both D and M at the same time. Therefore, while the slope is $-6$, that is, $w_1 = 6w_2$, or while one unit of diesel fuel is equal to 6 units of gasoline, both divisions D and M suggest the best combinations of the diesel fuel and the gasoline amounts.

While one unit of diesel fuel is equal to 3 units of gasoline, as it is represented by the line, $3x_1 + x_2 = t_2 \cong 61,237.2436$, the divisions M, Q and A have the best combinations of the diesel fuel and the gasoline amounts, simultaneously. Note that, Q lies on the line-segment MA. Similarly, while the slope of the line, $w_1x_1 + w_2x_2 = 0$, which passes the origin, is equal to $-0.\bar{3}$, moving the line through the heptagon, passes both A and I in the first connection with the heptagon, and hence, they are the best performers among other divisions while one unit of diesel fuel is equal to $33.\bar{3}\%$ of one unit of gasoline.

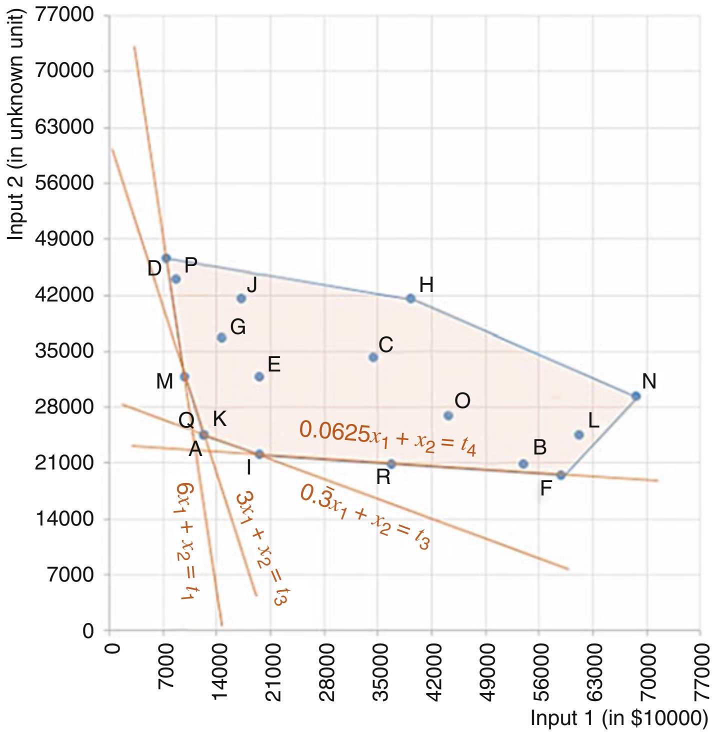


**Fig. 3.13** The heptagon sides with different slopes

The last line, $0.0625x_1 + x_2 = t_4 \cong 23,270.1526$, represents that I, R and F can be the best performers among the other divisions while one unit of the diesel fuel is equal to 6.25% of one unit of the gasoline. Note that, R lies on the line-segment IF.

The relative scores and the ranks of the divisions are different while the slope of the line is changed, and it is impossible to find a real value for the slope, $-w_1/w_2$, which introduces all the divisions on the frontiers in Figs. 3.8, 3.9, 3.10, 3.11, 3.12, and 3.13, as the best performers.

In other words, the divisions D, M, Q, A, I, R and F might have done the job right, (for instance, by applying the convexity approach), but it should not be supposed that D, M, Q, A, I, R and F have done the job well simultaneously, except when they lie on a line at the same time which is impossible in this example.

As can be seen in Fig. 3.14, every line which passes D with the slope less than equal $-6$, identifies D as the best performer, and for each slope the rank of the other divisions can also be different. M can also be the best performer while the slope of the line is greater than equal $-6$, and less than equal $-3$, as Fig. 3.15 illustrates. Q can only be the best performer while the slope of the line is $-3$.

As shown in Fig. 3.16, A is the best performer while the slope of the line is greater than equal $-3$, and less than equal $-0.\bar{3}$. It should not be forgotten that, every line which passes A, and has a slope between $-3$ and $-0.\bar{3}$, may rank divisions

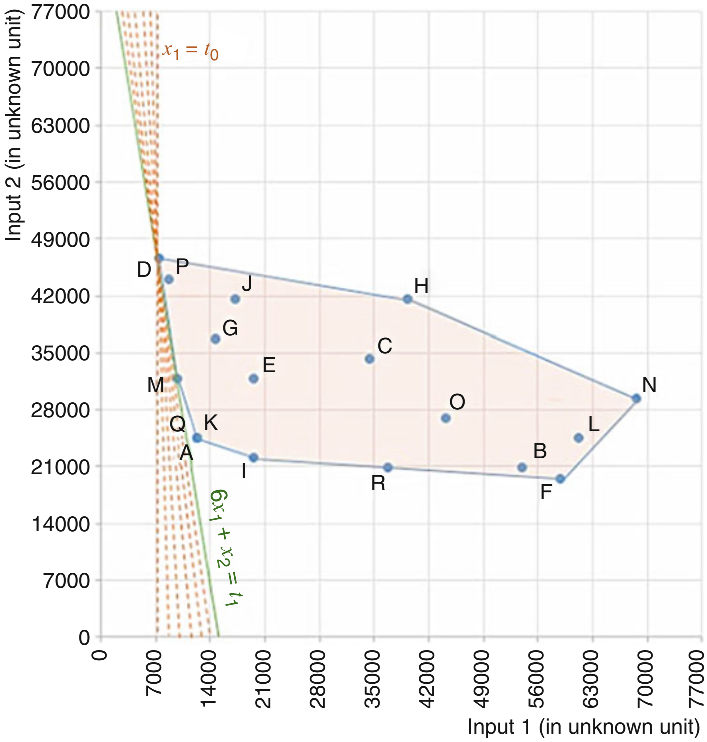


**Fig. 3.14** D is the best performer

differently. For instance, when the slope is $-3$, the rank of D is much higher than that of F, and when the slope is $-0.\bar{3}$, the rank of F is higher than that of D.

Similarly, Fig. 3.17 displays that I has the best rank among the divisions while the slope of the line belongs to the interval $\left[-0.\bar{3}, -0.0625\right]$. For the slope $-0.0625$, the divisions I, R and F have the best ranks, whereas P and D obtain the worst ranks. Furthermore, only the slope $-0.0625$ introduces R as the best performer among the other divisions.

While the slope of the line is greater than $-0.0625$ and lesser than 0, the best performer is F, as depicted in Fig. 3.18. Note that, since the weights, $w_1$ and $w_2$, are assumed as positive numbers, the slope of the line should be negative.

Table 3.10 summarizes the above results as well as the rank of each division. The first column in Table 3.10 displays the selected slopes and the other columns display the rank of each division according to the selected slope.

For instance, when the slope is $-6$, the third row in Table 3.10 illustrates that A has the fifth rank, D has the first rank, F has the sixteenth rank, and so on.

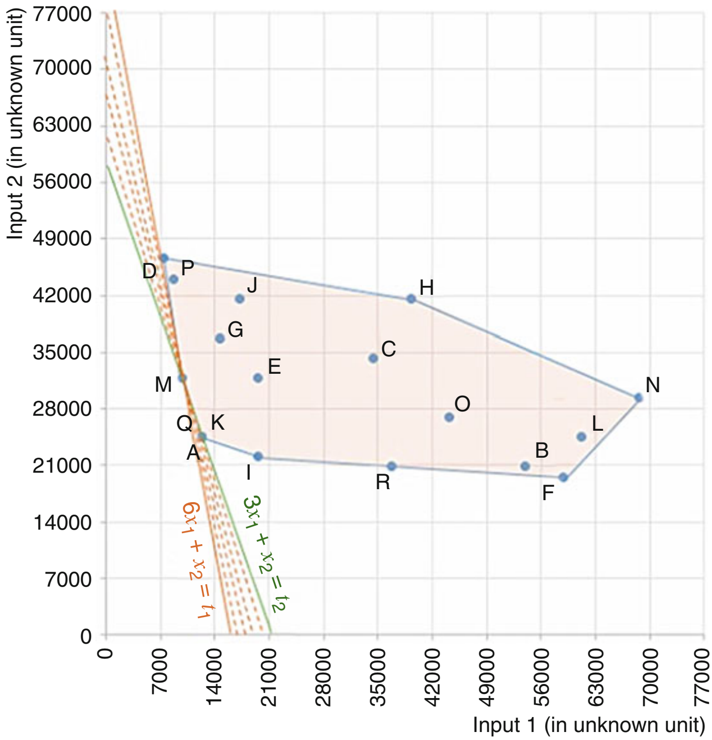


**Fig. 3.15** M is the best performer

#### 3.6.3.1 The Statistical Values of the Relative Scores

One may be eager to find the minimum, maximum, or average values of the relative scores of a division through the slopes on the interval $(-\infty, 0)$ or any other specified intervals, such as $(-a, -b)$, where $a, b \in \mathbb{R}_+$. This aim can be useful to decrease the risk of selecting the weights, and at the same time can be extremely harmful.

For instance, the maximum values of the relative scores of divisions should not be considered to rank divisions or to introduce them as those have done the jobs well. Table 3.11 illustrates the minimum and the maximum values of the relative scores of each division with six decimal digits as well as their highest and lowest ranks. For instance, while A gets the minimum value of its relative scores, its lowest rank is 5, and while B gets the maximum values of its relative scores, its highest rank is 2, (see Table 3.10 as well).

The last column in Table 3.11 shows the product of the lowest and the highest ranks for each division. These values may lead the environmentalist to estimate a relationship between the input factors in Table 3.4 with a reasonably lower risk. Indeed, the highest rank for A is 1, which means, A has done the job right, A wholly

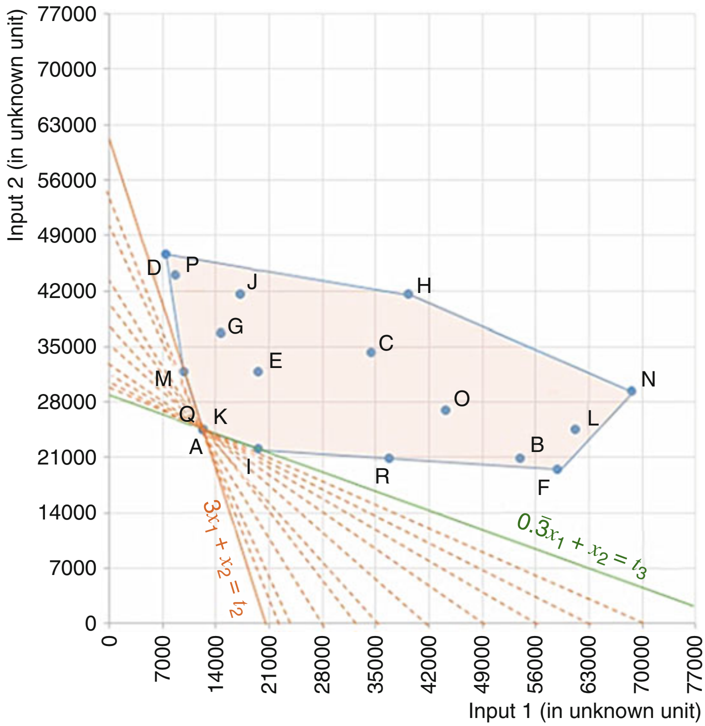


**Fig. 3.16** A is the best performer

dominates 9 divisions, and through all possible relationships between the input factors, the lowest rank for A is 5. Thus, the environmentalist can consider the division A measurement approximation to rank divisions, or introduce a slope in the interval $\left(-3, -0.\bar{3}\right)$ to discriminate divisions, as Fig. 3.16 displays.

Note that, the environmentalist should be very careful to interpret the outcomes in Table 3.11. For instance, a harmful interpretation is to propose the maximum (minimum, average, median and so on) value of the relative scores of a division as the relative score of that division among other maximum values in Table 3.11.

As Sect. 4.3 will mathematically explain, the relative score is meaningful while divisions are homogenous, and one simple condition of homogeneity is that each factor should have the same unit of measurement, (see Sect. 2.3 as well).

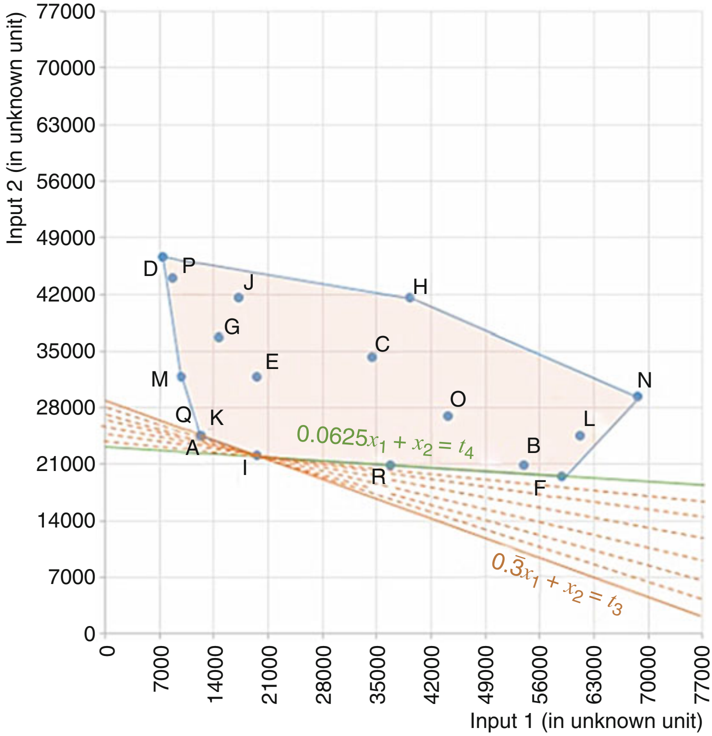


**Fig. 3.17** I is the best performer

## 3.7 Conclusion

In this chapter, an example of a set of homogenous divisions is considered while each division has two input factors. The corresponding factor in each division should have the same unit/worth/weight/price to allow measuring the relative scores. Several approximation methodologies are illustrated to deal with measuring the relative scores of divisions, while the relationships between the factors are unknown. The possible relationships between the factors are classified. The relative scores of divisions can logically be estimated, if the relationship between the factors is estimated. However, in real-life applications, the relationships between the factors are usually unknown or might be difficult or unreasonable to find. Thus, an approximation can be used to decrease the risk of finding the relationships between the factors. For instance, the wholly dominant approach can classify divisions, and introduce the divisions which have done the job right (regarding the wholly dominant approach); but it is not enough to find the divisions which have done the job well, to rank divisions, to regulate the values of each factor, and to measure the

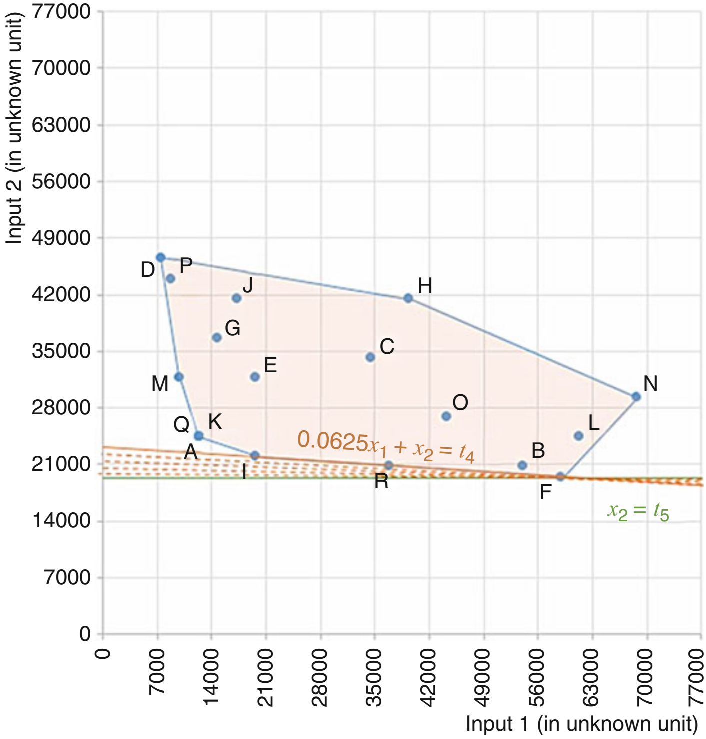


**Fig. 3.18** The range of the slopes

**Table 3.10** The ranks of the divisions by different ratio of the weights

| Slope | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $-\infty$ | 5 | 15 | 11 | 1 | 9 | 16 | 7 | 13 | 9 | 8 | 6 | 17 | 3 | 18 | 14 | 2 | 4 | 12 |
| –10 | 5 | 15 | 11 | 1 | 10 | 16 | 7 | 13 | 9 | 8 | 6 | 17 | 2 | 18 | 14 | 2 | 4 | 12 |
| **–6** | 5 | 15 | 11 | 1 | 10 | 16 | 7 | 13 | 8 | 9 | 6 | 17 | 1 | 18 | 14 | 3 | 4 | 12 |
| –5 | 4 | 15 | 12 | 2 | 10 | 16 | 7 | 13 | 8 | 9 | 5 | 17 | 1 | 18 | 14 | 6 | 3 | 11 |
| –4 | 3 | 15 | 12 | 5 | 9 | 16 | 7 | 13 | 8 | 10 | 4 | 17 | 1 | 18 | 14 | 6 | 2 | 11 |
| **–3** | 1 | 15 | 12 | 5 | 9 | 16 | 7 | 13 | 7 | 10 | 4 | 17 | 1 | 18 | 13 | 6 | 1 | 11 |
| –2 | 1 | 15 | 12 | 5 | 9 | 16 | 8 | 14 | 5 | 10 | 2 | 17 | 4 | 18 | 13 | 5 | 3 | 11 |
| –1 | 1 | 14 | 12 | 9 | 6 | 15 | 6 | 16 | 4 | 11 | 2 | 17 | 4 | 18 | 13 | 8 | 3 | 10 |
| –0.5 | 1 | 9 | 15 | 13 | 7 | 11 | 8 | 17 | 4 | 13 | 2 | 16 | 5 | 18 | 11 | 10 | 3 | 6 |
| **–0.3** | 2 | 7 | 13 | 16 | 9 | 8 | 11 | 18 | 1 | 15 | 3 | 12 | 6 | 17 | 10 | 14 | 4 | 5 |
| –0.2 | 2 | 7 | 13 | 17 | 9 | 6 | 12 | 18 | 1 | 15 | 3 | 11 | 8 | 14 | 9 | 16 | 4 | 5 |
| –0.1 | 4 | 7 | 13 | 18 | 11 | 3 | 14 | 17 | 1 | 15 | 5 | 8 | 10 | 12 | 9 | 16 | 6 | 2 |
| –0.08 | 5 | 4 | 13 | 18 | 11 | 3 | 14 | 16 | 1 | 15 | 6 | 8 | 10 | 12 | 9 | 16 | 7 | 2 |
| **–0.0625** | 5 | 4 | 13 | 18 | 11 | 1 | 14 | 16 | 1 | 15 | 6 | 8 | 10 | 12 | 9 | 17 | 7 | 1 |
| –0.05 | 5 | 4 | 13 | 18 | 11 | 1 | 14 | 16 | 3 | 15 | 6 | 8 | 10 | 11 | 9 | 17 | 7 | 2 |
| –0.01 | 5 | 3 | 13 | 18 | 12 | 1 | 14 | 16 | 4 | 15 | 6 | 8 | 11 | 10 | 9 | 17 | 7 | 2 |
| –0.0001 | 5 | 2 | 13 | 18 | 12 | 1 | 14 | 16 | 4 | 15 | 6 | 7 | 11 | 10 | 9 | 17 | 8 | 3 |
| $0^-$ | 5 | 2 | 13 | 18 | 11 | 1 | 14 | 15 | 4 | 15 | 7 | 5 | 11 | 10 | 9 | 17 | 8 | 3 |

**Table 3.11** The maximum and minimum values of the relative scores

| Division | N | Scores | | Rank | | |
|---|---|---|---|---|---|---|
| | | Min | Max | Lowest | Highest | Multiplying |
| A | 1 | 0.600000 | 1.000000 | 5 | 1 | 5 |
| B | 2 | 0.136364 | 0.962025 | 15 | 2 | 30 |
| C | 3 | 0.214286 | 0.638655 | 15 | 11 | 165 |
| D | 4 | 0.421053 | 1.000000 | 18 | 1 | 18 |
| E | 5 | 0.375000 | 0.744681 | 12 | 6 | 72 |
| F | 6 | 0.125000 | 1.000000 | 16 | 1 | 16 |
| G | 7 | 0.500000 | 0.757576 | 14 | 6 | 84 |
| H | 8 | 0.187500 | 0.527778 | 18 | 13 | 234 |
| I | 9 | 0.375000 | 1.000000 | 9 | 1 | 9 |
| J | 10 | 0.428571 | 0.657895 | 15 | 8 | 120 |
| K | 11 | 0.599988 | 0.999989 | 7 | 2 | 14 |
| L | 12 | 0.120000 | 0.821622 | 17 | 6 | 102 |
| M | 13 | 0.615385 | 1.000000 | 11 | 1 | 11 |
| N | 14 | 0.107143 | 0.690909 | 18 | 10 | 180 |
| O | 15 | 0.166667 | 0.783505 | 14 | 9 | 126 |
| P | 16 | 0.444444 | 0.948718 | 17 | 2 | 34 |
| Q | 17 | 0.601202 | 1.000000 | 8 | 1 | 8 |
| R | 18 | 0.200000 | 1.000000 | 12 | 1 | 12 |

relative scores of divisions. We will provide a philosophical discussion in the next chapter to clarify these statements. As a result, the concept of partially dominant is necessary to answer the mentioned objectives, and this concept requires some information about the weights/prices of the factors. In other words, the concepts of the wholly dominant approach, convexity approach, and radiate approach and so on are not enough to measure the relative scores of firms, and the measured scores by these approaches should not be called relative scores.

## 3.8 Exercise

3.1. In a fair competition, 11 homogenous Mathematics faculty members were able to answer a mathematic question correctly. Each member used different amounts of time (in minutes) and money (in $10 units) to answer the question. Table 3.12 illustrates the amounts of time and money which each faculty used.

(i) Suppose that the relationship between time and money is 1:2, that is, $w_1 = 2$ (the weight for time) and $w_2 = 1$ (the weight for money).

**Table 3.12** The property investors example

| Faculty | I | II | III | IV | V | VI | VII | IIX | IX | X | XI | XII |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Time | 252 | 240 | 168 | 180 | 780 | 240 | 312 | 270 | 300 | 210 | 270 | 276 |
| Money | 70.06 | 101.7 | 158.2 | 113 | 45.2 | 67.8 | 58.76 | 84.75 | 56.5 | 129.95 | 107.35 | 73.45 |

3.1.1. Measure the relative score of each member.
3.1.2. Sketch the location of each member in the Cartesian coordinate plane and graphically describe the results in the previous exercise.
3.1.3. Apply the wholly dominant approach and display the members which have done the job right.
3.1.4. Rank members and find how to regulate the factors of each member.

(ii) Suppose that the relationship between the factors in Table 3.12 is unknown.

3.1.5. Sketch the feasible area by applying the convexity approach.
3.1.6. Find the members who have done the job right.
3.1.7. Find the relative scores of members by the specified measurement approximation.
3.1.8. Classify all possible weights to introduce a member as the best performer, while it has done the job right.
3.1.9. Find the minimum and the maximum values of the relative scores of each member.
3.1.10. Measure the highest and the lowest rank that each member can obtain.
3.1.11. Specify the numbers of members which are dominated by each member which has done the job right.
3.1.12. Approximate the relationship between the factors and calculate the rank of members.

(iii) Show that if the members are supposed as non-homogenous, all the members in Table 3.12 can be introduced as those who have done the job right (well) at the same time.

3.2. Suppose that the data in Table 3.1 is given. Sketch the feasible area by applying the outer radiate and the convexity approaches, and write the mathematical equation to generate the feasible area linearly.

# Chapter 4
# The Optimization Approach

## 4.1 Introduction

In this chapter, we continue the discussion in Chap. 3, and gradually build optimization approaches and linear programming models, without pre-required knowledge in operations research. In order to solve a linear programming, Microsoft Excel Solver software and Microsoft Visual Basic software are used and the required instructions are comprehensively illustrated, so that no previous knowledge about the software are needed. We also discuss a different view for the Petroleum example with two output factors. Two important theorems are provided to explain when a score should be called a 'relative score'. In addition, two necessary types are added to Types 1–4 in Chap. 1, when the number of input factors and the number of output factors are more than 1. These two types provide a transparent view to rank and benchmark firms through the feasible area, as well as a direction for future arguments on these topics. At the end of this section, readers are prepared to calculate the performance evaluation of a set of homogenous firms and explain the related concepts visibly.

## 4.2 The Optimization Approach

Let's suppose that we have the data in Table 3.4 (data without unit of measurement). From Table 3.10, as also shown in Table 4.1, the best rank of D is 1 and the worst rank of D is 18 through selecting different relationships between the input factors or different slopes on the interval $(-\infty, 0)$. The environmentalist may desire to find the optimal values of the weights, $w_1$ and $w_2$, to calculate the maximum (minimum) relative score of a specified division and to find its highest



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(lowest) rank among other divisions. In this section, we provide the mathematical prerequisites to find the optimal weights.

Note that, this aim does not yield to find the division which has done the job well, but it is useful to find a division which has done the job right, (for instance while the convexity approach is applied).

### *4.2.1 The Maximum Value of the Relative Scores*

As discussed in Chap. 3, in order to measure the relative score of a division, Eq. 3.8 can be used, thus, the maximum value of the relative scores for each division, can be measured by the following equation, for $l = 1, 2, \ldots, 18$,

$$\max\left\{\frac{\frac{1}{x_{l1}w_1+x_{l2}w_2}}{\max\left\{\frac{1}{x_{i1}w_1+x_{i2}w_2} : i = 1, 2, \ldots, 18\right\}} : w_1 \geq 0, w_2 \geq 0\right\}. \tag{4.1}$$

Equation 4.1 is always less than equal to 1, for $l = 1, 2, .., 18$, and is equal to 1 for at least one of $l = 1, 2, .., 18$. A division with the score of 1 is introduced as the division which has done the job right.

Equation 4.1 can be simplified as the following equation:

$$\max\left\{\frac{\min\{x_{i1}w_1 + x_{i2}w_2 : i = 1, 2, \ldots, 18\}}{x_{l1}w_1 + x_{l2}w_2} : w_1 \geq 0, w_2 \geq 0\right\}. \tag{4.2}$$

Without loss of generality, assume that $\min\{w_1x_{i1} + w_2x_{i2} : i = 1, 2, \ldots, 18\} = 1$, thus, Eq. 4.2 is equivalent with Eq. 4.3.

$$1/\min\{x_{l1}w_1 + x_{l2}w_2 : \quad w_1 \geq 0, w_2 \geq 0, \quad \min\{x_{i1}w_1 + x_{i2}w_2 : \forall i\} = 1\}. \tag{4.3}$$

Furthermore, the equation $\min\{w_1x_{i1} + w_2x_{i2} : i = 1, 2, \ldots, 18\} = 1$ yields the following inequalities:

$$x_{i1}w_1 + x_{i2}w_2 \geq 1, \quad \text{for } i = 1, 2, \ldots, 18. \tag{4.4}$$

As a result, Eq. 4.2 can be rewritten as Eq. 4.5.

$$1/\min\{x_{l1}w_1 + x_{l2}w_2 : \quad w_1 \geq 0, w_2 \geq 0, \quad x_{i1}w_1 + x_{i2}w_2 \geq 1, \quad \forall i\}. \tag{4.5}$$

The denominator in Eq. 4.5 is a *Linear Programming* (LP) and can be written as the standard shape given by:

$$\begin{aligned}
&\min x_{l1}w_1 + x_{l2}w_2,\\
&\text{Subject to}\\
&x_{i1}w_1 + x_{i2}w_2 \geq 1, \quad \text{for } i = 1, 2, \ldots, 18,\\
&w_1 \geq 0,\\
&w_2 \geq 0.
\end{aligned} \tag{4.6}$$

An LP is a mathematical method to find the optimum value of a linear equation according to some conditions that each condition is in the form of linear equality or linear inequality. In the first line of Model 4.6, the equation, $w_1x_{l1}$ + $w_2x_{l2}$, is called the *objective* of LP and the other equations in the third, fourth and fifth lines are called the *constraints* of LP. The weights, $w_1$ and $w_2$ are called *variables* which their optimal values are desired. In Model 4.6, there are twenty constraints which generate an area, called *feasible area*, and let find the minimum value of the objective.

Similar to illustrations in Sect. 3.6.1, Model 4.6 can graphically be solved. At the first step, the feasible area is generated by the intersection of the constraints. After that, by sketching the objective (which is a line and passes the origin), and moving the line toward the feasible area, the first connection of the line with the feasible area (if any) is the optimal solution.

### *4.2.2 The Microsoft Excel Solver Software*

There are several software which can be used to solve an LP, such as: Microsoft Excel Solver, Mathematica, Lingo, GAMS, R software and so on. For example, the instructions to solve Model 4.6 by Microsoft Excel Solver 2013 software are given by:

1. Copy the four columns of Table 3.4 on an Excel sheet into the cells A1:D19, as Fig. 4.1 illustrates.
2. Label the four cells B21:E21 as 'Index', '$w_1$', '$w_2$' and 'Objective, respectively, and label cell E1 as 'Constraints'.
3. Assign number 1 to B22.
4. Assign the command '=Sumproduct(Index(C2:D19,B22,0),C22:D22)' into E22. Excel automatically calculates it and the value in E22 is 0. (Note that remove the quotation to use the command.)
5. Assign the command '=Sumproduct(C2:D2,C$22:D$22)' to E2.
6. Copy and paste cell E2 to cells E3:E19. As Fig. 4.1 displays, the two cells C22:D22 are not changed in the commands in cells E3:E19, whereas the cells C2:D2 in the commands are changed corresponding to each row.
7. From 'Data' in the toolbar menu, click on 'Solver' icon to open the 'Solver Parameters' window. Note that, if this icon is not shown, from 'File>Options>Add-Ins>Manage Excel Add-ins>Go…', tick 'Solver Add-in' and then 'OK'.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | Agency | N | Input 1 | Input 2 | | | | |
| 2 | A | 1 | 12247.44871 | 24494.89743 | | | | |
| 3 | B | 2 | 53888.77434 | 20820.66281 | | | | |
| 4 | C | 3 | 34292.8564 | 34292.8564 | | | | |
| 5 | D | 4 | 7348.469228 | 46540.30511 | | | | |
| 6 | E | 5 | 19595.91794 | 31843.36666 | | | | |
| 7 | F | 6 | 58787.75383 | 19595.91794 | | | | |
| 8 | G | 7 | 14696.93846 | 36742.34614 | | | | |
| 9 | H | 8 | 39191.83588 | 41641.32563 | | | | |
| 10 | I | 9 | 19595.91794 | 22045.40769 | | | | |
| 11 | J | 10 | 17146.4282 | 41641.32563 | | | | |
| 12 | K | 11 | 12247.69366 | 24495.14238 | | | | |
| 13 | L | 12 | 61237.24357 | 24494.89743 | | | | |
| 14 | M | 13 | 9797.958971 | 31843.36666 | | | | |
| 15 | N | 14 | 68585.7128 | 29393.87691 | | | | |
| 16 | O | 15 | 44090.81537 | 26944.38717 | | | | |
| 17 | P | 16 | 8573.2141 | 44090.81537 | | | | |
| 18 | Q | 17 | 12222.95382 | 24568.38212 | | | | |
| 19 | R | 18 | 36742.34614 | 20973.75592 | | | | |
| 20 | | | | | | | | |
| 21 | | | | | | | | |
| 22 | | | | | | | | |
| 23 | | | | | | | | |
| 24 | | | | | | | | |
| 25 | | | | | | | | |

**Fig. 4.1** The Excel sheet to solve Model 4.6

8. Assign ‘E22’ into ‘Set Objective’ and choose ‘Min’.
9. Assign ‘C22:D22’ into ‘By Changing Variable Cells’.
10. Click on ‘Add’ and assign E2:E19 into ‘Cell Reference’, then select ‘>=’, and assign 1 into ‘Constraint’.
11. Tick ‘Make Unconstrained Variables Non-Negative’.
12. Choose ‘Simplex LP’ from ‘Select a Solving Method’.
13. Click on ‘Solve’.
14. By changing the index value in cell B22, from 1–18, Model 4.6 is solved for divisions A–R (Fig. 4.2).

In order to solve Model 4.6 or Eq. 4.1, the inverse of the objective value in E22 should be calculated. The results of this process are the same as the results in the fourth column of Table 3.11, except the maximum value of the relative scores of division E which is almost equal to 0.7447 instead of 0.7403.

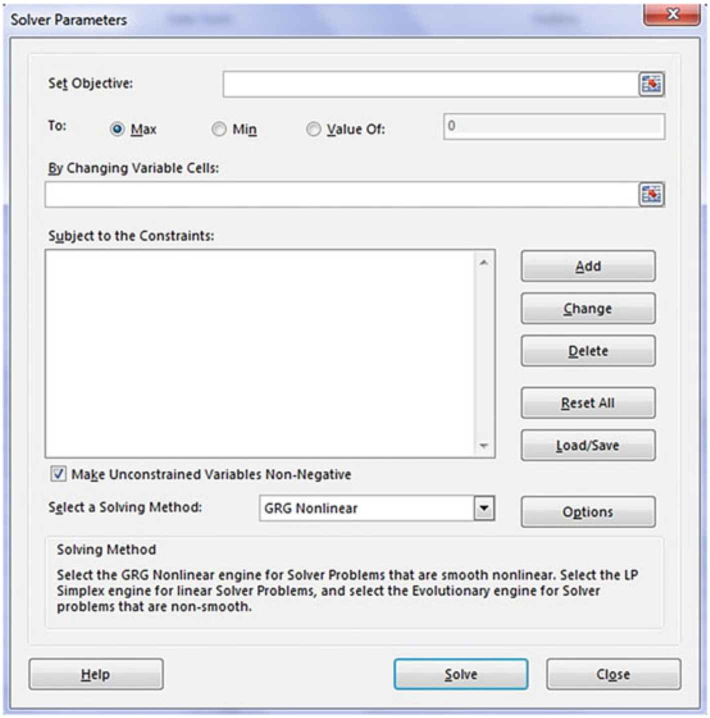



**Fig. 4.2** The Excel Solver window

### *4.2.3 The Microsoft Visual Basic Software*

Model 4.6 should be run 18 times in order to find the maximum of the relative scores of divisions and then the results should be inversed to solve Eq. 4.1. Since this process is not user-friendly, according to the following instructions, a Microsoft Visual Basic procedure can be added to the Excel sheet which let's calculate all the results by a click.

1. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window. Note that, if 'Developer' is not shown, from 'File>Options> Customize Ribbon>' Click on 'Developer' in the 'Main Tabs' in the right side of the opened window, and then 'OK'.
2. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet.
3. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened, as Fig. 4.3 illustrates.
4. From the toolbar menu, click on 'Tools> References...>' and tick 'Solver', and then 'OK'.

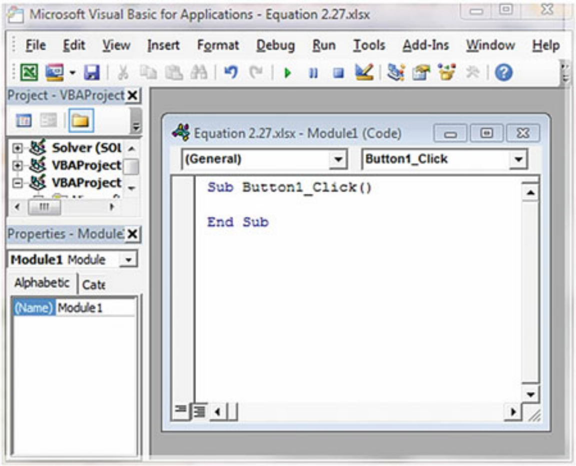



**Fig. 4.3** The Microsoft Visual Basic for Applications window

5. Inside the 'Microsoft Visual Basic for Applications' window, write the following commands between 'Sub Button1_Click ()' and 'End Sub', as Fig. 4.3 depicts.

```
Dim i As Integer
For i = 1 To 18
    Range("B22") = i
    SolverSolve Userfinish:=True
    Range("F" & i + 1) = 1/Range("E22")
Next i
```

6. Close the 'Microsoft Visual Basic for Applications' window.
7. Click on the small rectangle which was automatically made on the Excel sheet in the place which was clicked in step 2. The rectangle can be moved and renamed as well.
8. The results of solving Eq. 4.1 are represented in column F on the Excel sheet.

### *4.2.4 The Minimum Value of the Relative Scores*

Similar to calculating the maximum values of the relative scores of divisions by Eq. 4.1, the minimum values of the relative scores of divisions can be suggested by

Eq. 4.7, for $l = 1, 2, \ldots, 18$. If the previous technique to solve Eq. 4.1 is used, the optimal value is always 0.

$$\min\left\{\frac{\frac{1}{x_{l1}w_1+x_{l2}w_2}}{\max\left\{\frac{1}{x_{i1}w_1+x_{i2}w_2} : i = 1, 2, \ldots, 18\right\}} : w_1 \geq 0, w_2 \geq 0\right\}. \tag{4.7}$$

To deal with this problem, let's suppose, $w_1x_{l1} + w_2x_{l2} = 1$, thus, Eq. 4.7 is equivalent with the following equation.

$$\begin{aligned} &\min\nolimits_{1\leq i\leq 18}\{\\ &\quad \min\ x_{i1}w_1 + x_{i2}w_2,\\ &\quad \text{Subject to}\\ &\quad x_{l1}w_1 + x_{l2}w_2 = 1,\\ &\quad w_1 \geq 0,\\ &\quad w_2 \geq 0.\\ &\qquad\quad \}. \end{aligned} \tag{4.8}$$

In order to solve Model 4.8, the following instructions can be followed.

1. Copy the four columns of Table 3.4 on an Excel sheet into the cells A1:D19, as Fig. 4.1 illustrates.
2. Label the four cells B21:E21 as 'Index', '$w_1$', '$w_2$' and 'Objective', respectively.
3. Label E1 as 'Constraints'.
4. Assign number 1 to B22.
5. Assign the command '=Sumproduct(Index(C2:D19,B22,0),C22:D22)' into E22. (Note that, the command is without the quotation.)
6. Assign number 1 to B23.
7. Assign the command '=Sumproduct(Index(C2:D19,B23,0),C22:D22)' into E23.
8. Open 'Solver Parameters' window.
9. Assign 'E23' into 'Set Objective' and choose 'Min' (Fig. 4.4).
10. Assign 'C22:D22' into 'By Changing Variable Cells'.
11. Click on 'Add' and assign E22 into 'Cell Reference', then select '=', and assign 1 into 'Constraint'.
12. Tick 'Make Unconstrained Variables Non-Negative'.
13. Choose 'Simplex LP' from 'Select a Solving Method'.
14. Click on 'Solve'.
15. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.
16. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet.
17. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened, as Fig. 4.5 illustrates.

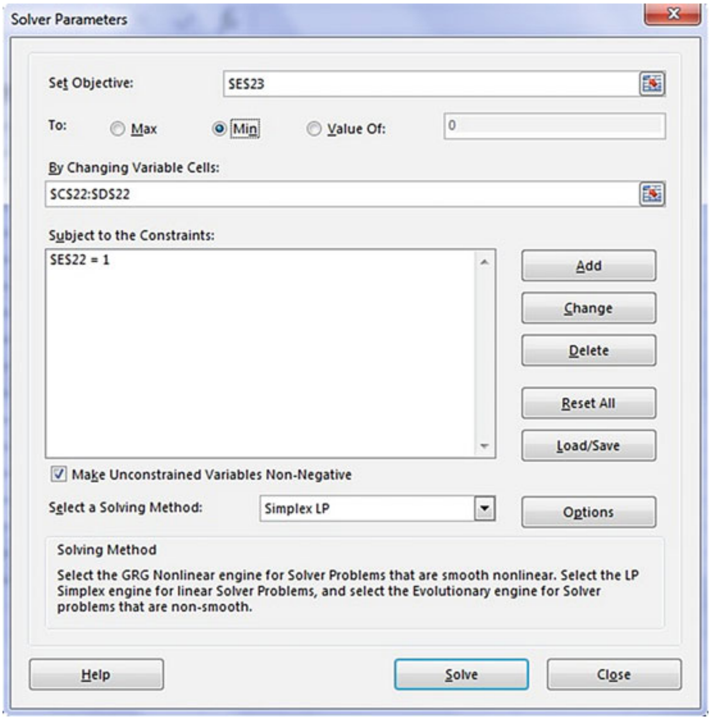
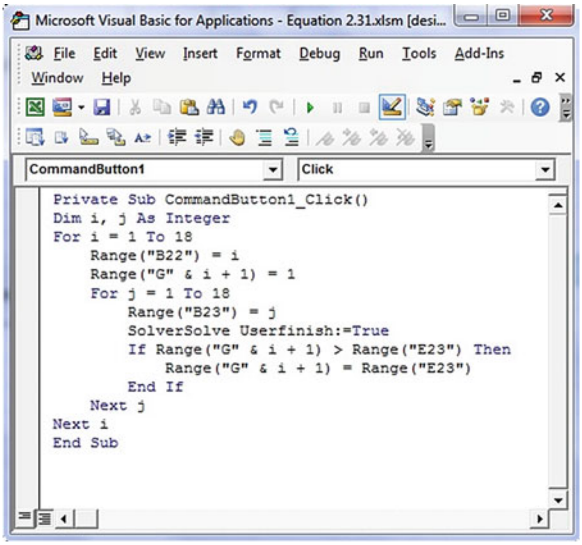



**Fig. 4.4** Solver parameters to solve Model 4.8

**Fig. 4.5** The macro to solve Model 4.8



18. From the toolbar menu, click on 'Tools> References...>' and make sure 'Solver' is ticked, and then 'OK'.
19. Inside of the 'Microsoft Visual Basic for Applications' window, write the following commands between 'Sub Button1_Click ()' and 'End Sub' as Fig. 4.5 depicts.

```
Dim i, j As Integer
For i = 1 To 18
    Range("B22") = i
    Range("G" & i + 1) = 1
    For j = 1 To 18
       Range("B23") = j
       SolverSolve Userfinish:=True
       If Range("G" & i + 1) &gt; Range("E23") Then
          Range("G" & i + 1) = Range("E23")
       End If
    Next j
Next i
```

20. Close the 'Microsoft Visual Basic for Applications' window.
21. Click on the small rectangle which was automatically made on the Excel sheet in the place which was created in the step 16.
22. The results of solving Model 4.8 are represented in the column G in the Excel Sheet, which are the same as the results in the third column in Table 3.11.

From the above instructions, the LP in Model 4.8 is solved $18 \times 18$ times, whereas the LP requires to be solved $18 \times 7$ times only, (see Exercise 2.3).

#### 4.2.4.1 Warnings About Misinterpretation

The maximum and the minimum values of the relative scores of each division, which are measured in Sects. 3.6.3 and 4.1, are unique according to the available data in Table 3.4 (the data with unknown units of measurement). These results can be used as a probable view to decrease the risk of selecting a relationship between the diesel fuel and the gasoline amounts. At the same time, the corresponding optimal weights for each division, measured in Sect. 4.2, are not unique, as Figs. 3.14, 3.15, 3.16, 3.17, and 3.18 illustrates. For example, see the optimal weights for F in the following figure.

As a result, there are two important warnings to focus on the maximum value of the relative scores of a division. The first warning is that the maximum value of the relative scores of a division is not a relative score for that division among other maximum values of the relative scores of divisions. The second warning is that, after the optimization to find the maximum value of the relative scores for a division, the environmentalist must not consider different relationship between the diesel fuel and the gasoline amount from one division to another.

In order to elucidate the above warnings, suppose that the optimal weights for each division are denoted by, $w_{l1}^*$ and $w_{l2}^*$, for $l = 1, 2, .., 18$. The relative scores and the ranks of divisions completely depend upon the selected weights, and can be different for each set of the optimal weights, $w_{l1}^*$ and $w_{l2}^*$, as illustrated in Tables 3.11 and 4.1.

For instance, the optimal weights, which obtain the maximum value of the relative scores for D, yield a very small relative score for F. In contrast, the optimal weights, which obtain the maximum value of the relative scores for F, as also shown in Fig. 4.6, yield a very small relative score for D. The divisions D and F have not done the job well at the same time, unless they are considered as non-homogenous divisions. However, a simple paradox for this harmful mistake is that the divisions in Table 3.4 (that is, data with unknown unit of measurement) must be homogenous to allow measurement of the relative scores by the following equation, as also explained in Eq. 3.8.

If the weights/worth/prices or the relationship between the factors in Table 3.4 can be different from one division to another, none of the methods in Sect. 3.6 are valid. The next section mathematically illustrates this transparent statement and the meaning of homogeneity.

The above illustrations can be expressed for the minimum, the average and other statistical values of the relative scores of divisions. In short, after finding the optimal values of the weights by Eq. 4.1, the environmentalist may prepare a table similar to Tables 4.1 and 3.11 to rank the divisions and calculate their relative scores regarding the selected non-unique optimal weights, $w_{l1}^*$ and $w_{l2}^*$, for $l = 1, 2, .., 18$. Nonetheless, all of these attempts should be made to decrease the risk of selecting a relationship between the diesel fuel and the gasoline amounts, in order to measure the relative scores of divisions in Table 3.4 (that is, data with unknown unit of measurement), as discussed in Sect. 3.6.3. At the end, the environmentalist should still select a set of weights to suggest the corresponding relative scores and ranks of divisions.

## 4.3 Homogeneity and the Relative Score

Can two homogenous divisions in Table 3.4 have different relationships between the diesel fuel and the gasoline amounts? Are the relative scores between these divisions meaningful when Eq. 3.8 is used? What are the conditions to use Eq. 3.8 to measure the relative score of a division among other divisions?

In order to answer the above questions, suppose that there are two divisions, labeled $A_1$ and $A_2$, and each $A_i$ ($i = 1, 2$) has two input factors with the positive real values, labeled $x_{i1}$ and $x_{i2}$. From Sect. 2.3, the unit and the worth (price/weight) of $x_{11}$ and $x_{21}$ should be the same, which means, one unit of $x_{11}$ should have exactly the same meaning with one unit of $x_{21}$ to allow displaying them on one axis in the

**Table 4.1** The ranks divisions by different set of weights

| $-w_1/w_2$ | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| −10 | 5 | 15 | 11 | **1** | 10 | 16 | 7 | 13 | 9 | 8 | 6 | 17 | 2 | 18 | 14 | 2 | 4 | 12 |
| −6 | 5 | 15 | 11 | **1** | 10 | 16 | 7 | 13 | 8 | 9 | 6 | 17 | 1 | 18 | 14 | 3 | 4 | 12 |
| −3 | 1 | 15 | 12 | **5** | 9 | 16 | 7 | 13 | 7 | 10 | 4 | 17 | 1 | 18 | 13 | 6 | 1 | 11 |
| −1 | 1 | 14 | 12 | **9** | 6 | 15 | 6 | 16 | 4 | 11 | 2 | 17 | 4 | 18 | 13 | 8 | 3 | 10 |
| −0.5 | 1 | 9 | 15 | **13** | 7 | 11 | 8 | 17 | 4 | 13 | 2 | 16 | 5 | 18 | 11 | 10 | 3 | 6 |
| −0.3 | 2 | 7 | 13 | **16** | 9 | 8 | 11 | 18 | 1 | 15 | 3 | 12 | 6 | 17 | 10 | 14 | 4 | 5 |
| −0.1 | 4 | 7 | 13 | **18** | 11 | 3 | 14 | 17 | 1 | 15 | 5 | 8 | 10 | 12 | 9 | 16 | 6 | 2 |
| −0.0625 | 5 | 4 | 13 | **18** | 11 | 1 | 14 | 16 | 1 | 15 | 6 | 8 | 10 | 12 | 9 | 17 | 7 | 1 |
| −0.01 | 5 | 3 | 13 | **18** | 12 | 1 | 14 | 16 | 4 | 15 | 6 | 8 | 11 | 10 | 9 | 17 | 7 | 2 |

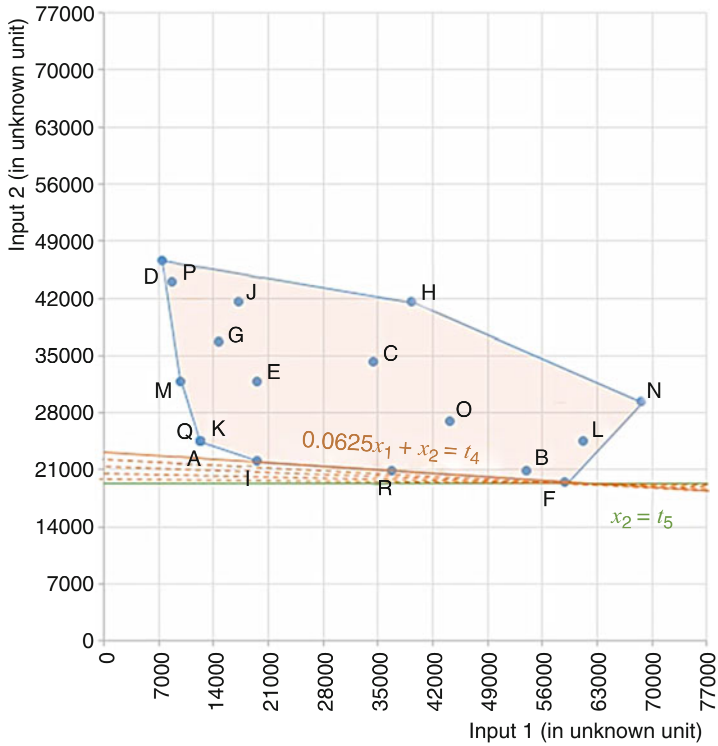



**Fig. 4.6** The optimal weights for division F

Cartesian coordinate plan. If $x_{11} < x_{21}$, the distance of $x_{11}$ to the origin should be smaller than the distance of $x_{21}$ to the origin. If $x_{11} > x_{21}$, the distance of $x_{11}$ to origin should be greater than the distance of $x_{21}$ to the origin. If $x_{11} = x_{21}$, the location of $x_{11}$ and $x_{21}$ should be the same on the axis. Similarly, the unit and the worth (price/weight) of $x_{12}$ and $x_{22}$ should be the same.

On the other hand, the units can be different for $x_{11}$ and $x_{12}$ (or $x_{21}$ and $x_{22}$) which do not let's add $x_{11}$ and $x_{12}$ together without introducing the relationship between $x_{11}$ and $x_{12}$. Since there are two values of input factors, $x_{i1}$ and $x_{i2}$, and they may have different units, a combination of the values of input factors is needed to provide an appropriate value to determine the used inputs of $A_i$ ($i = 1, 2$). One simple way to introduce a combination of the input factors is to consider a linear combination of the two input factors, that is, $w_{i1}x_{i1} + w_{i2}x_{i2}$, by introducing two positive real values (weights), $w_{i1}$ and $w_{i2}$ ($i = 1, 2$). Since a division which uses lesser values of input factors should have a greater score, the score of $A_i$ ($i = 1, 2$) is proposed by the inverse of a linear combination of the two input factors as follows:

$$\frac{1}{x_{i1}w_{i1} + x_{i2}w_{i2}}. \tag{4.9}$$

Without loss of generality, suppose that, $x_{11} = x_{21} = x_{12} = x_{22} = 1$ in \$. Thus, $A_1$ and $A_2$ have the same locations in the Cartesian coordinate plan; because $x_{11}$ and $x_{21}$ are in the same location on $x$-axis, $x_{21}$ and $x_{22}$ are in the same location on $y$-axis.

Assume that the relationship between the two input factors is different from one division to another, for instance, $w_{11} = w_{12}$ and $w_{21} = 3w_{22}$. This means, both input factors have the same weight (price/worth) for $A_1$, while for $A_2$ the weight (price/worth) of the first input factor is triple the weight (price/worth) of the second input factor. The scores of $A_1$ and $A_2$, measured by Eq. 4.9, are $1/(2w_{11})$ and $1/(4w_{21})$, respectively. If $w_{11} = w_{22} = 1$, the scores of $A_1$ and $A_2$, which have the same location in the Cartesian coordinate plan, are 0.5 and 0.25, respectively. As a result, the scores of $A_1$ and $A_2$ can be different, even if $A_1$ and $A_2$ have the same location in the Cartesian coordinate plan. *This transparent contradiction yields that the relationship between the input factors should not be different from one division to another.*

Indeed, if the relationship between $x_{11}$ and $x_{12}$ is different in comparison with the relationship between $x_{21}$ and $x_{22}$, the values of two input factors, $x_{11}$ and $x_{21}$ (or $x_{12}$ and $x_{22}$) cannot be compared and displayed on an axis in the Cartesian coordinate plan, unless the weights (price/worth), $w_{11}$ and $w_{21}$, (or $w_{12}$ and $w_{22}$) are known. In addition, no feasible area can be determined in Sect. 3.6 while the unknown relationship between $x_{11}$ and $x_{12}$ is supposed different with the unknown relationship between $x_{21}$ and $x_{22}$. The relative score in Eq. 3.8 as well as Eqs. 4.1 and 4.7 have two variables only, that is, the unknown relationship between $x_{11}$ and $x_{12}$ is the same as the unknown relationship between $x_{21}$ and $x_{22}$, which means $w_{11} = w_{21}$ and $w_{12} = w_{22}$.

From the above outcome, while data in Table 3.4 are given and the divisions, $A_1$, $A_2$, …, $A_{18}$, are called *homogenous*, measuring the relative scores of divisions by Eq. 3.8 are appropriate if the first (second) input factors of divisions, $x_{11}$, $x_{21}$, … and $x_{18,\ 1}$, ($x_{12}$, $x_{22}$, … and $x_{18,\ 2}$) have the same unit, worth, price, and weight. These conditions yield that the relationships between the input factors are the same from one division to another. Divisions are also compared according to the selected set of input factors only, that is, if another factor or information is added to Table 3.4 the divisions might not be known as homogenous.

In short, the assumption that 'the relationships between the input factors can be different from one division to another', is equivalent with the assumption that 'the units of the factors can be different from one division to another'. In both of these situations, sketching a feasible area is possible if the units of measurement or the relationships between the input factors are known. Otherwise, if one can assume that the amounts for the factors of division $A_1$ might have been measured in \$ unit and the amounts for the factors of division $A_2$ might have been measured in £ unit, sketching a feasible area is impossible, unless the values are replaced in the same unit, and we have the following theorems.

**Theorem 4.1** In order to measure the relative scores of a set of homogenous firms, the relationship between the input factors should not be different from one division to another, except the differences are known.

**Theorem 4.2** The maximum (minimum, average, median, and so on) value of the relative scores of a firm among a set of homogenous firms is not a relative score for that firm among the maximum (minimum, average, median, and so on) value of the relative scores of other firms.

## 4.4 Another View of the Petroleum Example

Instead of the EPA's view in the petroleum example in Sect. 3.2, suppose that a major oil company wants to measure the highest consumption of the two petroleum products. In this view, the best division is the division which consumed the greatest amounts of diesel fuel and gasoline. Thus, the factors are considered as output factors; similar to Eq. 3.2, a score can be provided for each division, which increases (decreases) while the value of Eq. 4.10 increases (decreases). The notations $y_1$ and $y_2$ represent the diesel fuel amount and the gasoline amount, respectively.

$$y_1\ (\$10000) + y_2(\$10000) \tag{4.10}$$

The relative score of division $l$ ($l = 1, 2, \ldots, 18$) is also given by:

$$\frac{y_{l1} + y_{l2}}{\max\{y_{i1} + y_{i2} : i = 1, 2, \ldots, 18\}}, \quad \text{for } l = 1, 2, \ldots, 18. \tag{4.11}$$

Table 4.2 illustrates the relative scores of divisions A–R and their ranks.

As can be seen, the ranks of divisions in Table 4.2 are completely opposite with the ranks in Table 3.2. Divisions A and K have the highest ranks in Table 3.2 and the lowest ranks in Table 4.2, respectively. Figure 4.7 depicts the location of each observation in Table 4.2 while the horizontal axis describes the amounts of diesel fuel in \$10,000 unit and the vertical axis describes the amounts of gasoline in \$10,000 unit. Equation 4.10 also represents the line, $y_1 + y_2 = r$, with the slope of $-1$. If $r = 0$, the line passes the origin, and by moving the line through the observations, the last connection of the line with the observations represents the maximum value of $r$, which displays division N. These outcomes are the same as the results in Table 4.2.

The practical points can similarly be proposed by an introduced approach such as the wholly dominant and the convexity approaches. Figure 4.8 depicts the area which is wholly dominated by C and the area which is wholly dominates C. Only division H wholly dominates C, because the coordinates of H satisfy the following

**Table 4.2** The relative scores of divisions A–R

| Division | N | Sum ($10,000) | Rank | Relative score |
|---|---|---|---|---|
| A | 1 | 36742.346140 | 18 | 0.375000 |
| B | 2 | 74709.437150 | 5 | 0.762500 |
| C | 3 | 68585.712800 | 7 | 0.700000 |
| D | 4 | 53888.774338 | 10 | 0.550000 |
| E | 5 | 51439.284600 | 12 | 0.525000 |
| F | 6 | 78383.671770 | 4 | 0.800000 |
| G | 7 | 51439.284600 | 12 | 0.525000 |
| H | 8 | 80833.161510 | 3 | 0.825000 |
| I | 9 | 41641.325630 | 15 | 0.425000 |
| J | 10 | 58787.753830 | 8 | 0.600000 |
| K | 11 | 36742.836040 | 17 | 0.375005 |
| L | 12 | 85732.141000 | 2 | 0.875000 |
| M | 13 | 41641.325631 | 14 | 0.425000 |
| N | 14 | 97979.589710 | 1 | 1.000000 |
| O | 15 | 71035.202540 | 6 | 0.725000 |
| P | 16 | 52664.029470 | 11 | 0.537500 |
| Q | 17 | 36791.335940 | 16 | 0.375500 |
| R | 18 | 57716.102060 | 9 | 0.589062 |

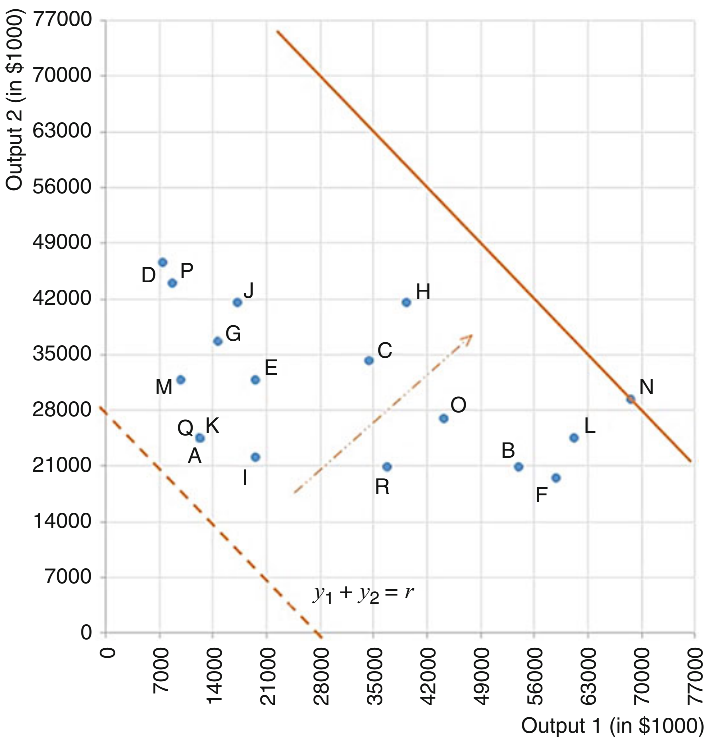



**Fig. 4.7** The line with the slope −1 which passes N

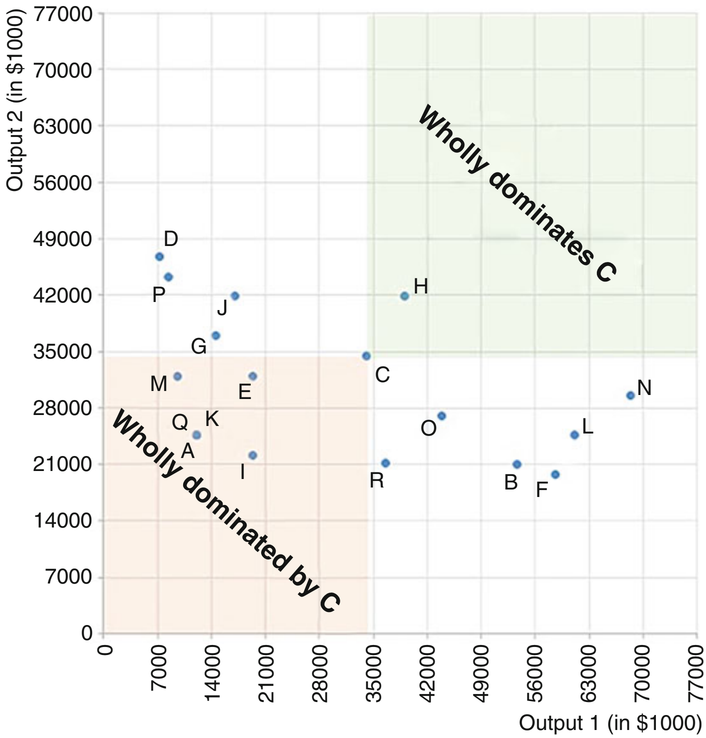



**Fig. 4.8** The area which wholly dominated by C

inequalities, while $y_1$ displays the first output factor and $y_2$ displays the second output factor.

$$34292.8564 \leq y_1, \quad 34292.8564 \leq y_2 \tag{4.12}$$

Division C also dominates divisions A, K, Q, M, I and E, because their coordinates satisfy the following inequalities.

$$34292.8564 \geq y_1, \quad 34292.8564 \geq y_2 \tag{4.13}$$

Figure 4.9 displays the area which is partially dominated by C and the area which partially dominates C. Indeed, every point below the line $y_1 + y_2 = 68585.7128$, is partially dominated by C, and every point above that line $y_1 + y_2 = 68585.7128$, partially dominates C. By applying the wholly dominant approach for each observation, the shaded area in Fig. 4.10 is generated.

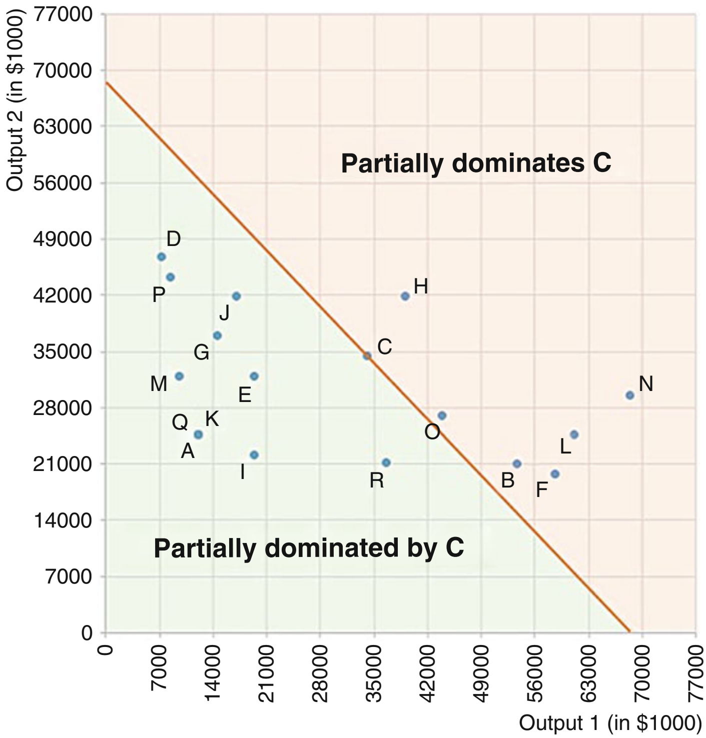



**Fig. 4.9** The area which partially dominated by C

The shaded area in Fig. 4.10 is also generated by the following equation, mathematically, where $y_{i1}$ and $y_{i2}$ are the coordinates of division $i$, for $i = 1$, 2, 3, . . ., 18.

$$\bigcup_{i=1}^{18} \left\{ \left(y_1', y_2'\right) : \ y_{i1} \geq y_1', \quad y_{i2} \geq y_2' \right\}. \tag{4.14}$$

**Theorem 4.3:** *Suppose that $A_i(y_{i1}, y_{i2})$, for $i = 1, 2, \ldots, 18$ denote the observations in Table 3.1. Equation 4.14 linearly yields the shaded area in Fig. 4.10.*

Similarly, the frontier in Fig. 4.10 illustrates that ‘regarding the wholly dominant approach, it is impossible to consume greater amounts of the diesel fuel and the gasoline in the petroleum example’. Therefore, the points on the frontier can be called as those have done the job right, and as illustrated, the meaning of doing the job right does not represent the meaning of doing the job well which is necessary to measure the relative scores of divisions.

In addition, if the convexity and the wholly dominant approaches are applied, the practical points are the shaded area in Fig. 4.11. The linear combination of the convexity and the wholly dominant approaches is illustrated by Eq. 4.15.

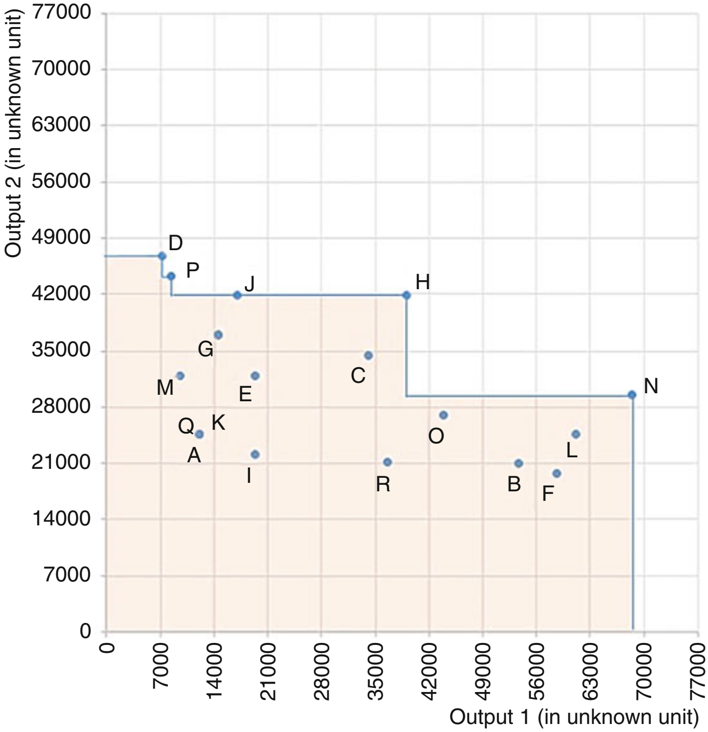



**Fig. 4.10** The wholly dominant approach

$$\left\{ \begin{array}{ll} (y'_1, y'_2): & \sum_{1=1}^{18} y_{i1}\lambda_i \geq y'_1, \quad \sum_{i=1}^{18} y_{i2}\lambda_i \geq y'_2, \\ & \sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \geq 0, \quad \text{for } i = 1, 2, \ldots, 18. \end{array} \right\}. \tag{4.15}$$

**Theorem 4.4** *Suppose that $A_i(y_{i1}, y_{i2})$, for $i = 1, 2, \ldots, 18$ denote the observations in Table 3.1. Equation 4.15 linearly yields the shaded area in Fig. 4.11.*

If the diesel fuel and the gasoline amounts are not in the same unit, two weights, $w_1$ and $w_2$, should be considered, and multiplied to the diesel fuel and the gasoline amounts, respectively. Thus, the relative score of division $l$ ($l = 1, 2, \ldots, 18$) can be measured by Eq. 4.16.

$$\frac{y_{l1}w_1 + y_{l2}w_2}{\max\{y_{i1}w_1 + y_{i2}w_2 : i = 1, 2, \ldots, 18\}}, \quad \text{for } l = 1, 2, \ldots, 18. \tag{4.16}$$

Similar to Sect. 3.6, the factors of each division can be selected as the units of measurement to calculate Eq. 4.16, while the weights, $w_1$ and $w_2$, are unknown. The relative scores and the ranks of divisions depend upon the selected weights, and

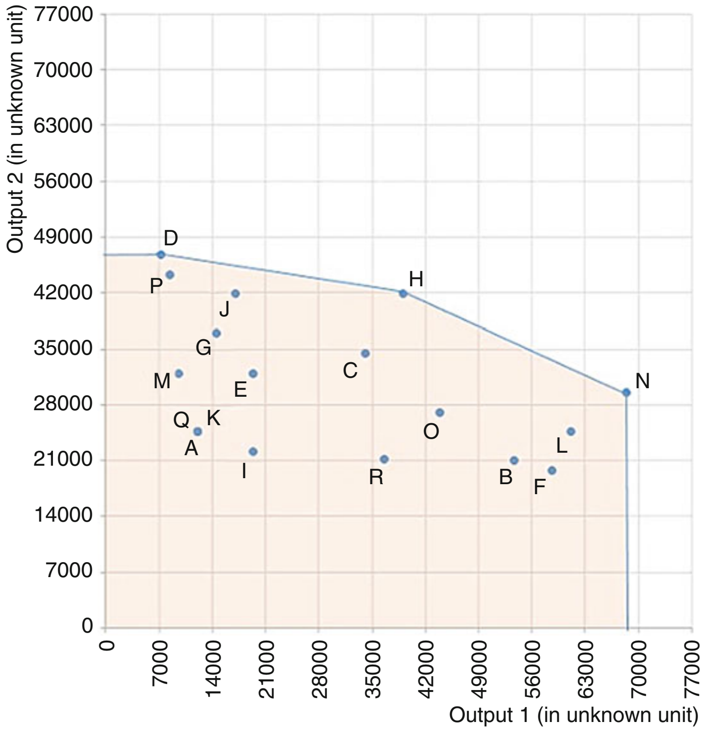



**Fig. 4.11** The convexity and the wholly dominant approaches

geometrically, while the slope of the line, $w_1y + w_2y_2 = r$, that is $-w_1/w_2$, is changed different relative scores and ranks for divisions are calculated. For instance, suppose that the data in Table 3.4 are given and the relationships between the two output factors are unknown. Assume that the convexity approach is applied.

Figure 4.12 illustrates that every line which passes N with the slope less than equal $-0.41\bar{6}$ identifies N as the best performer. Furthermore, for each slope, the ranks of other divisions can be different.

While the slope of the line is less than equal −0.15385, and greater than equal $-0.41\bar{6}$, H is the best performer, as Fig. 4.13 illustrates. As Fig. 4.14 depicts, D is also the best performer, while the slope of the line is greater than equal −0.15385 and less than equal to 0.

As a result, one or at most two of the three divisions, N, H and D, which have done the job right regarding the convexity approach, can be introduced as the best performers.

In addition, the relative scores and the ranks of divisions should be calculated after approximating the weights, $w_1$ and $w_2$.

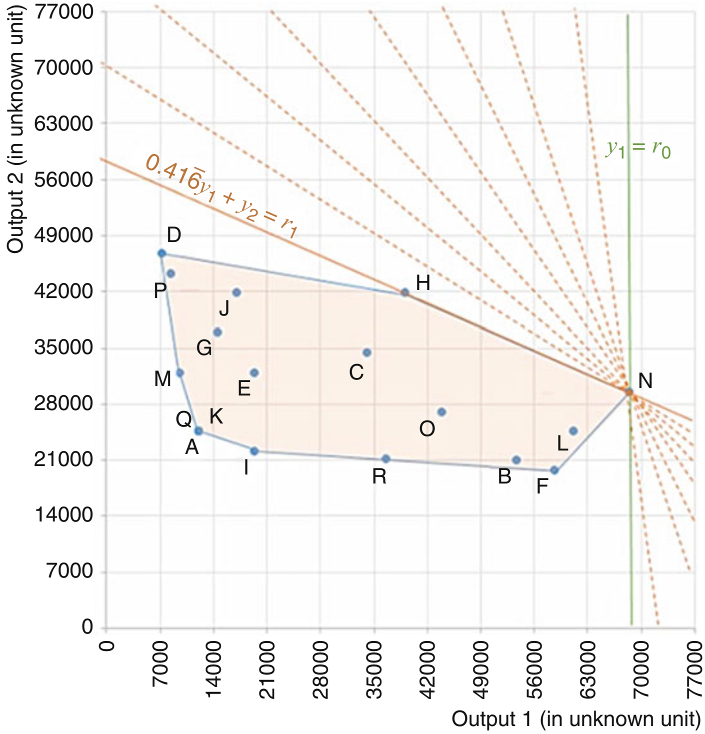



**Fig. 4.12** N is the best performer

Similar to Sect. 4.2, a linear programming can be introduced to measure the maximum value of the relative scores of each division. As illustrated, finding the maximum values of the relative scores of divisions can be useful (1) to find divisions which have done the job right according to an approach, (2) to classify the weights in order to approximate a specified set of weights and after that calculating the relative scores and the ranks of divisions.

In order to measure the relative scores, the weights cannot be different from one division to another, and the maximum value of the relative scores of a division is not a relative score among the other maximum values of the relative scores of divisions.

The maximum value of the relative scores of division $l$, for $l = 1, 2, \ldots, 18$, is given by

$$\max\left\{\frac{y_{l1}w_1 + y_{l2}w_2}{\max\{y_{i1}w_1 + y_{i2}w_2 : i = 1, 2, \ldots, 18\}} : w_1 \geq 0, w_2 \geq 0\right\}. \tag{4.17}$$

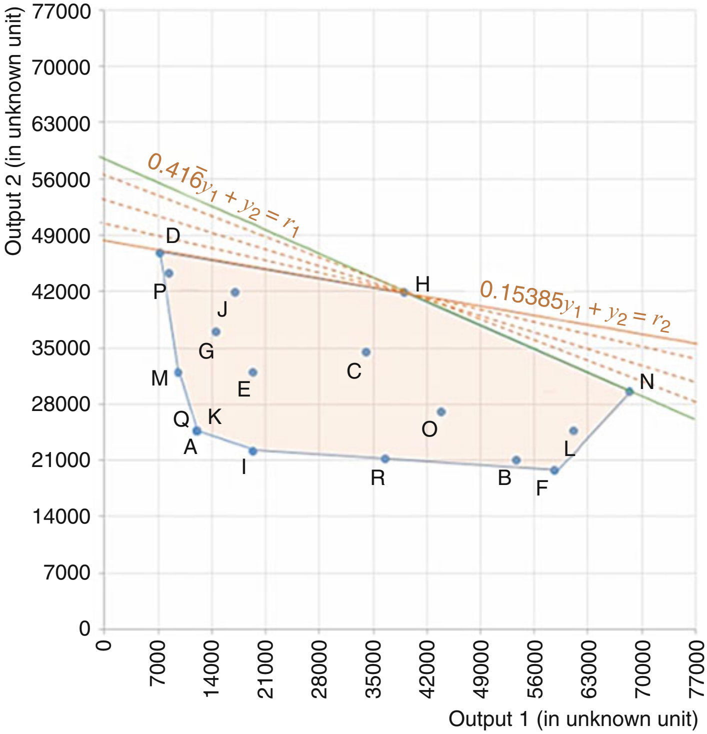



**Fig. 4.13** H is the best performer

Equation 4.17 yields the following LP which can easily be solved by Microsoft Excel Solver software.

$$
\begin{aligned}
&\max y_{l1}w_1 + 2y_{l2}w, \\
&\text{Subject to} \\
&y_{i1}w_1 + y_{i2}w_2 \leq 1, \quad \text{for } i = 1, 2, \ldots, 18, \\
&w_1 \geq 0, \\
&w_2 \geq 0.
\end{aligned}
\tag{4.18}
$$

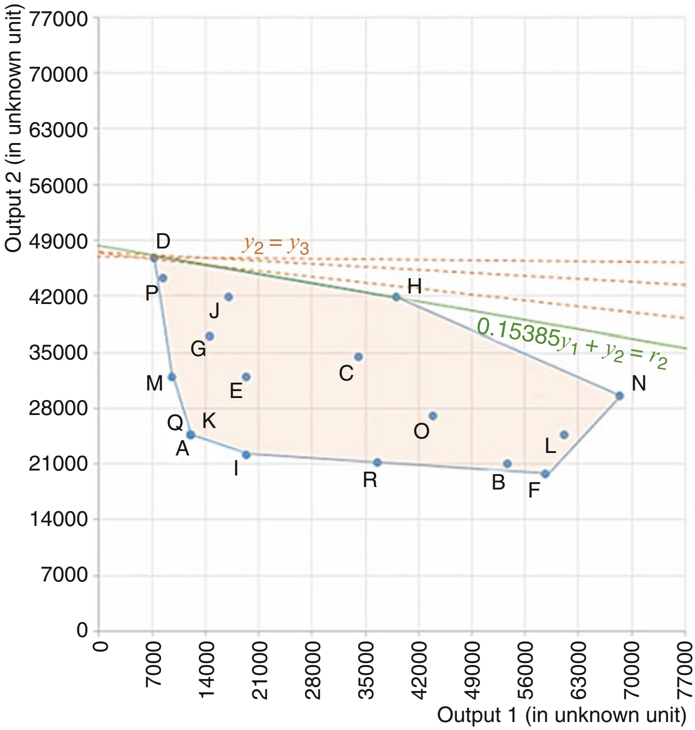



**Fig. 4.14** D is the best performer

Similarly, the minimum values of the relative scores of divisions can be calculated by Eq. 4.19, for $l = 1, 2, \ldots, 18$ and $i = 4$, 8 and 14.

$$1/\max_{i=4,8,14}\{ \max\ x_{i1}w_1 + x_{i2}w_2, \text{ Subject to } y_{l1}w_1 + y_{l2}w_2 = 1, w_1 \geq 0, w_2 \geq 0. \}. \tag{4.19}$$

Table 4.3 represents the results of Eqs. 4.17 and 4.18 as well as the highest and lowest ranks of each division while the maximum and the minimum values of their relative scores are calculated.

For instance, while A gets the maximum value of its relative scores, 0.553360, the highest rank of A is 14, and while A gets the minimum value of its relative scores, 0.178571, the lowest rank for A is 18. Note that, the ranks of divisions should not be measured from the third and the fourth columns in Table 4.3.

**Table 4.3** The maximum and minimum values of the relative scores

| Division | N | Scores | | Rank | | |
|---|---|---|---|---|---|---|
| | | Max | Min | Highest | Lowest | Multiply |
| A | 1 | 0.553360 | 0.178571 | 14 | 18 | 252 |
| B | 2 | 0.785714 | 0.447368 | 4 | 17 | 68 |
| C | 3 | 0.838028 | 0.500000 | 4 | 8 | 32 |
| D | 4 | 1.000000 | 0.107143 | 1 | 18 | 18 |
| E | 5 | 0.731225 | 0.285714 | 7 | 13 | 91 |
| F | 6 | 0.857143 | 0.421053 | 3 | 18 | 54 |
| G | 7 | 0.818182 | 0.214286 | 5 | 13 | 65 |
| H | 8 | 1.000000 | 0.571429 | 1 | 6 | 6 |
| I | 9 | 0.525692 | 0.285714 | 10 | 18 | 180 |
| J | 10 | 0.928854 | 0.250000 | 4 | 11 | 44 |
| K | 11 | 0.553366 | 0.178575 | 12 | 17 | 204 |
| L | 12 | 0.892857 | 0.526316 | 2 | 13 | 26 |
| M | 13 | 0.699605 | 0.142857 | 8 | 18 | 144 |
| N | 14 | 1.000000 | 0.631579 | 1 | 9 | 9 |
| O | 15 | 0.781690 | 0.578947 | 5 | 10 | 50 |
| P | 16 | 0.952569 | 0.125000 | 2 | 17 | 34 |
| Q | 17 | 0.554822 | 0.178214 | 11 | 17 | 187 |
| R | 18 | 0.625880 | 0.450658 | 7 | 17 | 119 |

By multiplying the highest and the lowest ranks of each division, as the last column in Table 4.3 illustrates, the amounts of division H can be suggested as the units of measurement. The relationships between the factors can also be introduced by selecting a slope between $-0.41\bar{6}$ and $-0.15385$, as Fig. 4.13 depicts.

In addition, the sum of the lowest rank of a division in Table 3.11 and the highest rank of that division in Table 4.3 is 19. The sum of the highest rank of a division in Table 3.11 and the lowest rank of that division in Table 4.3 is also 19, except for divisions Q and R which is 18.

## 4.5 Regulating the Amounts of Factors

In Table 3.2, A has the highest rank with the relative score of 1 and has done the job well in comparison with other divisions. As depicted in Fig. 4.15, A partially dominates all divisions and wholly dominates divisions, C, E, G, H, J, K, L, N and O. Therefore, each division is suggested to improve the amounts of input factors according to performance of A.

For instance, H consumed $269443871.70 more than A in diesel fuel and $171,464,282 more than A in gasoline. Thus, as Fig. 4.15 displays, H should plan to decrease the value of the first input factor by 269443871.70 and the value of the second input factor by 171,464,282 to have the same relative score as A.

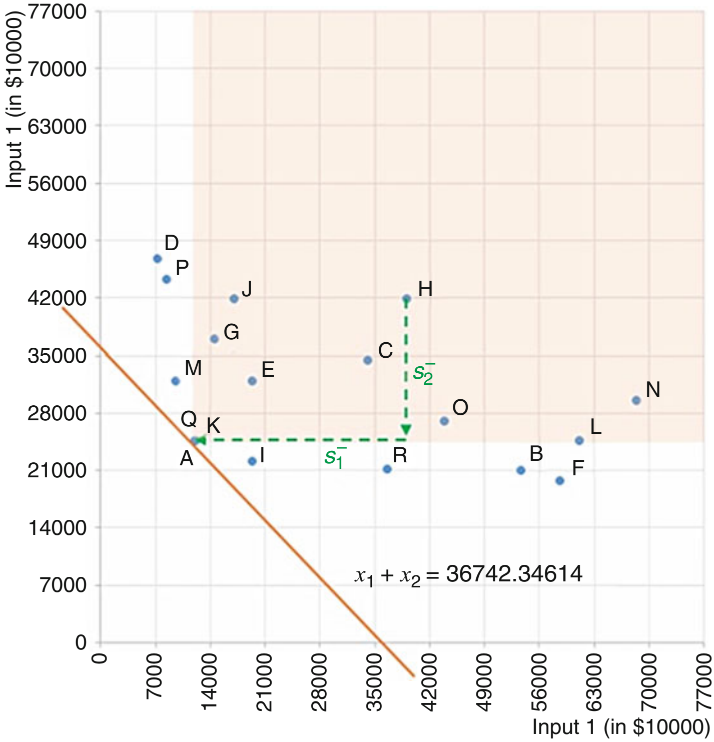



**Fig. 4.15** A is the best performer

In Fig. 4.15, the notations $s_1^-$ and $s_2^-$ illustrate the potential diminishing of the first and the second input factors, respectively.

H can also decrease the amounts of its factors by the radiate approach. As Fig. 4.16 illustrates, the intersection of the lines $x_1 + x_2 = 36742.34614$ and $1.0625x_1 - x_2 = 0$, which is (17814.470850, 18927.875290), can also be suggested to H. In other words, comparing H with A is the same as comparing H with H$'$ (or any other points with positive coordinates on the line, $x_1 + x_2 = 36742.34614$). H$'$ might not be practical, whereas A is practical and H can at least improve its input factors according to Fig. 4.16. In other words, the radiate approach may be useful to compare H with A, (while the line $x_1 + x_2 = 36742.34614$ is known), but it may not be useful to benchmark H toward A through the feasible area.

The same illustration can be expressed for divisions B, D, F, I, M, P, Q and R which are partially dominated by A. For instance, F can be suggested to F$'$, F$''$ and A (or any other points with positive coordinates on the line, $x_1 + x_2 = 36742.34614$).

As Fig. 4.17 illustrates, F can decrease the amount of diesel fuel to reach the line $x_1 + x_2 = 36742.34614$ at F$'$ with the coordinates (17146.4282, 19595.917940), which is not practical according to Figs. 3.5 and 3.6. F may also decrease the values of input factors by using the radiate approach to reach the location of F$''$

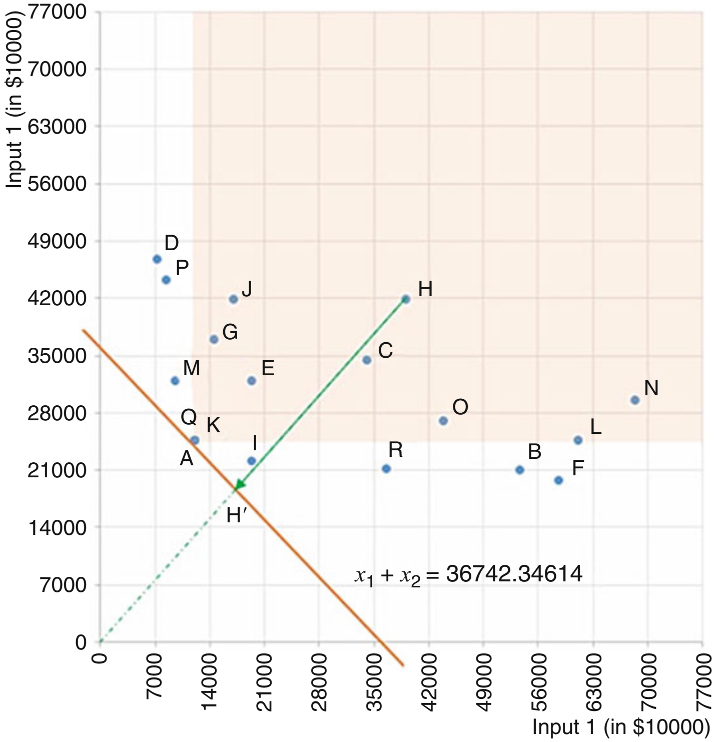



**Fig. 4.16** A is the best performer

(27556.759610, 9185.586534), which might not be practical according to Figs. 3.5 and 3.6. Nonetheless, F can reach to the location of A, by increasing the value of gasoline, as the red arrow in Fig. 4.17 displays, and then decreasing the amount of diesel fuel, as the green arrow in Fig. 4.17 displays. In other words, F should seriously decrease the amounts of diesel fuel; even if this plan requires increasing the amounts of gasoline in order to reach the location of A (which is certainly feasible).

If the wholly dominant or the convexity approaches are applied, the frontiers do not let F to reach the locations of $F'$ or $F''$ or any other points on the line, $x_1 + x_2 = 36742.34614$, so F has one chance only to reach the location of A. The same illustration can also be explained while the line $y_1 + y_2 = r$ is known. From such phenomenon the following types should be added to Types 1–4 in Sect. 1.2 in Chap. 1, in order to increase the values of Eqs. 3.2 and 4.10.

**Type 5** The value of a linear combination of input factors can be decreased by (1) decreasing the values of input factors or (2) increasing a small value of an input factor and decreasing a large value of another input factor.

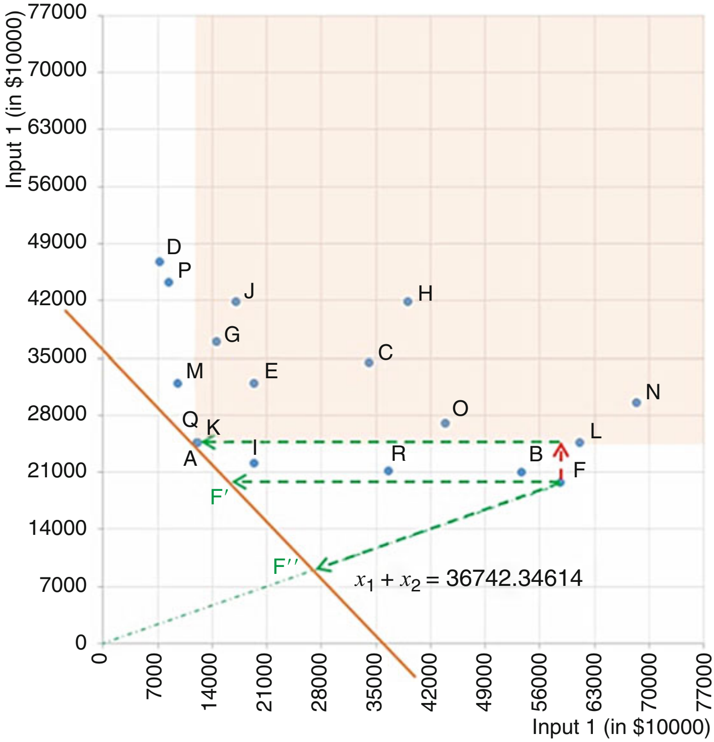



**Fig. 4.17** A is the best performer

**Type 6** The value of a linear combination of output factors can be increased by (1) increasing the values of output factors or (2) decreasing a small value of an output factor and increasing a large value of another output factor.

When the number of input and output factors is increased, Types 5 and 6 are very essential to regulate the values of factors according to a feasible area. Without considering these two types the measurement is not dependable for the concept of doing the job well. This topic is extended in Chaps. 3 and 4.

Note that, one may try to guide divisions to reach the frontiers which are approximated by one of the approaches in Sects. 2.2 and 3.4, in order to result that every division does the job right (regarding to the selected approach). At the same time, this harmful aim can mislead divisions, (1) to reach the points on the frontier which have not done the job well in comparison with most of other divisions; (2) the divisions which lie on the frontier are not suggested to regulate the amounts of their input factors in order to do the job well; (3) the problem to find the best performers and to rank divisions are not solved, which means squandering the study; (4) and the set of homogenous divisions can wrongly be recommended to be non-homogenous by considering different relationships between their factors, regardless of the practicability of such detrimental assumption, (see Sect. 4.3 as well).

## 4.6 Conclusion

In this chapter, the optimal values of the relative scores of divisions in the Petroleum example in Chap. 3 are measured, in order to outline all possible scenarios to introduce a relationship between the factors with the least risk. Another view of the Petroleum example is proposed while each division has two output factors. The similar discussions are illustrated to measure the relative scores of each division in which each division has two output factors. A philosophical discussion is presented to explain that the relative scores of divisions can be estimated if the relationship between the factors is firstly estimated. In other words, while the relationships between the factors are known in a set of homogenous divisions, the relative scores of divisions can logically be calculated. Since in real-life applications the relationships between the factors are usually unknown, an approximation of the weights/prices should be provided to measure the relative scores of divisions. As explained in Chap. 3, the wholly dominant approach, for instance, can introduce the divisions which have done the job right; but it cannot introduce the divisions which have done the job well. The wholly dominant approach should not blindly be used to rank divisions or regulate the values of each factor. The concept of partially dominant is essential to find the relative scores and this concept requires some information about the weights/prices of the factors.

## 4.7 Exercise

4.1 Prove that Model 4.8 requires to be solved $18 \times 7$ times only.

4.2 Solve the following equations for data in Table 3.1 and describe the results.

$$\begin{aligned} &\min \sum_{i=1}^{18} (w_1 x_{i1} + w_2 x_{i2}),\\ &\text{Subject to}\\ &w_1 x_{i1} + w_2 x_{i2} \geq 1,\\ &w_1 \geq 0,\\ &w_2 \geq 0. \end{aligned} \tag{4.20}$$

$$\begin{aligned} &\max \sum_{i=1}^{18} (w_1 y_{i1} + w_2 y_{i2}),\\ &\text{Subject to}\\ &w_1 y_{i1} + w_2 y_{i2} \leq 1,\\ &w_1 \geq 0,\\ &w_2 \geq 0. \end{aligned} \tag{4.21}$$

4.3 Find three reasons to select H as the best performer in Sect. 4.4.

4.4 Write a Visual Basic procedure to find the highest (lowest) rank of each division in Table 3.4.

4.5 Suppose that the EPA does not qualify a division which consumed more than 42,866 ($10,000) amounts of diesel fuel and 36,130 ($10,000) amounts of gasoline.

 4.5.1 Find divisions which have done the well job.

 4.5.2 Find divisions which have done the useful job, by the solution of Eq. 4.20, and describe the results.

# Chapter 5
# The Airport Example

## 5.1 Introduction

In this chapter, an example of eight Persian international airports is proposed to illustrate how to adjust the introduced concepts in the previous chapters when the number of input factors and the number of output factors are more than two. The mathematical methods, the linear programming models, and the computer programming are developed with detailed illustrations. Several different approaches are discussed and the strengths and shortcomings of each approach is gradually represented. At the end of this chapter, readers are prepared to generalize the concepts of doing the job right/well for a set of homogenous firms with multiple input factors and multiple output factors.

## 5.2 The Airport Example

*A researcher wants to compare eight homogenous airports, labeled A-H, according to seven factors, the area of airport (Hectare), the area of apron (Square meter), the area of terminal (Square meter), the area of runway (Square meter), the number of operating flights, the number of passengers' movements, the amount of air cargo (Metric ton), as* Table 5.1 *represents. Which airport does the job well?*

The first four factors, the areas of airport, apron, terminal and runway, are input factors, because they illustrate the infrastructure of airports and lesser values of these factors have worth. The last three factors, the numbers of operating flights and passengers' movements and the amounts of air cargo, are output factors, because they represent the business and production of airports and greater values of these factors have worth. From the discussion in the previous chapters, an airport which



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**Table 5.1** The data of 8 homogenous airports

| N | Airport | Area | Apron | Terminal | Runway | Flights | Passengers | Cargo |
|---|---|---|---|---|---|---|---|---|
| 1 | A | 1200 | 304,182 | 45,600 | 353,610 | 30,707 | 4,030,859 | 74,184 |
| 2 | B | 503 | 213,729 | 38,778 | 348,120 | 46,875 | 4,783,120 | 19,050 |
| 3 | C | 800 | 41,003 | 11,800 | 269,955 | 15,608 | 1,039,967 | 1587 |
| 4 | D | 1041 | 112,464 | 21,050 | 395,730 | 39,871 | 1,744,524 | 4919 |
| 5 | E | 1002 | 30,000 | 8000 | 192,330 | 4887 | 427,974 | 1574 |
| 6 | F | 478 | 63,000 | 23,000 | 389,115 | 41,088 | 2,165,572 | 5414 |
| 7 | G | 481 | 47,210 | 9300 | 268,995 | 19,010 | 971,313 | 3826 |
| 8 | H | 1346 | 503,274 | 76,370 | 421,305 | 129,153 | 11,709,741 | 39,556 |

uses lesser amounts of input factors and greater amounts of output factors has done the job well in comparison with another airport.

In order to provide a value for each airport to represent the used input factors, a linear combination of input factors can be introduced as follows:

$$x_{i1}w_1^- + x_{i2}w_2^- + x_{i3}w_3^- + x_{i4}w_4^- \tag{5.1}$$

In Eq. 5.1, $x_{ij}$ displays the value of $j^{\text{th}}$ input factor of the airport number $i$, and $w_j^-$ displays the weight/worth/price of the $j^{\text{th}}$ input factor to have a valid summation between the input factors, for $j = 1, 2, 3, 4$ and $i = 1, 2, 3, \ldots, 8$. Note that, the amount of $w_j^-$ should not be changed from one airport to another, unless the weights are known, and in such situation, the weights should be multiplied to data to illustrate that one unit of each factor from one airport has the same meaning to one unit of the corresponded factor from another airport. Likewise, Eq. 5.2 provides a value to represent the produced output factors for each airport.

$$y_{i1}w_1^+ + y_{i2}w_2^+ + y_{i3}w_3^+ \tag{5.2}$$

In Eq. 5.2, $y_{ik}$ displays the value of $k^{\text{th}}$ output factor of the airport number $i$, and $w_k^+$ displays the weight/worth/price of the $k^{\text{th}}$ output factor to have a valid summation in Eq. 5.2, for $k = 1, 2, 3$ and $i = 1, 2, 3, \ldots, 8$.

Similar to Chapter 1, by dividing Eq. 5.2 over Eq. 5.1, the following score can be provided to represent the discrimination between the airports.

$$\frac{y_{i1}w_1^+ + y_{i2}w_2^+ + y_{i3}w_3^+}{x_{i1}w_1^- + x_{i2}w_2^- + x_{i3}w_3^- + x_{i4}w_4^-} \tag{5.3}$$

When the value of Eq. 5.2, (that is, the numerator of Eq. 5.3), is increased or while the value of Eq. 5.1, (that is, the denominator of Eq. 5.3), is decreased, the value of Eq. 5.3 increases. The relative score of the airport number $l$ can be calculated by Eq. 5.4, for $l = 1, 2, 3, \ldots, 8$.

**Table 5.2** The scores of airports according to the known weights

| N | Airport | Score | Relative score | Rank |
|---|---|---|---|---|
| 1 | A | 0.9323747 | 0.4608393 | 5 |
| 2 | B | 1.2749310 | 0.6301526 | 2 |
| 3 | C | 0.6660562 | 0.3292076 | 7 |
| 4 | D | 0.9346243 | 0.4619512 | 4 |
| 5 | E | 0.3374253 | 0.1667772 | 8 |
| 6 | F | 1.0111172 | 0.4997589 | 3 |
| 7 | G | 0.7926462 | 0.3917765 | 6 |
| 8 | H | 2.0232100 | 1.0000000 | 1 |

$$\frac{\dfrac{y_{l1}w_1^+ + y_{l2}w_2^+ + y_{l3}w_3^+}{x_{l1}w_1^- + x_{l2}w_2^- + x_{l3}w_3^- + x_{l4}w_4^-}}{\max\left\{\dfrac{y_{i1}w_1^+ + y_{i2}w_2^+ + y_{i3}w_3^+}{x_{i1}w_1^- + x_{i2}w_2^- + x_{i3}w_3^- + x_{i4}w_4^-} : i = 1, 2, \ldots, 8.\right\}} \tag{5.4}$$

If the relationship between input factors and the relationship between output factors, (that is, the weights $w_j^-$ and $w_k^+$), are available, for $j = 1, 2, 3, 4$ and $k = 1, 2, 3$, the researcher can easily measure the relative score of each airport, and determine those that have done the job well.

For instance, suppose that $w_1^- = 10$, $w_2^- = 1$, $w_3^- = 100$, $w_4^- = 10$, $w_1^+ = 100$, $w_2^+ = 1$, and $w_3^+ = 10$. Table 5.2 illustrates the score and the relative score of each airport with seven decimal digits as well as the rank of airports according to the selected weights.

The relationships between the inputs (outputs) factors are not easily measured, and are almost always unknown. The geometric space of the locations of the airports has also 7 dimensions, whereas the researcher can sketch three-dimensional space at most. So, can the researcher deduce more information from data in Table 5.1 while the relationships between the factors are unknown?

## 5.3 The Wholly Dominant Approach

The researcher can mathematically introduce the practical points similar to the previous chapters. This aim is useful to find the airports which have done the job right, according to the wholly dominant approach. In this example, if the researcher applies this approach, none of the airports in Table 5.1 wholly dominate other airports. For instance, an airport can wholly dominate A, if the values of its factors satisfy the following inequalities:

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | N | Airport | Area | Apron | Terminal | Runway | Flights | Passengers | Cargo | |
| 2 | 1 | A | 1,200 | 304,182 | 45,600 | 353,610 | 30,707 | 4,030,859 | 74,184 | |
| 3 | 2 | B | 503 | 213,729 | 38,778 | 348,120 | 46,875 | 4,783,120 | 19,050 | |
| 4 | 3 | C | 800 | 41,003 | 11,800 | 269,955 | 15,608 | 1,039,967 | 1,587 | |
| 5 | 4 | D | 1,041 | 112,464 | 21,050 | 395,730 | 39,871 | 1,744,524 | 4,919 | |
| 6 | 5 | E | 1,002 | 30,000 | 8,000 | 192,330 | 4,887 | 427,974 | 1,574 | |
| 7 | 6 | F | 478 | 63,000 | 23,000 | 389,115 | 41,088 | 2,165,572 | 5,414 | |
| 8 | 7 | G | 481 | 47,210 | 9,300 | 268,995 | 19,010 | 971,313 | 3,826 | |
| 9 | 8 | H | 1,346 | 503,274 | 76,370 | 421,305 | 129,153 | 11,709,741 | 39,556 | |
| 10 | | | | | | | | | | |
| 11 | | Index1 | 1 | | Index2 | 1 | | | | |
| 12 | | | | | | | | | | Sum |
| 13 | | Conditions | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 7 |

**Fig. 5.1** The wholly dominant on the Excel sheet

$$\begin{aligned} &x_{i1} \leq 1200\; x_{i2} \leq 304,182\; x_{i3} \leq 45,600\; x_{i4} \leq 353,610 \\ &y_{i1} \geq 30,707\; y_{i2} \geq 4,030,859\; y_{i3} \geq 74,184 \end{aligned} \tag{5.5}$$

It is obvious that A wholly dominates itself. B does not wholly dominate A, because the data of B satisfies the first six inequalities in Eq. 5.5, but the value of third output factor of B does not satisfy the last inequality. C does not wholly dominate A, because the values of C's input factors satisfy the first four inequalities, whereas the values of C's output factors do not satisfy the last three inequalities in Eq. 5.5, and so on for other airports. From Eq. 5.5, the researcher can provide the following steps, using Microsoft Excel, to measure whether one airport in Table 5.1 wholly dominates another.

1. Copy the 9 columns of Table 5.1 on an Excel sheet into cells A1:I9, as Fig. 5.1 depicts.
2. Label B11 as 'Index1', E11 as 'Index2', B13 as 'Conditions', and J12 as 'Sum'.
3. Assign number 1 to C11 and F11.
4. Assign the following command (without quotations mark) into C13,

   '=IF(Index(C2:C9,$C11) > =Index(C2:C9,$F11), 1,0)'.

5. Copy C13 (by Ctrl + C), and paste it (by Ctrl + V) to D13, E13 and F13.
6. Assign the following command into G13,

   '=IF(Index(G2:G9,$C11) < =Index(G2:G9,$F11), 1, 0)'.

7. Copy G13 and then paste it to H13 and I13.
8. Assign the command '=Sum(C13:I13)' into J13.
9. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.

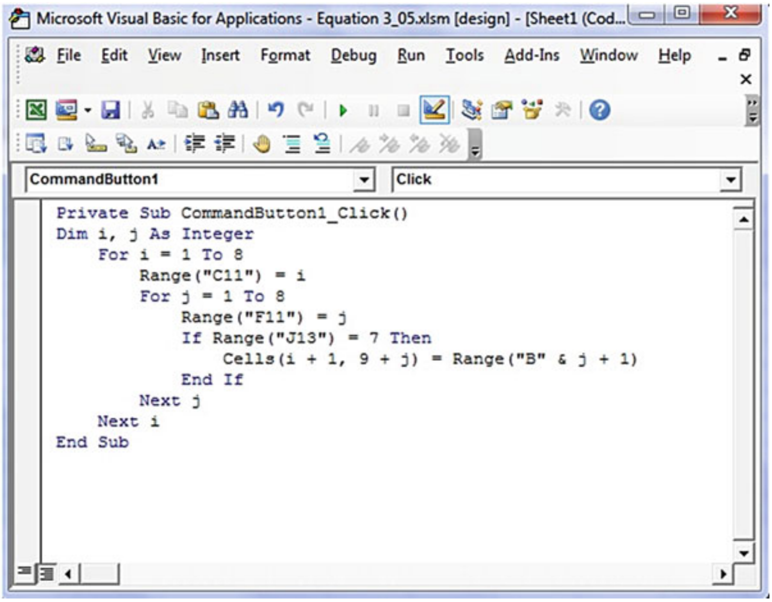



**Fig. 5.2** The Macro to find wholly dominated DMUs

10. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet (Fig. 5.2).
11. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.
12. Inside of the 'Microsoft Visual Basic for Applications' window, write the following commands between 'Sub Button1_Click ()' and 'End Sub'.

```
Dim i, j As Integer
For i = 1 To 8
   Range("C11") = i
   For j = 1 To 8
     Range("F11") = j
     If Range("J13") = 7 Then
       Cells(i + 1, 9 + j) = Range("B" & j + 1)
     End If
   Next j
Next i
```

13. Close the 'Microsoft Visual Basic for Applications' window.
14. Click on the small rectangle which was automatically made on the Excel sheet and created by step 10.
15. The results are represented to cells J2:Q9 and determine that every airport wholly dominates itself only (Fig. 5.3).

Similar to Eqs. 1.14 and 2.6, the following equation represents the practical points by the wholly dominant approach.

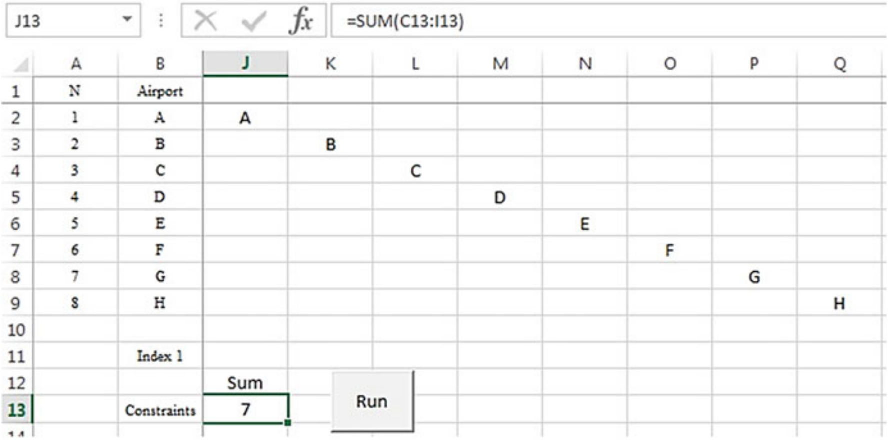



**Fig. 5.3** The results of the wholly dominant test

$$\bigcup_{i=1}^{8}\left\{(x_1', x_2', x_3', x_4', y_1', y_2', y_3') : \begin{array}{ll} x_{ij} \le x_j', & \text{for } j = 1, 2, 3, 4 \\ y_{ik} \ge y_k', & \text{for } k = 1, 2, 3 \end{array}\right\}. \tag{5.6}$$

In Eq. 5.6, the point with the coordinates $(x_1', x_2', x_3', x_4', y_1', y_2', y_3')$ is dominated by $(x_{i1}, x_{i2}, x_{i3}, x_{i4}, y_{i1}, y_{i2}, y_{i3})$ for some $i = 1, 2, \ldots, 8$, (that is, the index $i$, can be 1 (or 2, 3, ..., 8), or 1 and 2 (or 1 and 3, ..., 1 and 8), and so on for all of the 255 different cases).

There are 7 inequalities in Eq. 5.6, which can also be written as equalities (for $i = 1, 2, \ldots, 8$) given by:

$$x_{ij} + s_j^- = x_j' \quad \text{for} \quad j = 1, 2, 3, 4 \quad \& \quad y_{ik} - s_k^+ = y_k' \quad \text{for} \quad k = 1, 2, 3. \tag{5.7}$$

The variables $s_j^-$ and $s_k^+$ are non-negative real numbers, and are known as *slacks*, because they characterize the lack of performance. For instance, while A wholly dominates a point with the coordinates $(x_1', x_2', x_3', x_4', y_1', y_2', y_3')$, this can be interpreted by $(s_1^-, s_2^-, s_3^-, s_4^-, s_1^+, s_2^+, s_3^+)$, as the lack of performance of the point in comparison with the performance of A.

If one of the airports in Table 5.1 wholly dominates another, at least one of the slacks $(s_1^-, s_2^-, s_3^-, s_4^-, s_1^+, s_2^+, s_3^+)$ should be positive in Eq. 5.7. Since none of the airports in Table 5.1 wholly dominate others, there are no sets of positive slacks which can satisfy Eq. 5.7 for two different airports in Table 5.1. This statement can be proved by 56 times solving Eq. 5.7.

From another point of view, suppose that the point $(x_1', x_2', x_3', x_4', y_1', y_2', y_3')$ is replaced with the airport number $l$ coordinates ($l = 1, 2, \ldots, 8$) in Eq. 5.7, as Eq. 5.8 illustrates, for $i = 1, 2, \ldots, 8$.

$$x_{ij} + s_j^- = x_{lj} \quad \text{for} \quad j = 1, 2, 3, 4 \quad \& \quad y_{ik} - s_k^+ = y_{lk} \quad \text{for} \quad k = 1, 2, 3. \quad (5.8)$$

Equation 5.8 can also be expanded as follows, where $i = 1$:

$$\begin{aligned} x_{11} + s_1^- &= x_{l1}, \\ x_{12} + s_2^- &= x_{l2}, \\ x_{13} + s_3^- &= x_{l3}, \\ x_{14} + s_4^- &= x_{l4}, \\ y_{11} - s_1^+ &= y_{l1}, \\ y_{12} - s_2^+ &= y_{l2}, \\ y_{13} - s_3^+ &= y_{l3}. \end{aligned} \quad (5.9)$$

Similarly, the Eq. 5.8 can be expanded for $i = 2, 3, \ldots$, and 8. All these different situations can be combined as Eq. 5.10, by introducing the *binary multipliers* $\lambda_1, \lambda_2, \ldots, \lambda_8$, where $\lambda_i \in \{0, 1\}$, for $i = 1, 2, \ldots, 8$, and $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1$.

$$\begin{aligned} x_{11}\lambda_1 + x_{21}\lambda_2 + x_{31}\lambda_3 + x_{41}\lambda_4 + x_{51}\lambda_5 + x_{61}\lambda_6 + x_{71}\lambda_7 + x_{81}\lambda_8 + s_1^- &= x_{l1}, \\ x_{12}\lambda_1 + x_{22}\lambda_2 + x_{32}\lambda_3 + x_{42}\lambda_4 + x_{52}\lambda_5 + x_{62}\lambda_6 + x_{72}\lambda_7 + x_{82}\lambda_8 + s_2^- &= x_{l2}, \\ x_{13}\lambda_1 + x_{23}\lambda_2 + x_{33}\lambda_3 + x_{43}\lambda_4 + x_{53}\lambda_5 + x_{63}\lambda_6 + x_{73}\lambda_7 + x_{83}\lambda_8 + s_3^- &= x_{l3}, \\ x_{14}\lambda_1 + x_{24}\lambda_2 + x_{34}\lambda_3 + x_{44}\lambda_4 + x_{54}\lambda_5 + x_{64}\lambda_6 + x_{74}\lambda_7 + x_{84}\lambda_8 + s_4^- &= x_{l4}, \\ y_{11}\lambda_1 + y_{21}\lambda_2 + y_{31}\lambda_3 + y_{41}\lambda_4 + y_{51}\lambda_5 + y_{61}\lambda_6 + y_{71}\lambda_7 + y_{81}\lambda_8 - s_1^+ &= y_{l1}, \\ y_{12}\lambda_1 + y_{22}\lambda_2 + y_{32}\lambda_3 + y_{42}\lambda_4 + y_{52}\lambda_5 + y_{62}\lambda_6 + y_{72}\lambda_7 + y_{82}\lambda_8 - s_2^+ &= y_{l2}, \\ y_{13}\lambda_1 + y_{23}\lambda_2 + y_{33}\lambda_3 + y_{43}\lambda_4 + y_{53}\lambda_5 + y_{63}\lambda_6 + y_{73}\lambda_7 + y_{83}\lambda_8 - s_3^+ &= y_{l3}. \end{aligned} \quad (5.10)$$

If $\lambda_1 = 1$, then $\lambda_i = 0$ for $i = 2, 3, \ldots 8$, and Eq. 5.10 is the same as Eq. 5.9, and so on for other choices. Eq. 5.10 can also be compressed by using the notation '$\sum$' as follows:

$$\begin{aligned} \textstyle\sum_{i=1}^{8} x_{i1}\lambda_i + s_1^- &= x_{l1}, \\ \textstyle\sum_{i=1}^{8} x_{i2}\lambda_i + s_2^- &= x_{l2}, \\ \textstyle\sum_{i=1}^{8} x_{i3}\lambda_i + s_3^- &= x_{l3}, \\ \textstyle\sum_{i=1}^{8} x_{i4}\lambda_i + s_4^- &= x_{l4}, \\ \textstyle\sum_{i=1}^{8} y_{i1}\lambda_i - s_1^+ &= y_{l1}, \\ \textstyle\sum_{i=1}^{8} y_{i2}\lambda_i - s_2^+ &= y_{l2}, \\ \textstyle\sum_{i=1}^{8} y_{i3}\lambda_i - s_3^+ &= y_{l3}. \end{aligned} \quad (5.11)$$

In order to measure the optimal values of the slacks, Eq. 5.11 should be solved 8 times. In addition, the following model can measure the optimal values of the slacks for the airport number $l\,(l = 1, 2, \ldots, 8)$.

$$
\begin{aligned}
&\max \quad s_{l1}^- + s_{l2}^- + s_{l3}^- + s_{l4}^- + s_{l1}^+ + s_{l2}^+ + s_{l3}^+,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{8} x_{ij}\lambda_i + s_j^- = x_{lj}, \quad \text{for } j = 1, 2, 3, 4,\\
&\textstyle\sum_{i=1}^{8} y_{ik}\lambda_i - s_k^+ = y_{lk}, \quad \text{for } k = 1, 2, 3,\\
&\textstyle\sum_{i=1}^{8} \lambda_i = 1,\\
&\lambda_i \in \{0, 1\}, \quad \text{for } i = 1, 2, \ldots, 8,\\
&s_j^- \geq 0, \quad \text{for } j = 1, 2, 3, 4,\\
&s_k^+ \geq 0, \quad \text{for } k = 1, 2, 3.
\end{aligned}
\tag{5.12}
$$

Equation 5.12 has 23 constraints and 15 variables, and should be solved 8 times. The first two sets of constraints are the same as Eq. 5.11, and the other constraints are the conditions of binary *lambdas* and non-negative slacks. The following instructions illustrate how to solve Eq. 5.12 with the Microsoft Excel Solver 2013 software.

1. Copy the 9 columns of Table 5.1 on an Excel sheet into cells A1:I9, as Fig. 5.1 depicts.
2. Label B11 as 'Index', B13 as 'Left', B14 as 'Slacks', B15 as 'Right' and J1 as 'Lambdas'.
3. Assign number 1 to C11.
4. Assign the command '=Sumproduct(C2:C9,$J2:$J9) + C14' into C13.
5. Copy C13, and then, paste it into D13, E13 and F13.
6. Assign the command '=Sumproduct(G2:G9,$J2:$J9)-G14' into G13.
7. Copy G13, and then, paste it into the cells H13 and I13.
8. Assign the command '=Index(C2:C9,$C11)' into C15.
9. Copy C15, and then, paste it into cells D15, E15, F15, G15, H15 and I15.
10. Label J13 as 'Objective'.
11. Assign the command '=Sum(C14:I14)' into J14.
12. Assign the command '=Sum(J2:J9)' into J10.
13. Open 'Solver Parameters' window, from 'DATA' in toolbar menu.
14. Assign 'J14' into 'Set Objective' and choose 'Max'.
15. Assign 'C14:I14, J2:J9' into 'By Changing Variable Cells'.
16. Click on 'Add' and assign 'C13:I13' into 'Cell Reference', then select '=', and assign 'C15:I15' into 'Constraint' (Fig. 5.4).
17. Click on 'Add' and assign 'J2:J9' into 'Cell Reference', then select 'bin'.
18. Click on 'Add' and assign 'J10' into 'Cell Reference', then select '=', and assign '1' into 'Constraint'. Then click on 'OK' (Fig. 5.5).
19. Tick 'Make Unconstrained Variables Non-Negative'.
20. Choose 'Simplex LP' from 'Select a Solving Method'.


**Fig. 5.4** The change constraint in Excel Solver

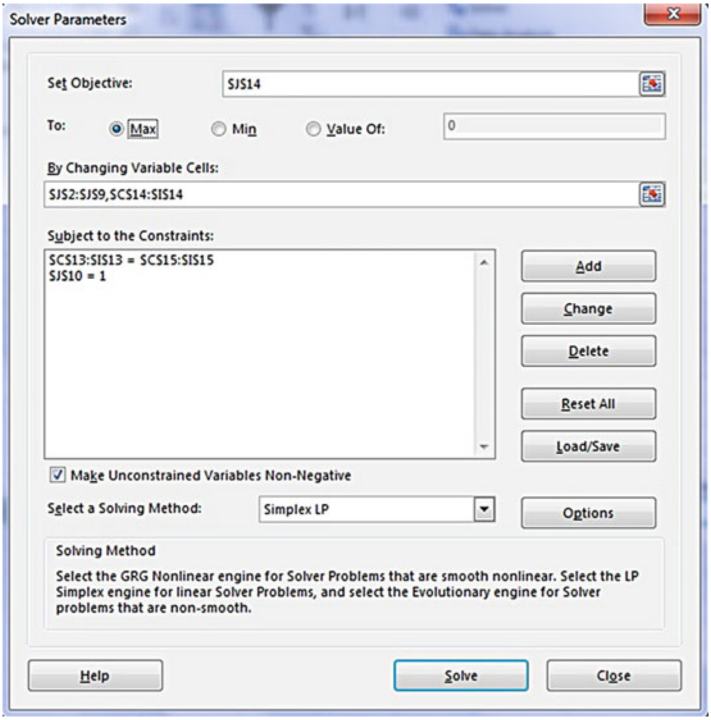



**Fig. 5.5** Setting Excel Solver to solve Eq. 5.12

21. Click on ‘Solve’.
22. Label K1 as ‘Objective’ and L1:S1 by A-H, respectively.
23. From ‘Developer’ in the toolbar menu, click on the ‘Insert’ icon to open the ‘Form Control’ window.
24. Click on the first icon, ‘Button (Form Control)’, and then click on a place on the Excel sheet.
25. In the opened window with the title ‘Assign Macro’, click on ‘New’. So, the ‘Microsoft Visual Basic for Applications’ window is opened.
26. From the toolbar menu, click on ‘Tools> References...>’ and make sure ‘Solver’ is ticked, and then ‘OK’, as shown in Figs. 5.6 and 5.7.

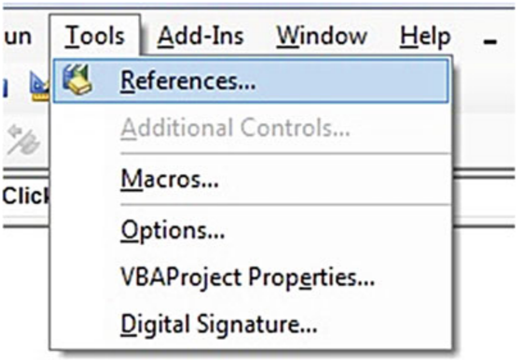



**Fig. 5.6** References in VBA

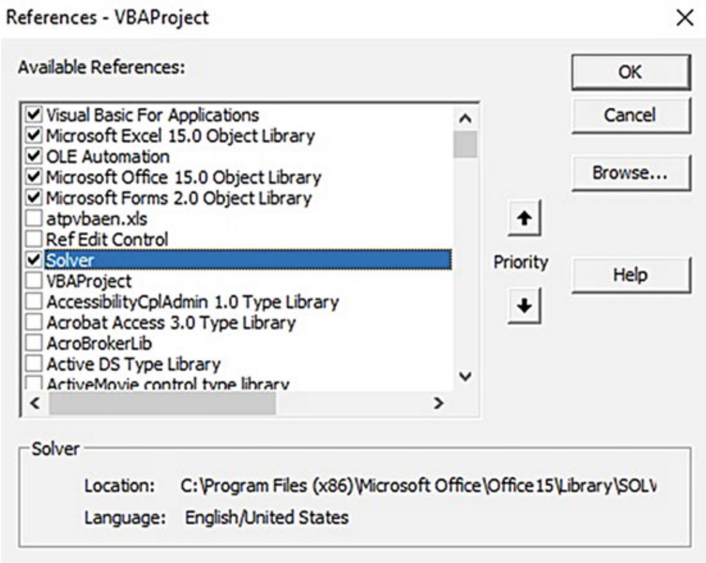



**Fig. 5.7** Selecting Solver in References in VBA

27. Inside of the ‘Microsoft Visual Basic for Applications’ window, write the following commands between ‘Sub Button1_Click ()’ and ‘End Sub’ as Fig. 5.8 depicts.

```
Dim i As Integer
For i = 1 To 8
   Range("C11") = i
   SolverSolve Userfinish:=True
   Range("K" & i + 1) = Range("J14")
   Range("J2:J9").Copy
   Range("L" & i + 1).Select
   Selection.PasteSpecial Transpose:=True
Next i
```

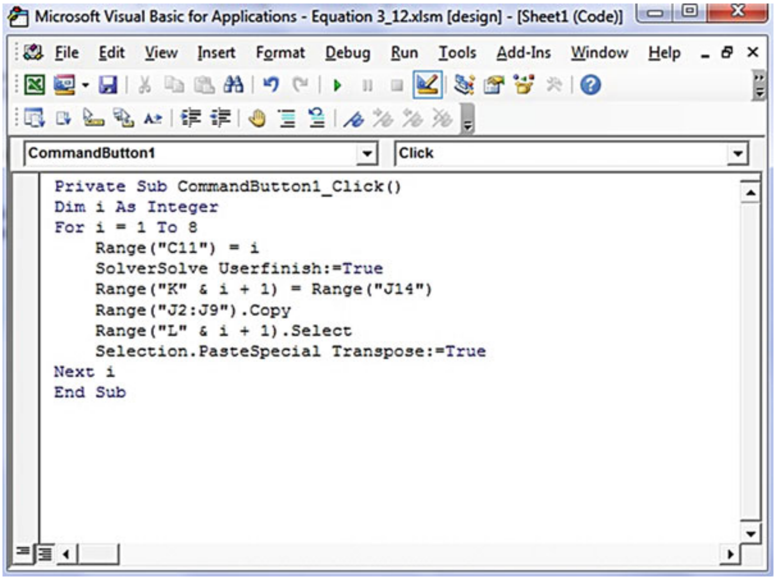



**Fig. 5.8** Setting VBA macro to solve Eq. 5.12

28. Close the 'Microsoft Visual Basic for Applications' window.
29. Click on the small rectangle which was automatically made on the Excel sheet by step 23.
30. The results of solving Eq. 5.12 are represented into cells L2:S9. Column K represents the values of the objective of Eq. 5.12 for each airport, and the cells K2:S9 illustrates the values of Lambdas for each airport. As the results indicate, the optimal values of slacks are 0 for each airport.

In short, none of the airports are wholly dominated by other airports, and all the airports in Table 5.1 are introduced as those that have done the job right, according to the wholly dominant approach.

## 5.4 The Convexity Approach

As explained in Chapters 1 and 2, while the convexity approach is applied for two observations A and B, the points which lie on the line-segment AB are generated. For instance, assume that the coordinates of the airports A and B in Table 5.1 are shown by $(x_{11}, x_{12}, x_{13}, x_{14}, y_{11}, y_{12}, y_{13})$ and $(x_{21}, x_{22}, x_{23}, x_{24}, y_{21}, y_{22}, y_{23})$, respectively. Eq. 5.13 illustrates applying the convexity approach for these two points, where $\lambda_1 + \lambda_2 = 1$, $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$.

$$\lambda_1(x_{11}, x_{12}, x_{13}, x_{14}, y_{11}, y_{12}, y_{13}) + \lambda_2(x_{21}, x_{22}, x_{23}, x_{24}, y_{21}, y_{22}, y_{23}) \tag{5.13}$$

Similar to Eqs. 1.22 and 2.7, the following equation mathematically describes the set of the practical points from data in Table 5.1 while the convexity approach is applied.

$$\left\{ (x'_1, x'_2, x'_3, x'_4, y'_1, y'_2, y'_3) : \begin{array}{l} x'_j = \sum_{i=1}^{8} x_{ij}\lambda_i, \quad \text{for } j = 1,2,3,4 \\ y'_k = \sum_{i=1}^{8} y_{ik}\lambda_i, \quad \text{for } k = 1,2,3 \\ \sum_{i=1}^{8} \lambda_i = 1, \ \ \lambda_i \geq 0, \ \ \text{for } i = 1,2,\ldots,8. \end{array} \right\}. \tag{5.14}$$

If $\lambda_1 = 1$, since $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1$ and $\lambda_i \geq 0$, for $i = 1$, 2, . . ., 8, hence, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 0$, and the linear combinations of the coordinates in Eq. 5.14 display airport A. This is the same while $\lambda_l = 1$, for $l \in \{1, 2, \ldots, 8\}$, which yields $\lambda_1 = 0$, for $i \in \{1, 2, \ldots, 8\} - \{l\}$, and the linear combinations of the coordinates in Eq. 5.14 display the airport number $l$ in Table 5.1.

If $\lambda_1 + \lambda_2 = 1$, where $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, then $\lambda_i = 0$, for $i \in \{3, 4, \ldots, 8\}$, and the linear combinations of the coordinates in Eq. 5.14 illustrate the line-segment AB, (the straight line between the points A and B which connects them in the seven-dimensional space). Similarly, if $\lambda_l + \lambda_{l'} = 1$, where $\lambda_l > 0$, $\lambda_{l'} > 0$, $l \neq l'$ and $l$, $l' \in \{1, 2, \ldots, 8\}$, then $\lambda_i = 0$, for $i \in \{1, 3, 4, \ldots, 8\} - \{l, l'\}$, and the linear combinations of the coordinates in Eq. 5.14 illustrate the line-segment which connects the airports number $l$ and $l'$, (see Eq. 5.13).

Correspondingly, every three airports in Table 5.1 which are non-collinear introduce a triangle disk, (inclusive the circumstance and the inside of the triangle). In other words, if $\lambda_l + \lambda_{l'} + \lambda_{l''} = 1$, where $\lambda_l > 0, \lambda_{l'} > 0, \lambda_{l''} > 0, l \neq l', l \neq l'', l' \neq l''$ and $l, l', l'' \in \{1, 2, \ldots, 8\}$, then $\lambda_i = 0$, for $i \in \{1, 3, 4, \ldots, 8\} - \{l, l', l''\}$, and similar to Fig. 1.30, the linear combinations of the coordinates in Eq. 5.14 represent a triangle disk, where the airports number $l$, $l'$ and $l''$ are the three corners of the triangle.

While the convexity approach is applied for four airports in Table 5.1, (for instance, if $\lambda_i > 0$ for $i = 1, 2, 3, 4$, and $\lambda_i = 0$ for $i = 5, 6, 7, 8$, so $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$), and the airports are not on the same plane, a triangular pyramid (three-dimensional hyper-triangle) is generated. Similarly, while the convexity approach is applied for five airports in Table 5.1 and the airports are not in the same three-dimensional space, a four-dimensional hyper-triangle is generated. Finally, if the convexity approach is applied for eight airports in Table 5.1 and the airports are not in the same six-dimensional space, a seven-dimensional hyper-triangle is generated. Note that, since each airport in Table 5.1 has seven dimensions only, the locations of the airports can, at most, be in a seven-dimensional space. In order to generate a seven-dimensional hyper-triangle, at least eight numbers of the airports should be available.

The convexity approach is useful to check whether there are any of the airports in Table 5.1, which the coordinates of one can be generated by a linear combination of the coordinates of other airports. This approach may decrease the number of airports which are introduced, as those that have done the job right, by applying the wholly dominant approach. Note that, the airports which are introduced as those that have done the job right are the same, either by applying the convexity approach or the combination of the convexity and the wholly dominant approaches. Eq. 5.15 represents the feasible area which is linearly generated by the convexity and the wholly dominant approaches.

$$\left\{ (x_1', x_2', x_3', x_4', y_1', y_2', y_3') : \begin{array}{r} x_j' \geq \sum_{i=1}^{8} x_{ij}\lambda_i, \quad \text{for } j = 1,2,3,4 \\ y_k' \leq \sum_{i=1}^{8} y_{ik}\lambda_i, \quad \text{for } k = 1,2,3 \\ \sum_{i=1}^{8} \lambda_i = 1, \ \ \lambda_i \geq 0, \ \ \text{for } i = 1,2, \ldots, 8. \end{array} \right\}. \tag{5.15}$$

Suppose that the airport number $l$ is selected and the convexity approach is applied. The conditions in Eq. 5.14 yield the following equations, where $\sum_{i=1}^{8} \lambda_i = 1,\ \ \lambda_i \geq 0,\ \ \text{for } i = 1,2, \ldots, 8.$

$$\begin{aligned} \sum_{i=1}^{8} x_{i1}\lambda_i &= x_{l1}, \\ \sum_{i=1}^{8} x_{i2}\lambda_i &= x_{l2}, \\ \sum_{i=1}^{8} x_{i3}\lambda_i &= x_{l3}, \\ \sum_{i=1}^{8} x_{i4}\lambda_i &= x_{l4}, \\ \sum_{i=1}^{8} y_{i1}\lambda_i &= y_{l1}, \\ \sum_{i=1}^{8} y_{i2}\lambda_i &= y_{l2}, \\ \sum_{i=1}^{8} y_{i3}\lambda_i &= y_{l3}. \end{aligned} \tag{5.16}$$

Equation 5.16 has at least one solution, which is, $\lambda_l = 1$ and $\lambda_i = 0$ for $i \in \{1, 2, \ldots, 8\} - \{l\}$. If there is not another solution for Eq. 5.16, this means that the airport number $l$ is at the corner of the feasible area, (which is generated by the convexity approach).

The same illustration can also be explained while the combination of the convexity and the wholly dominant approaches are applied and inequalities in Eq. 5.15 are considered. These inequalities can be written as equalities by introducing the non-negative variables $s_j^-$ and $s_k^-$, for $j = 1, 2, 3, 4$ and $k = 1, 2, 3$, similar to Eq. 5.11. While the optimal values of these slacks are 0 for an airport, this means that the airport lies on the frontier, and has done the job right, according to the convexity and the wholly dominant approaches. In order to find the optimal slacks, similar to Eq. 5.12, the following linear programming can be introduced.

$$\begin{aligned}
&\max \quad \sum_{j=1}^{4} s_j^- + \sum_{k=1}^{3} s_k^+, \\
&\text{Subject to} \\
&\sum_{i=1}^{8} x_{ij}\lambda_i + s_j^- = x_{lj} \quad \text{for } j = 1, 2, 3, 4, \\
&\sum_{i=1}^{8} y_{ik}\lambda_i - s_k^+ = y_{lk}, \quad \text{for } k = 1, 2, 3, \\
&\sum_{i=1}^{8} \lambda_i = 1, \\
&\lambda_i \geq 0, \quad \text{for } i = 1, 2, \ldots, 8, \\
&s_j^- \geq 0, \quad \text{for } j = 1, 2, 3, 4, \\
&s_k^+ \geq 0, \quad \text{for } k = 1, 2, 3.
\end{aligned} \tag{5.17}$$

The only differences between Eqs. 5.12 and 5.17 are the domains of lambdas, (that is, the multipliers in the linear combinations of the coordinates of the airports). While lambdas are binary, that is, $\lambda_i \in \{0, 1\}$, for $i = 1, 2, \ldots, 8$, Eq. 5.12 can introduce the airports which have done the job right by the wholly dominant approach, and where $\lambda_i \geq 0$, for $i = 1, 2, \ldots, 8$, Eq. 5.17 can introduce the airports which have done the job right by both of the convexity and the wholly dominant approaches.

Equation 5.17 is also solved in a similar way to Eq. 5.12. The optimal slacks for each airport in Table 5.1 are 0, (see Table 5.3), which illustrates that all the airports lie on the frontier, (which is introduced by the convexity approach).

Similar to Sect. 2.2.4.3, if the constraint, $\sum_{i=1}^{8} \lambda_i = 1$, is removed in Eq. 5.17, the feasible area is linearly generated by simultaneously applying the radiate, the convexity and the wholly dominant approaches, as Eq. 5.18 illustrates.

$$\begin{aligned}
&\max \quad \sum_{j=1}^{4} s_j^- + \sum_{k=1}^{3} s_k^+, \\
&\text{Subject to} \\
&\sum_{i=1}^{8} x_{ij}\lambda_i + s_j^- = x_{lj} \quad \text{for } j = 1, 2, 3, 4, \\
&\sum_{i=1}^{8} y_{ik}\lambda_i - s_k^+ = y_{lk}, \quad \text{for } k = 1, 2, 3, \\
&\lambda_i \geq 0, \quad \text{for } i = 1, 2, \ldots, 8, \\
&s_j^- \geq 0, \quad \text{for } j = 1, 2, 3, 4, \\
&s_k^+ \geq 0, \quad \text{for } k = 1, 2, 3.
\end{aligned} \tag{5.18}$$

**Table 5.3** The optimal objective in Eq. 5.17 for each airport

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Optimal objective | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Table 5.4** The optimal objective in Eq. 5.18 for each airport

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Optimal objective | 0 | 0 | 308138.75 | 0 | 541415.79 | 0 | 0 | 0 |

The results of solving Eq. 5.18 are represented in Table 5.4. Airports C and E are not introduced as those that have done the job right by solving Eq. 5.18, because the optimal slacks for these two airports are not 0. As Eq. 5.19 illustrates, C (3rd airport) is dominated by a linear combination of F (6th airport) and H (8th airport), regarding the selected approach, where $\lambda_6^* = 0.415026229, \lambda_8^* = 0.029519402$ and $\lambda_i^* = 0$, for $i = 1, 2, 3, 4, 5, 7$.

The notation, '*', in the variables $\lambda_i^*$, $s_j^{-*}$, and $s_k^{+*}$, for $i = 1, 2, 3, 4, 5, 7$, for $j = 1, 2, 3, 4$ and for $k = 1, 2$ displays the optimality of the solution.

$$\begin{aligned}
\sum_{i=1}^{8} x_{i1}\lambda_i^* &= x_{61}\lambda_6^* + x_{81}\lambda_8^* = 238.1156526 && \leq && x_{31}, \\
\sum_{i=1}^{8} x_{i2}\lambda_i^* &= x_{62}\lambda_6^* + x_{82}\lambda_8^* = 41,003 && \leq && x_{32}, \\
\sum_{i=1}^{8} x_{i3}\lambda_i^* &= x_{63}\lambda_6^* + x_{83}\lambda_8^* = 11,800 && \leq && x_{33}, \\
\sum_{i=1}^{8} x_{i4}\lambda_i^* &= x_{64}\lambda_6^* + x_{84}\lambda_8^* = 173,929.6027 && \leq && x_{34}, \\
\sum_{i=1}^{8} y_{i1}\lambda_i^* &= y_{61}\lambda_6^* + y_{81}\lambda_8^* = 20,865.11703 && \geq && y_{31}, \\
\sum_{i=1}^{8} y_{i2}\lambda_i^* &= y_{62}\lambda_6^* + y_{82}\lambda_8^* = 1,244,433.734 && \geq && y_{32}, \\
\sum_{i=1}^{8} y_{i3}\lambda_i^* &= y_{63}\lambda_6^* + y_{83}\lambda_8^* = 3414.621473 && \geq && y_{33}.
\end{aligned} \tag{5.19}$$

From Eq. 5.19, the optimal slacks are given by:

$$\begin{aligned}
s_1^{-*} &= x_{31} - \sum_{i=1}^{8} x_{i1}\lambda_i^* = 561.8843474, \\
s_2^{-*} &= x_{32} - \sum_{i=1}^{8} x_{i2}\lambda_i^* = 0, \\
s_3^{-*} &= x_{33} - \sum_{i=1}^{8} x_{i3}\lambda_i^* = 0, \\
s_4^{-*} &= x_{34} - \sum_{i=1}^{8} x_{i4}\lambda_i^* = 96,025.39734, \\
s_1^{+*} &= \sum_{i=1}^{8} y_{i1}\lambda_i^* - y_{31} = 5257.117027, \\
s_2^{+*} &= \sum_{i=1}^{8} y_{i2}\lambda_i^* - y_{32} = 204,466.7336, \\
s_3^{+*} &= \sum_{i=1}^{8} y_{i3}\lambda_i^* - y_{33} = 1827.621473.
\end{aligned} \tag{5.20}$$

Similarly, E is dominated by a linear combination of F and H, regarding the radiate, the convexity and the wholly dominant approaches, where $\lambda_6^* = 0.256518931$, $\lambda_8^* = 0.027498554$ and $\lambda_i^* = 0$, for $i = 1, 2, 3, 4, 5, 7$.

Applying the inner radiate, the convexity and the wholly dominant approaches yields the same outcomes that are measured by Eq. 5.18, while applying the outer radiate, the convexity and the wholly dominant approaches, yields the same outcomes that are measured by Eq. 5.17. Does this example prove that the results of the inner radiate, the convexity and the wholly dominant approaches are the same as the results of the radiate, the convexity and the wholly dominant approaches, or the results of the outer radiate, the convexity and the wholly dominant approaches are the same as the results of the convexity and the wholly dominant approaches?

As illustrated in Sect. 2.6, airports which have done the job right, according to an introduced approach, lie on the frontier of the feasible area which is generated by that approach. These points on the frontier should not be wrongly considered as those that have done the job well. Every relationship between the input factors (output factors) may display one of the airports which have done the job right as an airport which has done the job well. In other words, different relationships between the input factors (output factors) may introduce different airports which have done the job well as well as different relative scores and rank for airports. Nonetheless, finding the airports which have done the job right may lead us to approximate a relationship between the factors with a lower risk, in order to suggest airports which have done the job well.

## 5.5 The Partially Dominant Concept

Similar to the discussions in the previous chapters, the partially dominant concept is needed for discriminating between the airports in Table 5.1. Each airport in Table 5.1 has four input factors and three output factors, however, the linear combination of the input factors (Eq. 5.1) introduces one total input factor (or one virtual input factor), and the linear combination of the output factors (Eq. 5.2) introduces one total output factor (or one virtual output factor). Thus, similar to chapter 1, there is only one input factor and one output factor to measure the scores and the relative scores of the airports by Eqs. 5.3 and 5.4, respectively.

Suppose that the two first airports A and B are selected, when their coordinates are $(x_{11}, x_{12}, x_{13}, x_{14}, y_{11}, y_{12}, y_{13})$ and $(x_{21}, x_{22}, x_{23}, x_{24}, y_{21}, y_{22}, y_{23})$, respectively, and the weights $w_j^-$ and $w_k^+$ are available, for $j = 1, 2, 3, 4$ and $k = 1, 2, 3$. Assume that, $x_{i1}w_1^- + x_{i2}w_2^- + x_{i3}w_3^- + x_{i4}w_4^- = X_i$, and $y_{i1}w_1^+ + y_{i2}w_2^+ + y_{i3}w_3^+ = Y_i$, for $i = 1, 2$. Similar to Chapter 1, A partially dominates B if, and only if, the value of $Y_1/X_1$ is greater than the value of $Y_2/X_2$. The equation $Y_i/X_i$ is the same as Eq. 5.3 and can be compressed by $\left(\sum_{k=1}^{3} y_{ik}w_k^+\right)/\left(\sum_{j=1}^{4} y_{ij}w_j^-\right)$, for $i = 1, 2, \ldots, 8$.

When there was only one input factor and one output factor in Chapter 1, the partially dominant concept could be estimated from the combination of the radiate and the wholly dominant approaches. Is there a relationship between the introduced approaches and the partially dominant concept, when the number of factors increases? The next sections provide the requirements to answer this question. Of course, it is important to recognize that, when the weights, $w_j^-$ and $w_k^+$, for $j = 1, 2, 3, 4$ and $k = 1, 2, 3$, are unknown, the wholly dominant approach (or other introduced approaches) can only classify the airports which have done the job right, and cannot discriminate those that have done the job well, (see Definitions 1.1, 1.2 and 1.4). Therefore, similar to Chapters 1 and 2, different approximations can be proposed to estimate the best airports in Table 5.1, as is elucidated in the next section.

## 5.6 Unknown Units of Measurement

When the relationships between the factors or the weights, $w_j^-$ and $w_k^+$, for $j = 1, 2, 3, 4$ and $k = 1, 2, 3$, are unknown, this situation is equivalent to when the units of the factors are unknown. In this case, the researcher is required to specify the weights either by a questionnaire or one of the measurement approximations, and in both situations, the results are depended on the selected set of weights. Classifying different sets of weights and balancing them with real-life situations can be useful to make a fair decision. The following section describes several approaches for this aim.

### *5.6.1 The Extremum Measurement Approximation*

Suppose that the weights, $w_j^-$ and $w_k^+$, are introduced as $1/\min\{x_{ij} : x_{ij} \neq 0, \text{ for } i = 1, 2, \ldots, 8\}$ and $1/\max\{y_{ik} : i = 1, 2, \ldots, 8\}$, respectively, for $j = 1, 2, 3, 4$ and $k = 1, 2, 3$. It is obvious that the point with the coordinates of the minimum values of the input factors and the maximum values of the output factors, that is, (478, 30000, 8000, 192330, 129153, 11709741, 74184) wholly dominates every airport in Table 5.1. The researcher introduces the coordinates of this point as the units of measurement. After applying the introduced weights, the coordinates of the point are (1, 1, 1, 1, 1, 1, 1) in the unity scale.

As explained in Chapter 2, this approximation does not suppose whether this point is practical, and the method may be useful for special purposes.

Table 5.5 illustrates the data with five decimal digits, after applying the weights for the airports in Table 5.1.

In addition, Table 5.6 illustrates the scores and the relative scores of the airports with six decimal digits. The last column of Table 5.6 also represents the rank of airports in descending sort. As can be seen, airport H has the best relative score with this measurement. As an exercise the reader can examine the data in Tables 5.5 and 5.6.

**Table 5.5** The data where the units are the extremum values

| N | Airport | Area | Apron | Terminal | Runway | Flights | Passengers | Cargo |
|---|---|---|---|---|---|---|---|---|
| 1 | A | 2.510 | 10.139 | 5.700 | 1.839 | 0.238 | 0.344 | 1.000 |
| 2 | B | 1.052 | 7.124 | 4.847 | 1.810 | 0.363 | 0.408 | 0.257 |
| 3 | C | 1.674 | 1.367 | 1.475 | 1.404 | 0.121 | 0.089 | 0.021 |
| 4 | D | 2.178 | 3.749 | 2.631 | 2.058 | 0.309 | 0.149 | 0.066 |
| 5 | E | 2.096 | 1.000 | 1.000 | 1.000 | 0.038 | 0.037 | 0.021 |
| 6 | F | 1.000 | 2.100 | 2.875 | 2.023 | 0.318 | 0.185 | 0.073 |
| 7 | G | 1.006 | 1.574 | 1.163 | 1.399 | 0.147 | 0.083 | 0.052 |
| 8 | H | 2.816 | 16.776 | 9.546 | 2.191 | 1.000 | 1.000 | 0.533 |

**Table 5.6** The scores of airports by the extremum measurement

| N | Airport | Score | Relative score | Rank |
|---|---|---|---|---|
| 1 | A | 0.078361 | 0.969099 | 2 |
| 2 | B | 0.069315 | 0.857224 | 4 |
| 3 | C | 0.039036 | 0.482760 | 7 |
| 4 | D | 0.049362 | 0.610465 | 6 |
| 5 | E | 0.018760 | 0.232005 | 8 |
| 6 | F | 0.072023 | 0.890716 | 3 |
| 7 | G | 0.054797 | 0.677677 | 5 |
| 8 | H | 0.080860 | 1.000000 | 1 |

### *5.6.2 The Average Measurement Approximation*

The weights $w_j^-$ and $w_k^+$ can also be introduced as $1/\text{ave}\{x_{ij}: i = 1, 2, \ldots, 8\}$ and $1/\text{ave}\{y_{ik}: i = 1, 2, \ldots, 8\}$, respectively, for $j = 1, 2, 3, 4$ and $k = 1, 2, 3$. The point with the average values of the factors in Table 5.1, (that is, (856.375, 164,357.750, 29,237.250, 329,895.000, 40,899.875, 3,359,133.750, 18,763.750)), may wholly dominate an airport or be wholly dominated by an airport. From the average measurement approximation, the researcher introduces the average values of the factors as the units of measurement. Table 5.7 demonstrates the data with five decimal digits after applying this measurement approximation.

Table 5.8 illustrates the results with six decimal digits, as well as the rank of the airports, by the average measurement approximation. Airport H gets the highest rank followed by A, according to Table 5.8.

### *5.6.3 The Specified Airport Measurement Approximation*

The data of each airport can also be introduced as the units of measurement. In other words, for each $l = 1, 2, \ldots, 8$, the weights $w_j^-$ and $w_k^+$ can be selected as $1/x_{lj}$ and $1/y_{lk}$, respectively, for $j = 1, 2, 3, 4$ and $k = 1, 2, 3$. Table 5.9 illustrates the relative scores of airports by the specified airport measurement approximation.

Each row in Table 5.9 represents the relative scores of airports with four decimal digits according to the coordinates of the specified airport in that row. For instance, while the units of measurement are introduced as the coordinates of A, the relative score of A with four decimal digits is 0.5536 and it has fourth rank among other airports, according to Table 5.10. In this case, the best airports are introduced as H, B and F, with the relative scores equal to 1, 0.7414 and 0.6506, respectively.

The ranks of airports are not changed while the units of measurement are introduced as the coordinates of airports B-H. The relative scores of airports are different in the last seven rows in Table 5.9, except the relative score of A. Thus, the set of weights can be classified according to the coordinates of airports A and

**Table 5.7** The data where the units are the average values

| N | Airport | Area | Apron | Terminal | Runway | Flights | Passengers | Cargo |
|---|---|---|---|---|---|---|---|---|
| 1 | A | 1.401 | 1.851 | 1.560 | 1.072 | 0.751 | 1.200 | 3.954 |
| 2 | B | 0.587 | 1.300 | 1.326 | 1.055 | 1.146 | 1.424 | 1.015 |
| 3 | C | 0.934 | 0.249 | 0.404 | 0.818 | 0.382 | 0.310 | 0.085 |
| 4 | D | 1.216 | 0.684 | 0.720 | 1.200 | 0.975 | 0.519 | 0.262 |
| 5 | E | 1.170 | 0.183 | 0.274 | 0.583 | 0.119 | 0.127 | 0.084 |
| 6 | F | 0.558 | 0.383 | 0.787 | 1.180 | 1.005 | 0.645 | 0.289 |
| 7 | G | 0.562 | 0.287 | 0.318 | 0.815 | 0.465 | 0.289 | 0.204 |
| 8 | H | 1.572 | 3.062 | 2.612 | 1.277 | 3.158 | 3.486 | 2.108 |

**Table 5.8** The scores of airports by the average measurement approximation

| N | Airport | Score | Relative score | Rank |
|---|---|---|---|---|
| 1 | A | 1.0035367 | 0.9772942 | 2 |
| 2 | B | 0.8397745 | 0.8178144 | 3 |
| 3 | C | 0.3224993 | 0.3140660 | 7 |
| 4 | D | 0.4598475 | 0.4478225 | 6 |
| 5 | E | 0.1497273 | 0.1458119 | 8 |
| 6 | F | 0.6664531 | 0.6490253 | 4 |
| 7 | G | 0.4831805 | 0.4705453 | 5 |
| 8 | H | 1.0268522 | 1.0000000 | 1 |

H. Both A and H can get the maximum values of the relative scores by the coordinates of H and A, respectively, but the ranks of A and H are 4 and 2, when the coordinates of A and H are introduced as the units of measurement, respectively.

The above results are yielded without any additional information. For instance, the researcher can select the coordinates of A to introduce the relationships between the factors, and then, regulate the weights according to expert judgment.

### *5.6.4 The Optimization Measurement Approximation*

Table 5.11 illustrates the maximum and the minimum values of the relative scores of airports B-G in Table 5.9. Note that the scores in Table 5.11 are not relatively meaningful, (see Sects. 3.6.3.1 and 4.3, and Theorems 4.1 and 4.2), and illustrate that for each airport there is a relationship between the factors by a specified airport measurement approximation by which that airport can reach the maximum value of its relative scores, regardless of the relative scores of other airports corresponded to that relationship.

**Table 5.9** The relative scores by the specified airport measurement

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| A | 0.5536 | 0.7414 | 0.3188 | 0.4708 | 0.1280 | 0.6506 | 0.4425 | 1.0000 |
| B | 1.0000 | 0.8347 | 0.2464 | 0.3836 | 0.1064 | 0.6075 | 0.4114 | 0.9868 |
| C | 1.0000 | 0.5044 | 0.2010 | 0.2694 | 0.1363 | 0.3942 | 0.3457 | 0.5418 |
| D | 1.0000 | 0.5811 | 0.2100 | 0.2857 | 0.1250 | 0.4150 | 0.3444 | 0.6551 |
| E | 1.0000 | 0.6913 | 0.3950 | 0.4869 | 0.2253 | 0.6795 | 0.5592 | 0.7941 |
| F | 1.0000 | 0.6132 | 0.2048 | 0.2856 | 0.1116 | 0.4704 | 0.3559 | 0.6540 |
| G | 1.0000 | 0.6752 | 0.2893 | 0.3768 | 0.1556 | 0.5599 | 0.4519 | 0.7568 |
| H | 1.0000 | 0.7012 | 0.2026 | 0.3140 | 0.0998 | 0.4479 | 0.3220 | 0.8950 |

**Table 5.10** The ranks of airports by the specified airport measurement

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| A | 4 | 2 | 7 | 5 | 8 | 3 | 6 | 1 |
| B-H | 1 | 3 | 7 | 6 | 8 | 4 | 5 | 2 |

From Sects. 5.4 and 5.5, B can be introduced as an airport which has done the job right, for instance, according to the radiate, the convexity and the wholly dominant approaches. Thus, there should be a relationship between the factors which yields the maximum value of the relative scores of B as 1, when the corresponded relative scores for other airports are less than or equal to 1. The same situation can be illustrated for airports C-G from a specified approach to introduce the airports which have done the job right. Therefore, the researcher can find different relationships between the factors to classify the weights in order to decrease the risk of introducing a special set of weights or relationship between the factors to estimate the relative scores of the airports.

In order to find the maximum value of the relative scores of B, similar to Sect. 4.1, the following non-linear equation which is a fractional programming should be solved ($j = 1, 2, 3, 4$, $k = 1, 2, 3$ and $l = 2$). The fraction in Eq. 5.21 is the same as Eq. 5.4.

$$\max\left\{\frac{\dfrac{y_{l1}w_1^+ + y_{l2}w_2^+ + y_{l3}w_3^+}{x_{l1}w_1^- + x_{l2}w_2^- + x_{l3}w_3^- + x_{l4}w_4^-}}{\max\left\{\dfrac{y_{i1}w_1^+ + y_{i2}w_2^+ + y_{i3}w_3^+}{x_{i1}w_1^- + x_{i2}w_2^- + x_{i3}w_3^- + x_{i4}w_4^-} : i = 1, 2, \ldots, 8\right\}} : w_j^- \geq 0, w_k^+ \geq 0\right\}. \tag{5.21}$$

Without loss of generality, suppose that

$$\max\left\{\frac{y_{i1}w_1^+ + y_{i2}w_2^+ + y_{i3}w_3^+}{x_{i1}w_1^- + x_{i2}w_2^- + x_{i3}w_3^- + x_{i4}w_4^-} : i = 1, 2, \ldots, 8\right\} = 1. \tag{5.22}$$

**Table 5.11** The extremum values of the relative scores in Table 5.9

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| max | 1.0000 | 0.8347 | 0.3950 | 0.4869 | 0.2253 | 0.6795 | 0.5592 | 1.0000 |
| min | 0.5536 | 0.5044 | 0.2010 | 0.2694 | 0.0998 | 0.3942 | 0.3220 | 0.5418 |

Since the weights are optimized the above equation is equivalent with the following eight inequalities, for $i = 1, 2, \ldots, 8$.

$$\frac{y_{i1}w_1^+ + y_{i2}w_2^+ + y_{i3}w_3^+}{x_{i1}w_1^- + x_{i2}w_2^- + x_{i3}w_3^- + x_{i4}w_4^-} \le 1. \tag{5.23}$$

Hence, Eq. 5.21 is equivalent with the following fractional programming, for $l = 2$.

$$\begin{aligned}
&\max \frac{y_{l1}w_1^+ + y_{l2}w_2^+ + y_{l3}w_3^+}{x_{l1}w_1^- + x_{l2}w_2^- + x_{l3}w_3^- + x_{l4}w_4^-},\\
&\text{Subject to}\\
&\frac{y_{i1}w_1^+ + y_{i2}w_2^+ + y_{i3}w_3^+}{x_{i1}w_1^- + x_{i2}w_2^- + x_{i3}w_3^- + x_{i4}w_4^-} \le 1, \quad \text{for } i = 1, 2, \ldots, 8,\\
&w_j^+ \ge 0, \quad \text{for } j = 1, 2, 3, 4,\\
&w_k^- \ge 0, \quad \text{for } k = 1, 2, 3.
\end{aligned} \tag{5.24}$$

In Sect. 4.2.1, the Eq. 2.24 is solved while the linear combination of the output factors is 1 in Eq. 5.21. The Eq. 2.30 in Sect. 4.2.3 is also solved while the linear combination of the input factors is 1 in Eq. 5.21. These two situations introduce a linear programming to solve Eq. 5.21. The inequalities in the constraints of Eq. 5.24 can easily transformed to the following linear inequalities $y_{i1}w_1^+ + y_{i2}w_2^+ + y_{i3}w_3^+ \le x_{i1}w_1^- + x_{i2}w_2^- + x_{i3}w_3^- + x_{i4}w_4^-$, for $i = 1, 2, \ldots, 8$, where at least one of the terms $x_{ij}w_j^-$ $(j = 1, 2, 3, 4)$ is positive. In contrast, the fraction in the objective of Eq. 5.24 is not easily transformed into a linear equation without introducing some restrictions. In the following sections, several forms of restrictions are discussed.

#### 5.6.4.1 Form 1: Minimizing the Denominator

The first form to solve Eq. 5.24, for $l = 2$, is to minimize the denominator in the objective while the numerator is fixed. For such an aim, suppose that, $y_{21}w_1^+ + y_{22}w_2^+ + y_{23}w_3^+ = 1$, hence Eq. 5.25 for $l = 2$, (which is a linear programming), is yielded.

$$
\begin{aligned}
&\min x_{l1}w_1^- + x_{l2}w_2^- + x_{l3}w_3^- + x_{l4}w_4^-,\\
&\text{Subject to}\\
&y_{l1}w_1^+ + y_{l2}w_2^+ + y_{l3}w_3^+ = 1,\\
&y_{11}w_1^+ + y_{12}w_2^+ + y_{13}w_3^+ \le x_{11}w_1^- + x_{12}w_2^- + x_{13}w_3^- + x_{14}w_4^-,\\
&y_{21}w_1^+ + y_{22}w_2^+ + y_{23}w_3^+ \le x_{21}w_1^- + x_{22}w_2^- + x_{23}w_3^- + x_{24}w_4^-,\\
&y_{31}w_1^+ + y_{32}w_2^+ + y_{33}w_3^+ \le x_{31}w_1^- + x_{32}w_2^- + x_{33}w_3^- + x_{34}w_4^-,\\
&y_{41}w_1^+ + y_{42}w_2^+ + y_{43}w_3^+ \le x_{41}w_1^- + x_{42}w_2^- + x_{43}w_3^- + x_{44}w_4^-,\\
&y_{51}w_1^+ + y_{52}w_2^+ + y_{53}w_3^+ \le x_{51}w_1^- + x_{52}w_2^- + x_{53}w_3^- + x_{54}w_4^-,\\
&y_{61}w_1^+ + y_{62}w_2^+ + y_{63}w_3^+ \le x_{61}w_1^- + x_{62}w_2^- + x_{63}w_3^- + x_{64}w_4^-,\\
&y_{71}w_1^+ + y_{72}w_2^+ + y_{73}w_3^+ \le x_{71}w_1^- + x_{72}w_2^- + x_{73}w_3^- + x_{74}w_4^-,\\
&y_{81}w_1^+ + y_{82}w_2^+ + y_{83}w_3^+ \le x_{81}w_1^- + x_{82}w_2^- + x_{83}w_3^- + x_{84}w_4^-,\\
&w_j^+ \ge 0, \quad \text{for } j = 1,2,3,4,\\
&w_k^- \ge 0, \quad \text{for } k = 1,2,3.
\end{aligned}
\tag{5.25}
$$

From the first constraint in Eq. 5.25, $y_{21}w_1^+ + y_{22}w_2^+ + y_{23}w_3^+ = 1$, at least one of the $w_k^+(k = 1,2,3)$ is positive and from the third constraint which is $1 \le x_{21}w_1^- + x_{22}w_2^- + x_{23}w_3^- + x_{24}w_4^-$, at least one of the $w_j^-$ $(j = 1,2,3,4)$ is positive. These conditions yield to find a reasonable set of weights which let B reach the maximum value of its relative scores in the first form, while the corresponded relative scores for other airports are less than or equal to 1.

Note that the maximum value is measured by the inverse of the optimal value of the objective in Eq. 5.25.

Similar to what is depicted in Figs. 2.14, 2.18, 2.27 and 2.29 some of the weights might be 0, which represent how the location of B can be imagined in the feasible area. As explained earlier, there is no lack when some of the weights in Eq. 5.25 are zero; nonetheless, the researcher may introduce some additional constraints for one or all the weights, as long as the feasible area is available. For instance, the weights can be restricted with, $w_j^- > 10^{-9}$, for $j = 1, 2, 3, 4$ and $w_k^+ > 10^{-9}$, for $k = 1, 2, 3$, to avoid zero values in optimal solutions in Eq. 5.25 or any other purposes. However, introducing any restrictions for the weights may change the optimal solutions in Eq. 5.24 and suggest different ranking outcomes.

The same discussion can be elucidated for the airports C-G by, $l = 3, 4, \ldots, 7$, to find the maximum values of the relative scores of C-G. Such an attempt allows the researcher to find several sets of weights which may be useful to introduce a more reasonable set of weights to measure the relative scores of the airports.

In order to solve Eq. 5.24 for each airport in Form 1, with the Microsoft Excel Solver 2013 software, the following steps can be followed.

1. Copy the 9 columns of Table 5.1 on an Excel sheet into cells A1:I9, as Fig. 5.1 depicts.

2. Label B11 as 'Index', B13 as 'Weights', E11 as 'Objective', J1 as 'Constraints', J11 as 'Fixed numerator', K1 as 'Score', L1 as '$w_1^-$', M1 as '$w_2^-$', N1 as '$w_3^-$', O1 as '$w_4^-$', P1 as '$w_1^+$', Q1 as '$w_2^+$' and R1 as '$w_3^+$'.
3. Assign number 1 to C11.
4. Assign the command '=Sumproduct(Index(C2:F9,C11,0),C13:F13)' into F11, (without the quotation '').
5. Assign the command '=Sumproduct(Index(G2:I9,C11,0),G13:I13)' into J11.
6. Assign the following command into J2

   '=Sumproduct(G$13:I$13,G2:I2)-Sumproduct(C$13:F$13,C2:F2)'.

7. Copy J2 and then paste it into cells J3-J9.
8. Open 'Solver Parameters' window.
9. Assign 'F11' into 'Set Objective' and choose 'Min'.
10. Assign 'C13:I13' into 'By Changing Variable Cells'.
11. Click on 'Add' and assign 'J2:J9' into 'Cell Reference', then select '<=', and assign '0' into 'Constraint'.
12. Click on 'Add' and assign 'J11' into 'Cell Reference', then select '=', and assign '1' into 'Constraint'. Then click on 'OK'.
13. Tick 'Make Unconstrained Variables Non-Negative'.
14. Choose 'Simplex LP' from 'Select a Solving Method'.
15. Click on 'Solve'.
16. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window (Fig. 5.9).
17. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet.
18. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.
19. From the toolbar menu, click on 'Tools> References...>' and make sure 'Solver' is ticked, and then 'OK'.
20. Inside of the 'Microsoft Visual Basic for Applications' window, write the following commands between 'Sub Button1_Click ()' and 'End Sub' as Fig. 5.10 depicts.

```
Dim i As Integer
For i = 1 To 8
   Range("C11") = i
   SolverSolve Userfinish:=True
   Range("K" & i + 1) = 1/Range("F11")
   Range("C13:I13").Copy
   Range("L" & i + 1).Select
   Selection.PasteSpecial Paste:=xlPasteValues
Next i
```

21. Close the 'Microsoft Visual Basic for Applications' window.
22. Click on the small rectangle which was automatically made in the Excel sheet by step 17.

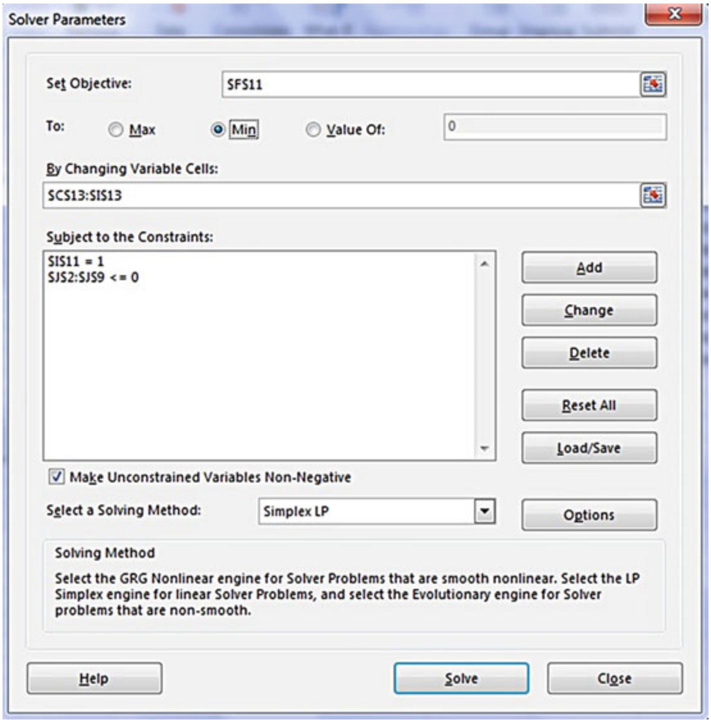



**Fig. 5.9** Setting Solver to solve Eq. 5.25

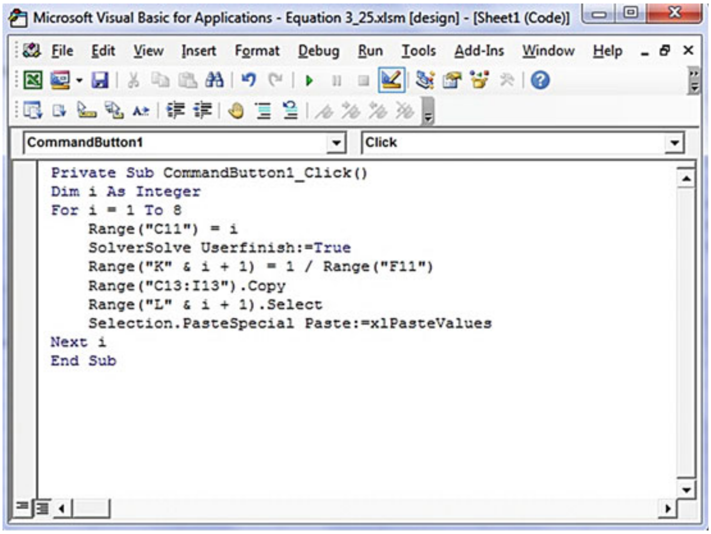



**Fig. 5.10** Setting VBA macro to solve Eq. 5.25

| | K | L | M | N | O | P | Q | R |
|---|---|---|---|---|---|---|---|---|
| 3 | Relative Score | $w^{*-}_1$ | $w^{*-}_2$ | $w^{*-}_3$ | $w^{*-}_4$ | $w^{*+}_1$ | $w^{*+}_2$ | $w^{*+}_3$ |
| 72 | 1.000000 | 0.0000000 | 0.0000033 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000135 |
| 23 | 1.000000 | 0.0002799 | 0.0000039 | 0.0000000 | 0.0000001 | 0.0000000 | 0.0000002 | 0.0000121 |
| )2 | 0.835695 | 0.0000000 | 0.0000148 | 0.0000501 | 0.0000000 | 0.0000000 | 0.0000010 | 0.0000000 |
| )6 | 1.000000 | 0.0000000 | 0.0000007 | 0.0000352 | 0.0000005 | 0.0000251 | 0.0000000 | 0.0000000 |
| 72 | 0.509123 | 0.0000000 | 0.0000378 | 0.0001036 | 0.0000000 | 0.0000000 | 0.0000019 | 0.0001148 |
| 0 | 1.000000 | 0.0000000 | 0.0000047 | 0.0000000 | 0.0000018 | 0.0000243 | 0.0000000 | 0.0000000 |
| 28 | 1.000000 | 0.0000000 | 0.0000053 | 0.0000806 | 0.0000000 | 0.0000323 | 0.0000004 | 0.0000000 |
| 0 | 1.000000 | 0.0000000 | 0.0000018 | 0.0000000 | 0.0000002 | 0.0000000 | 0.0000001 | 0.0000000 |

Run

**Fig. 5.11** Results of solving Eq. 5.25

**Table 5.12** The maximum values of the relative scores by Form 1

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Max | 1 | 1 | 0.835695 | 1 | 0.509123 | 1 | 1 | 1 |

23. The maximum scores of the relative scores of airports by solving Eq. 5.25 are represented into cells K2:K9 and the corresponded sets of weights are displayed in cells L2:R9, as shown in Fig. 5.11.

Tables 5.12 and 5.13 illustrate the results of Eq. 5.24 in Form 1 for each airport, where $l = 1, 2, \ldots, 8$.

The outcomes in Table 5.12 are equivalent with the outcomes by Eq. 5.18 in Table 5.4. As a result, there may be a relationship between these two approaches which is examined in Sect. 5.8.

From each set of weights in Table 5.13, the researcher can measure the relative scores of each airport, as described in Table 5.14. Note that the results in Table 5.13 are written with seven decimal digits, which should be noticed while the data in Table 5.14 are manually examined.

The second column in Table 5.14 illustrates the relative scores of A while the corresponded set of optimal weights in each row of Table 5.13 is selected. For instance, A reaches the maximum value of its relative scores by the sets of weights in the first, second, and fifth rows in Table 5.13.

Furthermore, the data in diameter of Table 5.14 is bolded, which are the maximum values of the relative scores of the airports. For instance, the sixth column in Table 5.14 illustrates that the maximum value of the relative scores of E is 0.509123,

**Table 5.13** The optimal weights from Eq. 5.25

| Airport | $w_1^{-*}$ | $w_2^{-*}$ | $w_3^{-*}$ | $w_4^{-*}$ | $w_1^{+*}$ | $w_2^{+*}$ | $w_3^{+*}$ |
|---|---|---|---|---|---|---|---|
| A | 0.0000000 | 0.0000033 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000135 |
| B | 0.0002799 | 0.0000039 | 0.0000000 | 0.0000001 | 0.0000000 | 0.0000002 | 0.0000121 |
| C | 0.0000000 | 0.0000148 | 0.0000501 | 0.0000000 | 0.0000000 | 0.0000010 | 0.0000000 |
| D | 0.0000000 | 0.0000007 | 0.0000352 | 0.0000005 | 0.0000251 | 0.0000000 | 0.0000000 |
| E | 0.0000000 | 0.0000378 | 0.0001036 | 0.0000000 | 0.0000000 | 0.0000019 | 0.0001148 |
| F | 0.0000000 | 0.0000047 | 0.0000000 | 0.0000018 | 0.0000243 | 0.0000000 | 0.0000000 |
| G | 0.0000000 | 0.0000053 | 0.0000806 | 0.0000000 | 0.0000323 | 0.0000004 | 0.0000000 |
| H | 0.0000000 | 0.0000018 | 0.0000000 | 0.0000002 | 0.0000000 | 0.0000001 | 0.0000000 |

**Table 5.14** The relative scores of the airports according to Table 5.13

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| A | **1.000000** | 0.365473 | 0.158703 | 0.179344 | 0.215133 | 0.352372 | 0.332303 | 0.322278 |
| B | 1.000000 | **1.000000** | 0.457432 | 0.445629 | 0.212062 | 1.000000 | 0.591513 | 1.000000 |
| C | 0.571886 | 0.901926 | **0.835695** | 0.617719 | 0.487713 | 1.000000 | 0.802968 | 1.000000 |
| D | 0.388439 | 0.701716 | 0.690976 | **1.000000** | 0.314404 | 1.000000 | 0.988322 | 1.000000 |
| E | 1.000000 | 0.937047 | 0.783263 | 0.606552 | **0.509123** | 1.000000 | 0.835855 | 1.000000 |
| F | 0.359595 | 0.695643 | 0.557843 | 0.778668 | 0.243291 | **1.000000** | 0.652905 | 1.000000 |
| G | 0.490425 | 0.801803 | 0.785152 | 0.863864 | 0.407952 | 1.000000 | **1.000000** | 1.000000 |
| H | 0.553437 | 0.898266 | 0.720104 | 0.537756 | 0.409206 | 1.000000 | 0.616393 | **1.000000** |

**Table 5.15** The ranks of airports according to Table 5.14

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| A | 1 | 2 | 8 | 7 | 6 | 3 | 4 | 5 |
| B | 1 | 1 | 6 | 7 | 8 | 1 | 5 | 1 |
| C | 7 | 3 | 4 | 6 | 8 | 1 | 5 | 1 |
| D | 7 | 5 | 6 | 1 | 8 | 1 | 4 | 1 |
| E | 1 | 4 | 6 | 7 | 8 | 1 | 5 | 1 |
| F | 7 | 4 | 6 | 3 | 8 | 1 | 5 | 1 |
| G | 7 | 5 | 6 | 4 | 8 | 1 | 1 | 1 |
| H | 6 | 3 | 4 | 7 | 8 | 1 | 5 | 1 |

which is measured from the optimal set of weights for E in the fifth row of Table 5.13. Airports F and H reach the maximum values of their relative scores with all corresponded sets of weights for B-H in Table 5.13. These results are almost equivalent with the results in Table 5.10 to classify the airports into two different classes. All airports B-H have the lowest values of their relative scores in Table 5.14, while the corresponded optimal set of weights for A is selected.

Table 5.15 also illustrates the ranks of airports according to the scores in Table 5.14. For instance, while E reaches its maximum value of its relative scores, the highest rank for E is 8, (that is, the lowest rank among the other airports), whereas E has the 6th rank by the corresponded set of optimal weights for A. These results also support the discussions in Sects. 3.6.3.1 and 4.3.

#### 5.6.4.2 Form 2: Maximizing the Numerator

The second form to solve Eq. 5.24 is to maximize the numerator while the denominator is fixed. Similar to the previous section, assume that $x_{21}w_1^- + x_{22}w_2^- + x_{23}w_3^- + x_{24}w_4^- = 1$, thus, Eq. 5.26 is yielded to find the maximum value of the relative scores of B ($l = 2$) in Form 2.

$$\begin{aligned}
&\max \; y_{l1}w_1^+ + y_{l2}w_2^+ + y_{l3}w_3^+,\\
&\text{Subject to}\\
&x_{l1}w_1^- + x_{l2}w_2^- + x_{l3}w_3^- + x_{l4}w_4^- = 1,\\
&\textstyle\sum_{k=1}^{3} y_{ik}w_k^+ \le \sum_{j=1}^{4} x_{ij}w_j^-, \quad \text{for } i = 1, 2, \ldots, 8,\\
&w_j^+ \ge 0, \quad \text{for } j = 1, 2, 3, 4,\\
&w_k^- \ge 0, \quad \text{for } k = 1, 2, 3.
\end{aligned} \tag{5.26}$$

From the constraint, $x_{21}w_1^- + x_{22}w_2^- + x_{23}w_3^- + x_{24}w_4^- = 1$, at least one of the weights $w_j^- (j = 1, 2, 3, 4)$, is positive, and since the objective of Eq. 5.26 is maximized, at least one of the weights, $w_k^- (k = 1, 2, 3)$, is also positive. Therefore, reasonable sets of weights and the corresponded scores can be calculated for airport B. The same illustration can be discussed for other airports by replacing the value of

**Table 5.16** The maximum values of the relative scores by Form 2

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Max Score | 1 | 1 | 0.835695 | 1 | 0.509123 | 1 | 1 | 1 |

$l = 1, 3, \ldots, 8$. Table 5.16 illustrates the optimal values of the objective in Eq. 5.24 for each airport in Form 2. Note that these scores are not relatively meaningful, as discussed in Theorems 4.1 and 4.2.

The results in Table 5.16 are the same as the results in Table 5.13, which gives us a mathematical direction to prove whether this event is a random outcome or a real statement, (see Exercise 5.6). The optimal weights corresponded to the scores in Table 5.16 are illustrated in Table 5.17. Only the corresponded sets of optimal weights for C, E and G are different in Tables 5.11 and 5.16. The relative scores from these three different sets of optimal weights are represented in Table 5.18.

As can be seen, the relative scores of F and H followed by A, in Tables 5.13 and 5.17, are 1 for almost all of the optimal sets of weights. Are there some sets of weights which show that, for instance, only G has the score of 1 among other airports?

The answer to the above question is positive because, from Sect. 5.4 and the results in Table 5.4, G lies on the corner of the feasible area, therefore, there should be some sets of weights which let none of the airports get the score of 1, except G. For instance, suppose that the airports can, at most, get the relative scores of 0.9, while the relative score of G is 1. Eq. 5.27 can be solved for each airport number $l = 1, 2, \ldots, 8$, which yield the eight sets of weights represented in Table 5.19, to display that only G gets the relative score of 1.

$$\begin{aligned}
&\max \sum_{k=1}^{3} y_{lk} w_k^+, \\
&\text{Subject to} \\
&\sum_{j=1}^{4} x_{lj} w_j^- = 1, \\
&\sum_{k=1}^{3} y_{ik} w_k^+ \le 0.9^* \sum_{j=1}^{4} x_{ij} w_j^-, \quad \text{for } i = 1, 2, \ldots, 6, 8, \\
&\sum_{k=1}^{3} y_{7k} w_k^+ = \sum_{j=1}^{4} x_{7j} w_j^-, \\
&w_j^+ \ge 0, \quad \text{for } j = 1, 2, 3, 4, \\
&w_k^- \ge 0, \quad \text{for } k = 1, 2, 3.
\end{aligned} \tag{5.27}$$

The value 0.9 in the constraints in Eq. 5.27, can also be varied from an airport to another, however, there might not be a feasible region sometimes. In addition, the constraint $\sum_{k=1}^{3} y_{7k} w_k^+ = \sum_{j=1}^{4} x_{7j} w_j^-$ in Eq. 5.27 yields that the relative scores of G become as 1, for $l = 1, 2, \ldots, 8$.

Table 5.20 illustrates the relative scores of each airport according to the sets of weights in Table 5.19. Each column of Table 5.20 displays the relative scores of an airport while the corresponded sets of weights are changed in each row. For instance,

**Table 5.17** The optimal weights from Eq. 5.25

| Airport | $w_1^{-*}$ | $w_2^{-*}$ | $w_3^{-*}$ | $w_4^{-*}$ | $w_1^{+*}$ | $w_2^{+*}$ | $w_3^{+*}$ |
|---|---|---|---|---|---|---|---|
| A | 0.0000000 | 0.0000033 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000135 |
| B | 0.0002799 | 0.0000039 | 0.0000000 | 0.0000001 | 0.0000000 | 0.0000002 | 0.0000121 |
| C | 0.0000000 | 0.0000123 | 0.0000418 | 0.0000000 | 0.0000000 | 0.0000008 | 0.0000000 |
| D | 0.0000000 | 0.0000007 | 0.0000352 | 0.0000005 | 0.0000251 | 0.0000000 | 0.0000000 |
| E | 0.0000000 | 0.0000193 | 0.0000528 | 0.0000000 | 0.0000000 | 0.0000010 | 0.0000585 |
| F | 0.0000000 | 0.0000047 | 0.0000000 | 0.0000018 | 0.0000243 | 0.0000000 | 0.0000000 |
| G | 0.0000000 | 0.0000016 | 0.0000758 | 0.0000008 | 0.0000526 | 0.0000000 | 0.0000000 |
| H | 0.0000000 | 0.0000018 | 0.0000000 | 0.0000002 | 0.0000000 | 0.0000001 | 0.0000000 |

**Table 5.18** The relative scores of the airports according to Table 5.17

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| C | 0.571886 | 0.901926 | **0.835695** | 0.617719 | 0.487713 | 1.000000 | 0.802968 | 1.000000 |
| E | 1.000000 | 0.937047 | 0.783263 | 0.606552 | **0.509123** | 1.000000 | 0.835855 | 1.000000 |
| G | 0.382243 | 0.692360 | 0.695594 | 1.000000 | 0.316845 | 1.000000 | **1.000000** | 0.980743 |

**Table 5.19** The optimal weights from Eq. 5.27

| Airport | $w_1^{-*}$ | $w_2^{-*}$ | $w_3^{-*}$ | $w_4^{-*}$ | $w_1^{+*}$ | $w_2^{+*}$ | $w_3^{+*}$ |
|---|---|---|---|---|---|---|---|
| A | 0.0000000 | 0.0000008 | 0.0000168 | 0.0000000 | 0.0000084 | 0.0000000 | 0.0000087 |
| B | 0.0000000 | 0.0000009 | 0.0000210 | 0.0000000 | 0.0000095 | 0.0000000 | 0.0000099 |
| C | 0.0000000 | 0.0000011 | 0.0000808 | 0.0000000 | 0.0000361 | 0.0000001 | 0.0000036 |
| D | 0.0000000 | 0.0000010 | 0.0000422 | 0.0000000 | 0.0000217 | 0.0000000 | 0.0000069 |
| E | 0.0000000 | 0.0000045 | 0.0001082 | 0.0000000 | 0.0000488 | 0.0000001 | 0.0000508 |
| F | 0.0000000 | 0.0000010 | 0.0000408 | 0.0000000 | 0.0000210 | 0.0000000 | 0.0000067 |
| G | 0.0000000 | 0.0000023 | 0.0000961 | 0.0000000 | 0.0000494 | 0.0000000 | 0.0000157 |
| H | 0.0000000 | 0.0000002 | 0.0000113 | 0.0000000 | 0.0000058 | 0.0000000 | 0.0000038 |

**Table 5.20** The relative scores of airports according to Table 5.19

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| A | **0.900000** | 0.684045 | 0.629362 | 0.856466 | 0.346768 | 0.900000 | **1.000000** | 0.853715 |
| B | 0.900000 | **0.723237** | 0.646924 | 0.851103 | 0.360683 | 0.900000 | **1.000000** | 0.900000 |
| C | 0.449748 | 0.674425 | **0.681449** | 0.900000 | 0.335492 | 0.900000 | **1.000000** | 0.900000 |
| D | 0.529919 | 0.621964 | 0.649687 | **0.900000** | 0.318531 | 0.900000 | **1.000000** | 0.827013 |
| E | 0.900000 | 0.723237 | 0.646924 | 0.851103 | **0.360683** | 0.900000 | **1.000000** | 0.900000 |
| F | 0.529919 | 0.621964 | 0.649687 | 0.900000 | 0.318531 | **0.900000** | **1.000000** | 0.827013 |
| G | 0.529919 | 0.621964 | 0.649687 | 0.900000 | 0.318531 | 0.900000 | **1.000000** | 0.827013 |
| H | 0.770955 | 0.687224 | 0.636122 | 0.900000 | 0.331506 | 0.900000 | **1.000000** | **0.900000** |

the eighth column in Table 5.20 illustrates that the relative scores of G is 1 for all the corresponded sets of optimal weights for the airports A-H.

From the sets of optimal weights in Table 5.19, none of the airports A-H get the relative scores of more than 0.9, except G. The bolded scores in diameter of Table 5.20 illustrate the maximum values of the relative scores of each airport, according to the sets of optimal weights in Table 5.19.

The minimum value of the relative scores of A in Table 5.20 is 0.449748 which is measured while the corresponded optimal set of weights for C is selected. This outcome may encourage the researcher to find the minimum value of the relative scores of A while G has the relative score of 1, and so on for other purposes.

As a result, while the weights are unknown, a variety set of weights can be introduced to measure the relative scores of airports and each set of weights covers a special purpose. The worst outcome in such a situation is that the researcher can intentionally suggest some sets of weights to introduce an airport as the best performer in comparison with other airports, similar to the results in Table 5.20.

#### 5.6.4.3 Form 3: Mix of Forms 1 and 2

Equation 5.28 is linearly solved for airport number $l\,(l = 1, 2, \ldots, 8)$ in Forms 1 and 2, in the previous sections. The denominator in Eq. 5.28 is minimized in Form 1 while the numerator is fixed, and in Form 2 the numerator is maximized while the denominator is fixed. Now, how can Eq. 5.28 linearly be solved while simultaneously the numerator is maximized and the denominator is minimized in the objective of Eq. 5.28? Are the measured maximum values of the relative scores for airports changed by such an approach?

$$\begin{aligned}
&\min\left(\textstyle\sum_{k=1}^{3} y_{lk} w_k^+ \big/ \sum_{j=1}^{4} x_{lj} w_j^-\right),\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{3} y_{ik} w_k^+ \le \sum_{j=1}^{4} x_{ij} w_j^-, \quad \text{for } i = 1, 2, \ldots, 8,\\
&w_j^+ \ge 0, \quad \text{for } j = 1, 2, 3, 4,\\
&w_k^- \ge 0, \quad \text{for } k = 1, 2, 3.
\end{aligned} \tag{5.28}$$

In order to answer the above question, Form 3 can be introduced, that is, simultaneously maximizing the numerator and minimizing the denominator of the objective in Eq. 5.28. One popular way for such an aim is to replace the equation $\sum_{k=1}^{3} y_{lk} w_k^+ / \sum_{j=1}^{4} x_{lj} w_j^-$ in the objective of Eq. 5.28 with $\sum_{k=1}^{3} y_{lk} w_k^+ - \sum_{j=1}^{4} x_{lj} w_j^-$, as Eq. 5.29 illustrates.

$$
\begin{aligned}
&\min \left(\textstyle\sum_{k=1}^{3} y_{lk} w_k^+ - \sum_{j=1}^{4} x_{lj} w_j^-\right),\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{3} y_{ik} w_k^+ \le \sum_{j=1}^{4} x_{ij} w_j^-, \quad \text{for } i = 1, 2, \ldots, 8,\\
&w_j^- \ge 0, \quad \text{for } j = 1, 2, 3, 4,\\
&w_k^+ \ge 0, \quad \text{for } k = 1, 2, 3.
\end{aligned}
\tag{5.29}
$$

The first term of the objective in Eq. 5.29, that is, $\max \sum_{k=1}^{3} y_{lk} w_k^+$, shows maximizing the numerator of the objective in Eq. 5.28. The second term in the objective of Eq. 5.29, that is, $\max\left(-\sum_{j=1}^{4} x_{lj} w_j^-\right)$, is equivalent with $-\min \sum_{j=1}^{4} x_{lj} w_j^-$, which shows minimizing the denominator of the objective in Eq. 5.28. If the constraint, $\sum_{k=1}^{3} y_{lk} w_k^+ = 1$, is added to Eq. 5.29, Form 1 is represented. If the constrain, $\sum_{j=1}^{4} x_{lj} w_j^- = 1$, is added to Eq. 5.29, Form 2 is displayed. The corresponded sets of optimal weights while one of the constraints, $\sum_{k=1}^{3} y_{lk} w_k^+ = 1$ or $\sum_{j=1}^{4} x_{lj} w_j^- = 1$, is added to Eq. 5.29 yield the same maximum values of the relative scores for each airport, which are represented in Tables 5.12 and 5.15. Can the researcher deduce that Eq. 5.29 solves Eq. 5.28? Are the optimal solutions for Eq. 5.29, optimal for Eq. 5.28 as well?

As can be seen, from the constraints, $\sum_{k=1}^{3} y_{ik} w_k^+ - \sum_{j=1}^{4} x_{ij} w_j^- \le 0$, the maximum value of the objective in Eq. 5.29 is always 0 for each airport. In other words, the vector (0, 0, 0, 0, 0, 0, 0) is a solution for Eq. 5.29, because it satisfies all the constraints and let the objective become non-negative. But this solution is not valid for Eq. 5.28. At least one of the corresponded weights to input (output) factors should be positive to have a valid fraction in the objective of Eq. 5.28. How can the researcher deal with this issue?

One risky way to deal with this question is to introduce a positive value, such as $0.0000001 = 10^{-7}$ or smaller/larger, and replace the last two sets of constraints in Eq. 5.29 with the constraints $w_j^- \ge 10^{-7}$, for $j = 1, 2, 3, 4$, and $w_k^+ \ge 10^{-7}$, for $k = 1$, 2, 3, as Eq. 5.30 displays. Do these negligible restrictions on the weights in Eq. 5.30 yield different sets of optimal weights and different maximum values of the relative scores for Eq. 5.28?

$$
\begin{aligned}
&\max \left(\textstyle\sum_{k=1}^{3} y_{lk} w_k^+ - \sum_{j=1}^{4} x_{lj} w_j^-\right),\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{3} y_{ik} w_k^+ / \sum_{j=1}^{4} x_{ij} w_j^-, \quad \text{for } i = 1, 2, \ldots, 8,\\
&w_j^+ \ge 0.0000001, \quad \text{for } j = 1, 2, 3, 4,\\
&w_k^- \ge 0.0000001, \quad \text{for } k = 1, 2, 3.
\end{aligned}
\tag{5.30}
$$

As illustrated earlier, Eq. 5.30 should be solved for each airport number $l$ ($l = 1, 2, \ldots, 8$). The following steps can be used to solve Eq. 5.30 by the Microsoft Excel Solver 2013 software.

F11 =SUMPRODUCT(INDEX(G2:I9,C11,0),G13:I13)-SUMPRODUCT(INDEX(C2:F9,C11,0),C13:F13)

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | N | Airport | Area | Apron | Terminal | Runway | Flights | Passenger | Cargo | Constraints |
| 2 | 1 | A | 1,200 | 304,182 | 45,600 | 353,610 | 30,707 | 4,030,859 | 74,184 | 0.000 |
| 3 | 2 | B | 503 | 213,729 | 38,778 | 348,120 | 46,875 | 4,783,120 | 19,050 | 0.000 |
| 4 | 3 | C | 800 | 41,003 | 11,800 | 269,955 | 15,608 | 1,039,967 | 1,587 | 0.000 |
| 5 | 4 | D | 1,041 | 112,464 | 21,050 | 395,730 | 39,871 | 1,744,524 | 4,919 | 0.000 |
| 6 | 5 | E | 1,002 | 30,000 | 8,000 | 192,330 | 4,887 | 427,974 | 1,574 | 0.000 |
| 7 | 6 | F | 478 | 63,000 | 23,000 | 389,115 | 41,088 | 2,165,572 | 5,414 | 0.000 |
| 8 | 7 | G | 481 | 47,210 | 9,300 | 268,995 | 19,010 | 971,313 | 3,826 | 0.000 |
| 9 | 8 | H | 1,346 | 503,274 | 76,370 | 421,305 | 129,153 | ######## | 39,556 | 0.000 |
| 10 | | | | | | | | | | |
| 11 | | Index1 | 1 | | Objective | 0.00000 | | | | |
| 12 | | | | | | | | | | |
| 13 | | Weights | | | | | | | | |
| 14 | | | | | | | | | | |

**Fig. 5.12** Setting Excel to solve Eq. 5.30

1. Copy the 9 columns of Table 5.1 on an Excel sheet into cells A1:I9, as Fig. 5.12 depicts.
2. Label B11 as ‘Index’, B13 as ‘Weights’, E11 as ‘Objective’, J1 as ‘Constraints’.
3. Assign number 1 to C11.
4. Assign the following command to F11 (without the quotations ‘’):

   ‘=Sumproduct(Index(G2:I9,C11,0),G13:I13)- Sumproduct(Index(C2:F9, C11,0), C13:F13)’.

5. Assign the following command into J2

   ‘=Sumproduct(G$13:I$13,G2:I2)-Sumproduct(C$13:F$13,C2:F2)’.

6. Copy J2 and then paste it into cells J3-J9.
7. Open ‘Solver Parameters’ window.
8. Assign ‘F11’ into ‘Set Objective’ and choose ‘Max’.
9. Assign ‘C13:I13’ into ‘By Changing Variable Cells’.
10. Click on ‘Add’ and assign ‘J2:J9’ into ‘Cell Reference’, then select ‘<=’, and assign ‘0’ into ‘Constraint’.
11. Click on ‘Add’ and assign ‘C13:I13’ into ‘Cell Reference’, then select ‘>=’, and assign ‘0.0000001’ into ‘Constraint’. Then click on ‘OK’.
12. Tick ‘Make Unconstrained Variables Non-Negative’.
13. Choose ‘Simplex LP’ from ‘Select a Solving Method’ (Fig. 5.13).
14. Click on ‘Solve’.
15. From ‘Developer’ in the toolbar menu, click on the ‘Insert’ icon to open the ‘Form Control’ window.

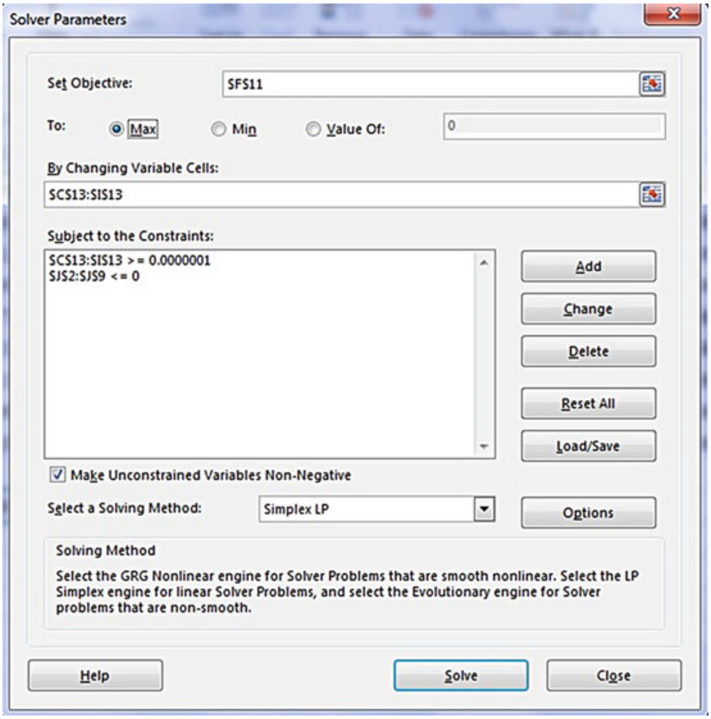



**Fig. 5.13** Setting Solver to solve Eq. 5.30

16. Click on the first icon, ‘Button (Form Control)’, and then click on a place on the Excel sheet.
17. In the opened window with the title ‘Assign Macro’, click on ‘New’. So, the ‘Microsoft Visual Basic for Applications’ window is opened.
18. From the toolbar menu, click on ‘Tools> References...>’ and make sure ‘Solver’ is ticked, and then ‘OK’.
19. Inside of the ‘Microsoft Visual Basic for Applications’ window, write the following commands between ‘Sub Button1_Click ()’ and ‘End Sub’ as Fig. 5.14 depicts.

```
For i = 1 To 8
   Range("C11") = i
   SolverSolve Userfinish:=True
   Range("K" & i + 1) = Range("F11")
   Range("C13:I13").Copy
   Range("L" & i + 1).Select
   Selection.PasteSpecial Paste:=xlPasteValues
Next i
```

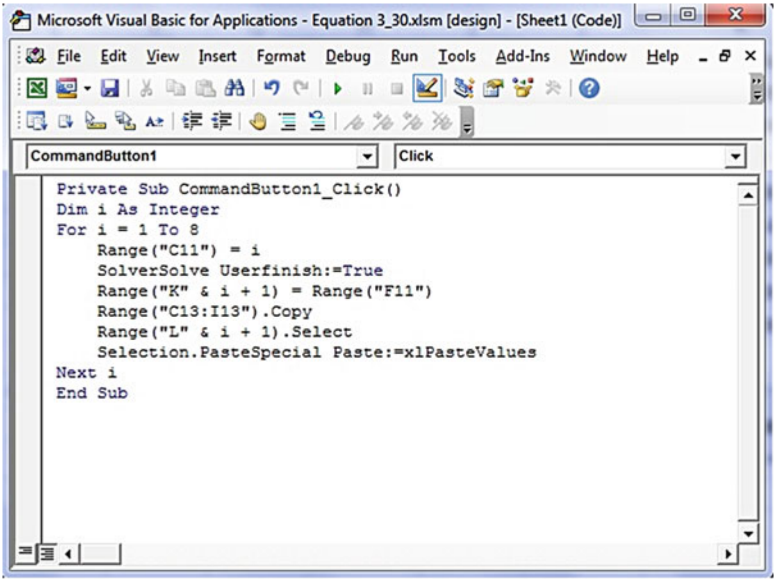



**Fig. 5.14** Setting VBA to solve Eq. 5.30

20. Close the 'Microsoft Visual Basic for Applications' window.
21. Click on the small rectangle which was automatically made in the Excel sheet by step 17.
22. The optimal objective of airports by solving Eq. 5.30 in Form 3 are represented into cells K2:K9 and the corresponded sets of weights are displayed in cells L2:R9, as shown in Fig. 5.15.

Table 5.21 illustrates the optimal weights when Eq. 5.30 is solved for each airport.

The relative scores corresponded to the sets of weights in Table 5.21 are also represented in Table 5.22. In addition, Table 5.23 compares the maximum value of the relative scores of each airport, which are calculated by Forms 1, 2 and 3 in Tables 5.13, 5.17 and 5.21, respectively.

The maximum scores of the relative scores of C and E in Form 3 are less than the corresponded scores in Forms 1 and 2, respectively. This phenomenon is not yielded in every example, and it is possible that a measured score by Form 3 becomes greater than corresponded measured scores by Forms 1 and 2.

Form 3 by Eqs. 5.29 and 5.30 may not represent the optimal solution of Eq. 5.28. In other words, there are two important differences between Eqs. 5.28 and 5.30. The most important difference is that the objective in Eq. 5.28 is not equivalent with the objective in Eq. 5.30. In order to give a counterexample to prove this statement, suppose that two sets of numbers, such as, $\{1, 1.6\}$ and $\{0.01, 0.02\}$, are given. As

| K | L | M | N | O | P | Q | R |
|---|---|---|---|---|---|---|---|
| Optimal Objective | $w^{*-}_1$ | $w^{*-}_2$ | $w^{*-}_3$ | $w^{*-}_4$ | $w^{*+}_1$ | $w^{*+}_2$ | $w^{*+}_3$ |
| 0.000000 | 0.0000001 | 0.0000027 | 0.0000001 | 0.0000002 | 0.0000001 | 0.0000001 | 0.0000066 |
| 0.000000 | 0.0005407 | 0.0000008 | 0.0000001 | 0.0000001 | 0.0000001 | 0.0000001 | 0.0000001 |
| -0.030814 | 0.0000001 | 0.0000018 | 0.0000029 | 0.0000001 | 0.0000001 | 0.0000001 | 0.0000001 |
| 0.000000 | 0.0000001 | 0.0000089 | 0.0004608 | 0.0000051 | 0.0003147 | 0.0000001 | 0.0000001 |
| -0.054142 | 0.0000001 | 0.0000018 | 0.0000029 | 0.0000001 | 0.0000001 | 0.0000001 | 0.0000001 |
| 0.000000 | 0.0000001 | 0.0000022 | 0.0000001 | 0.0000002 | 0.0000001 | 0.0000001 | 0.0000001 |
| 0.000000 | 0.0000001 | 0.0000015 | 0.0000289 | 0.0000001 | 0.0000141 | 0.0000001 | 0.0000001 |
| 0.000000 | 0.0000001 | 0.0000022 | 0.0000001 | 0.0000002 | 0.0000001 | 0.0000001 | 0.0000001 |

Run

**Fig. 5.15** The results of solving Eq. 5.30

can be seen, $1 \div 1.6 = 0.625$ which is greater than $0.01 \div 0.02 = 0.5$, whereas $1 - 1.6 = -0.6$ which is less than $0.01 - 0.02 = -0.01$. Therefore, if there are only these two sets, as the solutions for Eqs. 5.28 and 5.30, then the first set is the only optimal solution for Eq. 5.28 and the second set is the only optimal solution for Eq. 5.30. In addition, the magnitude between the linear combination of the output factors, $\sum_{k=1}^{3} y_{lk} w_k^+$, and the linear combination of the input factors, $\sum_{j=1}^{4} x_{lj} w_j^-$, is not measured by Form 3. For instance, both $12 - 10$ and $4 - 2$ are 2, whereas 12/10 and 4/2 are 1.2 and 2, respectively. It is now clear that the weights in Table 5.21 are the optimal solutions for Eq. 5.30 and they may not be optimal for Eq. 5.28, that is, the results in Table 5.22 may not be the optimal solutions for Eq. 5.28.

Another difference between Eqs. 5.28 and 5.30 is that the last two sets of constraints in Eq. 5.30 are different from the corresponded sets of constraints in Eq. 5.28. This means that all the weights in Eq. 5.30 are restricted to become as greater than equal to 0.0000001, and none of the weights can select 0. In contrast, at least one of the corresponded weights to the input factors should be positive in Eq. 5.28.

For instance, if all the weights in Eq. 5.28 are restricted to be greater than 0.0000001 in Form 1, the maximum scores of the airports C and E will be less than what are represented in Table 5.14. Besides, if only these two constraints, $w_3^- \geq 0.0000001$ and $w_2^+ \geq 0.0000001$, are added to Eq. 5.29 in comparison with Eq. 5.30, the maximum value of the relative scores for C is suggested as the same as that in Forms 1 and 2. As a result, even introducing a very small value to restrict all or some of the weights $w_j^-$ and $w_k^+$ to become as positive, the optimal solutions for Eq. 5.29 are changed. For instance, if $w_1^- \geq 0.0000001$, $w_4^- \geq 0.0000001$ and $w_1^+ \geq 0.0000001$ in Eq. 5.29, the maximum value of the relative scores of E is 0.084289685. If it is supposed that at least one of the weights

**Table 5.21** The optimal weights from Eq. 5.30

| Airport | $w_1^{-*}$ | $w_2^{-*}$ | $w_3^{-*}$ | $w_4^{-*}$ | $w_1^{+*}$ | $w_2^{+*}$ | $w_3^{+*}$ |
|---|---|---|---|---|---|---|---|
| A | 0.0000001 | 0.0000027 | 0.0000001 | 0.0000002 | 0.0000001 | 0.0000001 | 0.0000066 |
| B | 0.0005407 | 0.0000008 | 0.0000001 | 0.0000001 | 0.0000001 | 0.0000001 | 0.0000001 |
| C | 0.0000001 | 0.0000018 | 0.0000029 | 0.0000001 | 0.0000001 | 0.0000001 | 0.0000001 |
| D | 0.0000001 | 0.0000089 | 0.0004608 | 0.0000051 | 0.0003147 | 0.0000001 | 0.0000001 |
| E | 0.0000001 | 0.0000018 | 0.0000029 | 0.0000001 | 0.0000001 | 0.0000001 | 0.0000001 |
| F | 0.0000001 | 0.0000022 | 0.0000001 | 0.0000002 | 0.0000001 | 0.0000001 | 0.0000001 |
| G | 0.0000001 | 0.0000015 | 0.0000289 | 0.0000001 | 0.0000141 | 0.0000001 | 0.0000001 |
| H | 0.0000001 | 0.0000022 | 0.0000001 | 0.0000002 | 0.0000001 | 0.0000001 | 0.0000001 |

**Table 5.22** The relative scores of airports according to Table 5.21

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| A | **1.000000** | 0.934199 | 0.681446 | 0.541001 | 0.434820 | 1.000000 | 0.666773 | 1.000000 |
| B | 0.441500 | **1.000000** | 0.213928 | 0.256997 | 0.074100 | 0.630168 | 0.304583 | 1.000000 |
| C | 0.569457 | 0.898465 | **0.774307** | 0.582481 | 0.445186 | 1.000000 | 0.707161 | 1.000000 |
| D | 0.394683 | 0.706860 | 0.698431 | **1.000000** | 0.320265 | 1.000000 | 1.000000 | 1.000000 |
| E | 0.569457 | 0.898465 | 0.774307 | 0.582481 | **0.445186** | 1.000000 | 0.707161 | 1.000000 |
| F | 0.559752 | 0.896541 | 0.718009 | 0.542720 | 0.407510 | **1.000000** | 0.620546 | 1.000000 |
| G | 0.467305 | 0.774981 | 0.756396 | 0.905093 | 0.379317 | 1.000000 | **1.000000** | 1.000000 |
| H | 0.559752 | 0.896541 | 0.718009 | 0.542720 | 0.407510 | 1.000000 | 0.620546 | **1.000000** |

**Table 5.23** The maximum scores of the relative scores by Forms 1–3

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Form 1 | 1 | 1 | 0.835695 | 1 | 0.509123 | 1 | 1 | 1 |
| Form 2 | 1 | 1 | 0.835695 | 1 | 0.509123 | 1 | 1 | 1 |
| Form 3 | 1 | 1 | 0.774307 | 1 | 0.445186 | 1 | 1 | 1 |

$w_j^-$, for $j = 1, 2, 3, 4$, and at least one of the weights $w_k^+$, for $k = 1, 2, 3$, should be positive, there are 105 different situations to introduce the positive weights in Eq. 5.29, and each situation may introduce different maximum values for the relative scores of the airports.

Note that, every positive real number, such as $r \in \mathbb{R}_+$, can be used instead of 0.0000001 in Eq. 5.30. Because, if $\left(w_1^{-*}, w_2^{-*}, w_3^{-*}, w_4^{-*}, w_1^{+*}, w_2^{+*}, w_3^{+*}\right)$ is an optimal solution for Eq. 5.30, then an optimal solution for the following equation is $10^7 r \times \left(w_1^{-*}, w_2^{-*}, w_3^{-*}, w_4^{-*}, w_1^{+*}, w_2^{+*}, w_3^{+*}\right)$, which let us calculate the same relative score for airport number $l\,(l = 1, 2, \ldots, 8)$.

$$
\begin{aligned}
&\max\ \left(\textstyle\sum_{k=1}^{3} y_{lk} w_k^+ - \sum_{j=1}^{4} x_{lj} w_j^-\right),\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{3} y_{ik} w_k^+ / \sum_{j=1}^{4} x_{ij} w_j^-, \quad \text{for } i = 1, 2, \ldots, 8,\\
&w_j^+ \geq r, \quad \text{for } j = 1, 2, 3, 4,\\
&w_k^- \geq r, \quad \text{for } k = 1, 2, 3.
\end{aligned}
\tag{5.31}
$$

In short, Form 3 covers both Forms 1 and 2 while the same constraints in Eqs. 5.25 and 5.26 are considered for Eq. 5.29 as well. None of the introduced forms may solve Eq. 5.24. Eq. 5.30 estimates Eq. 5.24 without assessing the magnitude of the linear combination of the output factors in comparison with the linear combination of the input factors in the objective of Eq. 5.24. Different restrictions to introduce positive weights may also change the optimal solutions for Eqs. 5.24 and 5.29. On the other hand, the main idea and the exact motivation for all these efforts is to find several different sets of weights which may lead the researcher to introduce a set of weights in order to measure the relative scores of the airports with a lower risk. In other words, these optimization approaches neither provide the relative scores for the airports nor introduce the airports which have done the job well. The provided scores by Forms 1–3 are not relatively meaningful and should not be used to rank or benchmark the airports. In order to rectify these problems, a stronger methodology different with these Forms 1–3 will be illustrated in the next chapter.

### *5.6.5 Minimum Value of the Relative Scores*

Similar to Chapter 2, the minimum values of the relative scores of the airports can be measured. For each airport number $l\,(l = 1, 2, \ldots, 8)$, the following fractional programming should be solved, (where $j = 1, 2, 3, 4$, and $k = 1, 2, 3$), in order to measure the minimum values of the relative scores for that airport. Note that the minimum value of the relative scores of an airport is not a relative score for that airport in comparison with the minimum values of the relative scores of other airports.

$$\min\left\{\frac{\sum\limits_{k=1}^{3} y_{lk}w_k^+ / \sum_{j=1}^{4} x_{lj}w_j^-}{\max\left\{\sum_{k=1}^{3} y_{ik}w_k^+ / \sum_{j=1}^{4} x_{ij}w_j^- \ : \ i = 1, 2, \ldots, 8\right\}} : w_j^- \geq 0, w_k^+ \geq 0\right\}. \tag{5.32}$$

Without loss of generality, assume that $\left(\sum_{k=1}^{3} y_{lk}w_k^+ / \sum_{j=1}^{4} x_{lj}w_j^-\right) = 1$, for $l$ $(l = 1, 2, \ldots, 8)$, thus the following equation should be solved for $i = 1, 2, \ldots, 8$.

$$\begin{aligned}
&\min \left(\textstyle\sum_{k=1}^{3} y_{ik}w_k^+ / \sum_{j=1}^{4} x_{ij}w_j^-\right),\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{3} y_{lk}w_k^+ = \sum_{j=1}^{4} x_{lj}w_j^-,\\
&w_j^- \geq 0, \quad \text{for } j = 1, 2, 3, 4,\\
&w_k^+ \geq 0, \quad \text{for } k = 1, 2, 3.
\end{aligned} \tag{5.33}$$

One way to solve Eq. 5.33 linearly, for $i = 1, 2, \ldots, 8$, is to assume $\sum_{j=1}^{4} x_{ij}w_j^- = 1$, as Eq. 5.34 illustrates. Hence the optimal solutions of the objective in Eq. 5.34 are measured for $i = 1, 2, \ldots, 8$, and subsequently the inverse of the maximum solution is the minimum value of the relative scores of $l\,(l = 1, 2, \ldots, 8)$. Note that, Eq. 5.34 should be solved 48 times to represent the minimum values of the relative scores of the airports A-H.

$$\begin{aligned}
&\min \textstyle\sum_{k=1}^{3} y_{ik}w_k^+,\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{3} y_{lk}w_k^+ = \sum_{j=1}^{4} x_{lj}w_j^-,\\
&\textstyle\sum_{j=1}^{4} x_{ij}w_j^- = 1,\\
&w_j^- \geq 0, \quad \text{for } j = 1, 2, 3, 4,\\
&w_k^+ \geq 0, \quad \text{for } k = 1, 2, 3.
\end{aligned} \tag{5.34}$$

The following steps can be used to solve Eq. 5.34 by the Microsoft Excel Solver software.

1. Copy the 9 columns of Table 5.1 on an Excel sheet into cells A1:I9, as Fig. 5.16 depicts.
2. Label B11 as 'Index1', E11 as 'Index2', H11 as 'Ratio', B13 as 'Weights', B15 as 'Objective', E15 as 'Constraint', H15 as 'Constraint', J1 as 'Reference Set

C15 =SUMPRODUCT(INDEX(G2:I9,F11,0),G13:I13)

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 1 | N | Airport | Area | Apron | Terminal | Runway | Flights | Passengers | Cargo |
| 2 | 1 | A | 1,200 | 304,182 | 45,600 | 353,610 | 30,707 | 4,030,859 | 74,184 |
| 3 | 2 | B | 503 | 213,729 | 38,778 | 348,120 | 46,875 | 4,783,120 | 19,050 |
| 4 | 3 | C | 800 | 41,003 | 11,800 | 269,955 | 15,608 | 1,039,967 | 1,587 |
| 5 | 4 | D | 1,041 | 112,464 | 21,050 | 395,730 | 39,871 | 1,744,524 | 4,919 |
| 6 | 5 | E | 1,002 | 30,000 | 8,000 | 192,330 | 4,887 | 427,974 | 1,574 |
| 7 | 6 | F | 478 | 63,000 | 23,000 | 389,115 | 41,088 | 2,165,572 | 5,414 |
| 8 | 7 | G | 481 | 47,210 | 9,300 | 268,995 | 19,010 | 971,313 | 3,826 |
| 9 | 8 | H | 1,346 | 503,274 | 76,370 | 421,305 | 129,153 | 11,709,741 | 39,556 |
| 10 | | | | | | | | | |
| 11 | | Index1 | 1 | | Index2 | 6 | | Ratio | 0.154785 |
| 12 | | | | | | | | | |
| 13 | | Weights | 0.00000000 | 0.00001587 | 0.00000000 | 0.00000000 | 0.00015724 | 0.00000000 | 0.00000000 |
| 14 | | | | | | | | | |
| 15 | | Objective | 6.4605661 | | Constraint | 1 | | Constraint | 0.000000 |
| 16 | | | | | | | | | |

**Fig. 5.16** Setting Excel to solve Eq. 5.34

Number', K1 as 'Optimal Score', L1 as '$w_1^{-*}$', M1 as '$w_2^{-*}$', N1 as '$w_3^{-*}$', O1 as '$w_4^{-*}$', P1 as '$w_1^{+*}$', Q1 as '$w_2^{+*}$' and R1 as '$w_3^{+*}$'.

3. Assign number 1 to C11 and F11.
4. Assign the command '=Sumproduct(Index(G2:I9,F11,0),G13:I13)' into C15, (without the quotation '').
5. Assign the command '=Sumproduct(Index(C2:F9,F11,0),C13:F13)' into F15.
6. Assign the following command into I15

   '= Sumproduct(Index (G2:I9,C11,0),G13:I13)- Sumproduct(Index (C2:F9, C11,0),C13:F13)'.

7. Assign the following command into I11

   '= Sumproduct(Index(C2:F9,F11,0),C13:F13)/Sumproduct(Index (G2:I9, F11,0),G13:I13)'.

8. Open 'Solver Parameters' window (Fig. 5.17).
9. Assign 'C15' into 'Set Objective' and choose 'Max'.
10. Assign 'C13:I13' into 'By Changing Variable Cells'.
11. Click on 'Add' and assign 'F15' into 'Cell Reference', then select '=', and assign '1' into 'Constraint'.
12. Click on 'Add' and assign 'I15' into 'Cell Reference', then select '=', and assign '0' into 'Constraint'. Then click on 'OK'.
13. Tick 'Make Unconstrained Variables Non-Negative'.
14. Choose 'Simplex LP' from 'Select a Solving Method'.
15. Click on 'Solve'.
16. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.

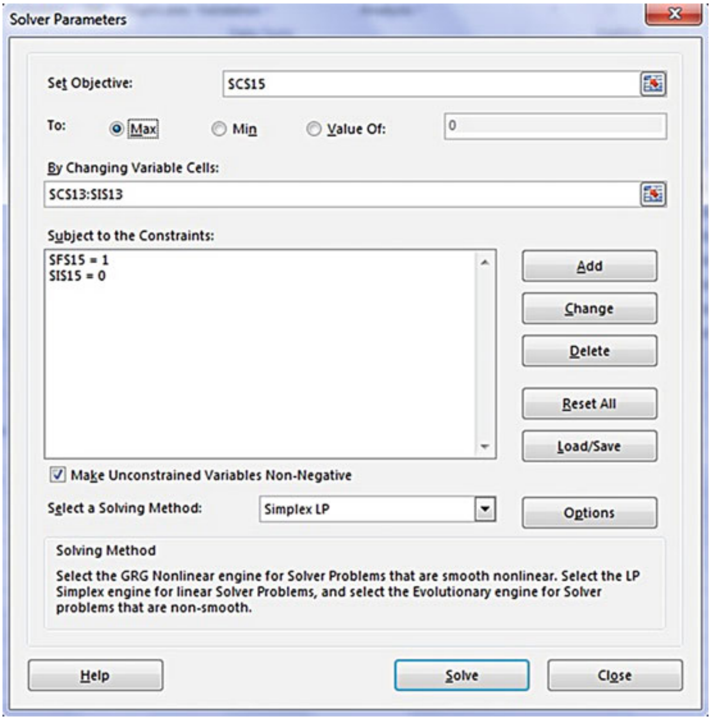



**Fig. 5.17** Setting Solver to solve Eq. 5.34

17. Click on the first icon, ‘Button (Form Control)’, and then click on a place on the Excel sheet.
18. In the opened window with the title ‘Assign Macro’, click on ‘New’. So, the ‘Microsoft Visual Basic for Applications’ window is opened.
19. From the toolbar menu, click on ‘Tools> References...>’ and make sure ‘Solver’ is ticked, and then ‘OK’.
20. Inside of the ‘Microsoft Visual Basic for Applications’ window, write the following commands between ‘Sub Button1_Click ()’ and ‘End Sub’ as Fig. 5.18 depicts.

```
Dim i, j As Integer
For i = 1 To 8
   Range("C11") = i
   Range("K" & i + 1) = 1
   For j = 1 To 8
      Range("F11") = j
      SolverSolve Userfinish:=True
      If Range("I11") &lt; Range("K" & i + 1) Then
        Range("K" & i + 1) = Range("I11")
```

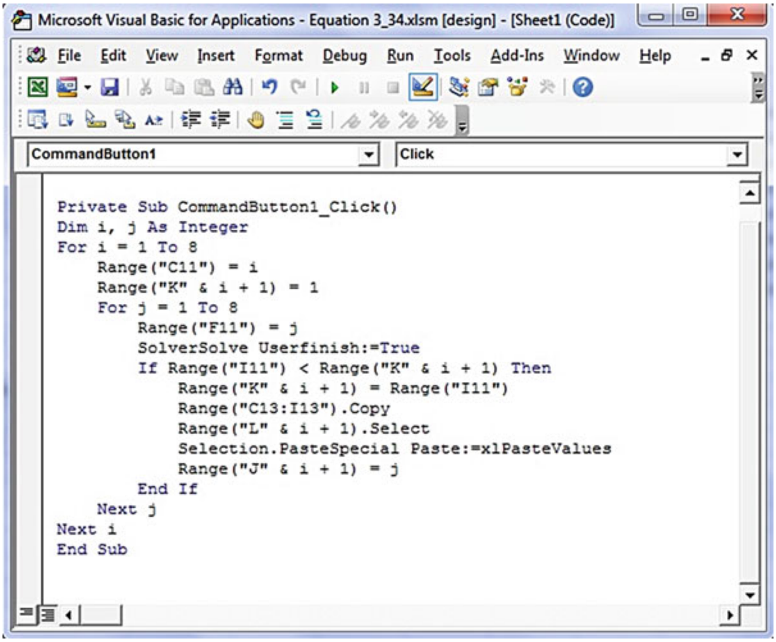



**Fig. 5.18** Setting VBA to solve Eq. 5.34

```
            Range("C13:I13").Copy
            Range("L" & i + 1).Select
            Selection.PasteSpecial Paste:=xlPasteValues
            Range("J" & i + 1) = j
        End If
    Next j
Next i
```

21. Close the 'Microsoft Visual Basic for Applications' window.
22. Click on the small rectangle which was automatically made in the Excel sheet by step 17.
23. The optimal score by solving Eq. 5.34 are represented into cells K2:K9 and the corresponded sets of weights are displayed in cells L2:R9. Column J also shown the reference set, the airport that the related minimum score is derived, for each airport (Fig. 5.19).

Table 5.24 illustrates the minimum values of the relative scores for each airport according to Eq. 5.34.

Moreover, airport A reaches the minimum value of its relative scores while the relative score of F is 1, and airports B-H reach the minimum values of their relative scores while A has the relative score of 1. These outcomes can classify the airports into two groups, as mentioned in Table 5.10.

| J | K | L | M | N | O | P | Q | R |
|---|---|---|---|---|---|---|---|---|
| j | MinScore | $w^-_1$ | $w^-_2$ | $w^-_3$ | $w^-_4$ | $w^+_1$ | $w^+_2$ | $w^+_3$ |
| 6 | 0.154785 | 0.0000000 | 0.0000159 | 0.0000000 | 0.0000000 | 0.0001572 | 0.0000000 | 0.0000000 |
| 1 | 0.260844 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000028 | 0.0000000 | 0.0000000 | 0.0000517 |
| 1 | 0.028022 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000028 | 0.0000000 | 0.0000000 | 0.0004810 |
| 1 | 0.059251 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000028 | 0.0000000 | 0.0000000 | 0.0002275 |
| 1 | 0.025410 | 0.0008333 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0005305 |
| 1 | 0.066322 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000028 | 0.0000000 | 0.0000000 | 0.0002033 |
| 1 | 0.067798 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000028 | 0.0000000 | 0.0000000 | 0.0001988 |
| 1 | 0.318379 | 0.0000000 | 0.0000000 | 0.0000219 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000423 |

Run

**Fig. 5.19** The results of solving Eq. 5.34

**Table 5.24** The minimum values of the relative scores of airports

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Min | 0.15479 | 0.26084 | 0.02802 | 0.05925 | 0.02541 | 0.06632 | 0.06780 | 0.31838 |

## 5.7 Conclusion

In this chapter, the performance of eight airports, in which each airport has four input factors and three output factors, is evaluated. The introduced concepts in the previous chapters are illustrated and settled for the cases of multiple input factors and multiple output factors. The pros and cons of several mathematical programming are illustrated and the instructions to solve each linear programming with the Microsoft Excel Solver 2013 software are represented. The concept of doing the job well, which is the most important concept to evaluate the airports, is transparently illustrated and it is featured that, while the units of measurement prices/weights/worth of the factors are not available, the decision making is not pure, and at least some user judgment should be provided to rank the airport accurately. The concept of doing the job right depends upon the approaches which are used to create the practical points. In contrast, the concept of doing the job well depends upon the introduced restrictions for the weights. Therefore, while the units of measurement/weights/worth/prices of the factors are unknown, decision making is not pure, but there are several ways to estimate the results, as discussed in this chapter. In order to have accurate results, we need to know how practical points should be generated and how the weights should be restricted by expert judgment.

## 5.8 Exercises

5.1. Add the point, (856, 164,358, 29,237, 329,895, 40,900, 3,359,134, 18,764), as the 9th airport (labeled I) in Table 5.1. Then,

5.1.1. Has airport I done the job right, by

5.1.1.1. the wholly dominant approach (using Eq. 5.12)?
5.1.1.2. the convexity and the wholly dominant approaches (using Eq. 5.17)?
5.1.1.3. the radiate, the convexity and the wholly dominant approaches (using Eq. 5.18)?
5.1.1.4. the inner (outer) radiate, the convexity and the wholly dominant approaches?

5.1.2. Find the rank of airport I using,

5.1.2.1. the extremum measurement approximation.
5.1.2.2. the average measurement approximation.
5.1.2.3. the specified measurement approximation.
5.1.2.4. the extremum measurement approximation, find the rank of airport I.

5.1.3. Find the rank and the maximum value of the relative scores of airport I by:

5.1.3.1. Form 1.
5.1.3.2. Form 2.
5.1.3.3. Form 3 (Eq. 5.30).
5.1.3.4. Form 3 (Eq. 5.31, when $r = 1$).

5.1.4. Find the minimum value of the relative score of airport I by Forms 1, 2 and 3.

5.2. Prove that the introduced airports which have done the job right by applying the convexity approach or the combination of the convexity and the wholly dominant approaches are the same.
5.3. Solve Eq. 5.29 by the Microsoft Excel Solver software and the Microsoft Visual Basic software while at least one of the each input factor and one of the each output factor are positive.
5.4. Find the rank of airport B when it has the minimum value of its relative scores.

# Chapter 6
# The Delta Neighborhood

## 6.1 Introduction

In this chapter, the relative scores, under certain conditions, are measured for the example of eight homogenous airports from the previous chapter. As discussed in the previous chapters, each supposed condition can result different relative scores and ranks for airports. In other words, the relative scores of firms are completely dependent upon the supposed relationship between factors or the supposed weight/worth/price for each factor. The optimization approach is applied while the weights/prices of factors are the same for the airports and one of the conditions such as decreasing the overall input values, increasing overall output values or both decreasing the input values and increasing the output values are assumed. After that, a delta neighborhood of a firm is introduced to find a bridge between the concepts 'doing the job rights' and 'doing the job well'. Mathematical arguments are provided as well as computer programming with detailed illustrations. At the end of this chapter, readers are completely prepared to utilize the performance evaluation of a set of homogenous firms with multiple input factors and multiple output factors.

## 6.2 One Set of Weights

The relative scores of the airports depend on the introduced set of weights, and thus, different sets of weights may introduce different airports as those that have done the job well. Studying about the extremum values of the relative scores is useful to provide several sets of weights in order to nominate the airports which might have done the job well. But, it is usually difficult to select a set of weights from the extremum values of the relative scores and, for this aim, several linear programming should also be solved. This may motivate the researcher to find a method to provide



D. Khezrimotlagh, Y. Chen, *Decision Making and Performance Evaluation Using Data Envelopment Analysis*, International Series in Operations Research & Management Science 269, https://doi.org/10.1007/978-3-319-76345-3_6

only one set of weights to suggest the relative scores for all airports simultaneously; for instance, similar to average estimation measurement in Sect. 5.6.2. However, none of the methods are pure while the weights are not available, and each method may only cover a special purpose which should transparently be noticed. In this section, several different methods are illustrated.

Since the score of each airport number $i\,(i = 1, 2, \ldots, 8)$ is measured by Eq. 5.3, without loss of generality, suppose that each score is less than equal to 1, that is, $\sum_{k=1}^{3} y_{ik} w_k^+ / \sum_{j=1}^{4} x_{ij} w_j^- \leq 1$, for $i = 1, 2, \ldots, 8$. This assumption is always possible. Hence, the following eight inequalities are yielded:

$$\sum_{k=1}^{3} y_{ik} w_k^+ \leq \sum_{j=1}^{4} x_{ij} w_j^-, \quad \text{for} \quad i = 1, 2, \ldots, 8. \tag{6.1}$$

*Method 1* For the first method to find a set of weights, which satisfies all the inequalities in Eq. 6.1, let's suppose that, $\sum_{k=1}^{3} y_{ik} w_k^+ \geq 1$, for $i = 1, 2, \ldots,$ 8. This assumption always yields positive weights for at least one of the output factors and one of the input factors. Now, assume that, the researcher would like to measure the minimum value of the linear combination of the total value of each input factor of the airports, that is, $\sum_{i=1}^{8} \sum_{j=1}^{4} (x_{ij} w_j^-) = \sum_{j=1}^{4} (\sum_{i=1}^{8} x_{ij}) w_j^-$. The following linear programming lets us calculate a set of weights to measure the relative scores of all airports simultaneously.

$$\begin{aligned} &\min \sum_{i=1}^{8} \sum_{j=1}^{4} x_{ij} w_j^-, \\ &\text{Subject to} \\ &\sum_{k=1}^{3} y_{ik} w_k^+ \leq \sum_{j=1}^{4} x_{ij} w_j^-, \quad \text{for} \quad i = 1, 2, \ldots, 8, \\ &\sum_{k=1}^{3} y_{ik} w_k^+ \geq 1, \quad \text{for} \quad i = 1, 2, \ldots, 8, \\ &w_j^- \geq 0, \quad \text{for} \quad j = 1, 2, 3, 4, \\ &w_k^+ \geq 0, \quad \text{for} \quad k = 1, 2, 3. \end{aligned} \tag{6.2}$$

1. Copy the 9 columns of Table 5.1 on an Excel sheet into cells A1:I9, as Fig. 6.1 depicts.
2. Label B11 as 'Weights', J1 as 'Y Constraints', K1 as 'X Constraints', and L1 as 'Relative Scores'.
3. Assign the following command into J2 (without the quotation '')

   '=Sumproduct(G2:I2,G$11:I$11)'.
4. Copy J2 and paste it into J3:J9.
5. Assign the following command into K2 (without the quotation '')

   '=Sumproduct(C2:F2,C$11:F$11)'.

K9 =SUMPRODUCT(C9:F9,$C$11:$F$11)

| | A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | N | Airport | Area | Apron | Terminal | Runway | Flights | Passengers | Cargo | Y Constraints | X Constraints | Relative Scores |
| 2 | 1 | A | 1,200 | 304,182 | 45,600 | 353,610 | 30,707 | 4,030,859 | 74,184 | 16.235574 | 16.235574 | 1.000000 |
| 3 | 2 | B | 503 | 213,729 | 38,778 | 348,120 | 46,875 | 4,783,120 | 19,050 | 11.343714 | 12.105815 | 0.937047 |
| 4 | 3 | C | 800 | 41,003 | 11,800 | 269,955 | 15,608 | 1,039,967 | 1,587 | 2.172979 | 2.774265 | 0.783263 |
| 5 | 4 | D | 1,041 | 112,464 | 21,050 | 395,730 | 39,871 | 1,744,524 | 4,919 | 3.904316 | 6.436905 | 0.606552 |
| 6 | 5 | E | 1,002 | 30,000 | 8,000 | 192,330 | 4,887 | 427,974 | 1,574 | 1.000000 | 1.964161 | 0.509123 |
| 7 | 6 | F | 478 | 63,000 | 23,000 | 389,115 | 41,088 | 2,165,572 | 5,414 | 4.767140 | 4.767140 | 1.000000 |
| 8 | 7 | G | 481 | 47,210 | 9,300 | 268,995 | 19,010 | 971,313 | 3,826 | 2.298698 | 2.750117 | 0.835855 |
| 9 | 8 | H | 1,346 | 503,274 | 76,370 | 421,305 | 129,153 | 11,709,741 | 39,556 | 26.957764 | 26.957764 | 1.000000 |
| 10 | | | | | | | | | | | 73.991742 | |
| 11 | | Weights | 0.000000000 | 0.000037842 | 0.000103613 | 0.000000000 | 0.000000000 | 0.000001914 | 0.000114845 | | | |

**Fig. 6.1** Setting Excel to solve Eq. 6.2

6. Copy K2 and paste it into K3:K9.
7. Assign the command '=Sum(K2:K9)' into K10.
8. Assign the command '=J2/K2' into L2.
9. Copy L2 and paste it into L3:L9.
10. Open 'Solver Parameters' window.
11. Assign 'K10' into 'Set Objective' and choose 'Min'.
12. Assign 'C11:I11' into 'By Changing Variable Cells'.
13. Click on 'Add' and assign 'C11:I11' into 'Cell Reference', then select '>=', and assign '0' into 'Constraint'.
14. Click on 'Add' and assign 'J2:J9' into 'Cell Reference', then select '<=', and assign 'K2:K9' into 'Constraint'.
15. Click on 'Add' and assign 'J2:J9' into 'Cell Reference', then select '>=', and assign '1' into 'Constraint'. Then click on 'OK'.
16. Tick 'Make Unconstrained Variables Non-Negative'.
17. Choose 'Simplex LP' from 'Select a Solving Method'.
18. Click on 'Solve' (Fig. 6.2).

Table 6.1 illustrates the relative scores of the airports in Method 1 with four decimal digits. Three airports A, F and H have the relative scores of 1 and E has the least score.

The weights in Eq. 6.2 can also be restricted to positive values, for example by $w_j^- \geq 1$, $w_k^+ \geq 1$. Table 6.2 represents the relative scores for airports with four decimal digits, where the weights are greater than 1.

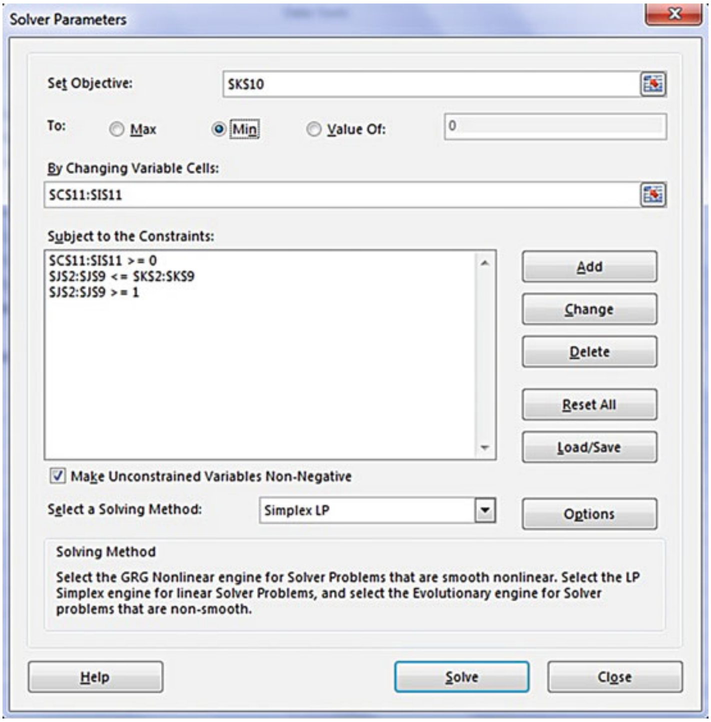



**Fig. 6.2** Setting Solver to solve Eq. 6.2

**Table 6.1** The results by Eq. 6.2

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Relative score | 1.0000 | 0.9370 | 0.7833 | 0.6066 | 0.5091 | 1.0000 | 0.8359 | 1.0000 |

**Table 6.2** The results by Eq. 6.2 where w's are greater than 1

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Relative score | 0.5695 | 0.8985 | 0.7743 | 0.5825 | 0.4452 | 1.0000 | 0.7072 | 1.0000 |

The relative score of A is sharply changed while the weights are restricted to become greater than 1. As illustrated earlier, any restrictions on the weights may change the results.

*Method 2* The previous method does not simultaneously measure the strengths and weaknesses of all of the input and output factors. Hence, the researcher can replace the objective of Eq. 6.2 with $\sum_{i=1}^{8}\left(\sum_{j=1}^{4} x_{ij}w_j^- - \sum_{k=1}^{3} y_{ik}w_k^+\right)$, as Eq. 6.3 illustrates.

**Table 6.3** The results by Eq. 6.3

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Relative score | 1.0000 | 0.8587 | 0.7422 | 0.8068 | 0.4409 | 1.0000 | 1.0000 | 1.0000 |

**Table 6.4** The results by Eq. 6.4

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Relative score | 1.0000 | 1.0000 | 0.1485 | 0.2995 | 0.0535 | 0.6145 | 0.3347 | 1.0000 |

$$\begin{aligned}
&\min \textstyle\sum_{i=1}^{8}\left(\sum_{j=1}^{4} x_{ij}w_j^- - \sum_{k=1}^{3} y_{ik}w_k^+\right),\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{3} y_{ik}w_k^+ \le \sum_{j=1}^{4} x_{ij}w_j^-, \quad \text{for} \quad i = 1, 2, \ldots, 8,\\
&\textstyle\sum_{k=1}^{3} y_{ik}w_k^+ \ge 1, \quad \text{for} \quad i = 1, 2, \ldots 8,\\
&w_j^- \ge 0, \quad \text{for} \quad j = 1, 2, 3, 4,\\
&w_k^+ \ge 0, \quad \text{for} \quad k = 1, 2, 3.
\end{aligned} \tag{6.3}$$

Equation 6.3 maximizes the linear combinations of the total values of each output factor of the airports and minimizes the linear combinations of the total values of each input factor of the airports. The lower bound for the objective in Eq. 6.3 is 0, and the model is always feasible. The results of Eq. 6.3 are illustrated in Table 6.3. There are four airports A, F, G and H with the relative score of 1 where airports E and C have the least relative scores, respectively.

*Method 3* For the third method suppose that, $\sum_{j=1}^{4} x_{ij}w_j^- \le 1$, for $i = 1, 2, \ldots, 8$, where the maximum value of $\sum_{i=1}^{8} \sum_{k=1}^{3} y_{ik}w_j^+$ is interested, by Eq. 6.4. Similar to the previous method, at least one of the corresponded weights to the input factors and one of the corresponded weights to the output factors are positive. Moreover, the constraints $\sum_{j=1}^{4} x_{ij}w_j^- \le 1$, for $i = 1, 2, \ldots, 8$, do not let Eq. 6.4 become unbounded and the model is always feasible.

$$\begin{aligned}
&\max \textstyle\sum_{i=1}^{8} \sum_{k=1}^{3} y_{ik}w_k^+,\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{3} y_{ik}w_k^+ \le \sum_{j=1}^{4} x_{ij}w_j^-, \quad \text{for} \quad i = 1, 2, \ldots, 8,\\
&\textstyle\sum_{j=1}^{4} x_{ik}w_j^- \le 1, \quad \text{for} \quad i = 1, 2, \ldots 8,\\
&w_j^- \ge 0, \quad \text{for} \quad j = 1, 2, 3, 4,\\
&w_k^+ \ge 0, \quad \text{for} \quad k = 1, 2, 3.
\end{aligned} \tag{6.4}$$

As can be seen, the results in Table 6.4 are different from those in Tables 6.1, 6.2 and 6.3. A negligible difference to introduce a constraint may yield different results. Eq. 6.4

**Table 6.5** The results by Eq. 6.5 where w's are greater than $10^{-9}$

| Airport | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Relative score | 1.0000 | 0.9347 | 0.7327 | 0.5772 | 0.4711 | 1.0000 | 0.7507 | 1.0000 |

maximizes the linear combinations of the total values of each output factor of the airports. Hence, similar to Method 2, the following method can also be introduced.

*Method 4* The objective in Eq. 6.4 can be replaced by the following equation, $\sum_{i=1}^{8}\sum_{k=1}^{3} y_{ik}w_k^+ - \sum_{j=1}^{4} x_{ij}w_j^-$, as Eq. 6.5 illustrates.

$$\begin{aligned} &\max\sum_{i=1}^{8}\left(\sum_{k=1}^{3} y_{ik}w_k^+ - \sum_{j=1}^{4} x_{ij}w_j^-\right),\\ &\text{Subject to}\\ &\sum_{k=1}^{3} y_{ik}w_k^+ \le \sum_{j=1}^{4} x_{ij}w_j^-, \quad \text{for} \quad i = 1,2,\ldots,8,\\ &\sum_{j=1}^{4} x_{ij}w_j^- \le 1, \quad \text{for} \quad i = 1,2,\ldots 8,\\ &w_j^- \ge 0, \quad \text{for} \quad j = 1,2,3,4,\\ &w_k^+ \ge 0, \quad \text{for} \quad k = 1,2,3. \end{aligned} \tag{6.5}$$

The optimal weights for the above equation are always 0 and there is a need to restrict the weights to become as positive. For example, suppose that the weights are greater than a very small positive real number such as $0.000000001 = 10^{-9}$ in Eq. 6.5. The relative scores of airports are illustrated in Table 6.5.

In short, there are many different methods to introduce one set of weights to measure the relative scores of airports, and the results of each method may suggest different relative scores for each airport. While the weights are unknown or the relationship between input factors and/or the relationship between output factors are not available, decision making cannot be precise. Nonetheless, studying about possible relationships between factors is useful and can at least provide a transparent view to let decision makers find strengths and weaknesses of airports in each specified approach.

## 6.3 Relationship Between Two Approaches

As represented in Sect. 5.4, Eq. 5.18 illustrates that C and E are dominated by a linear combination of F and H. In previous section, it is also shown that there are not any sets of positive weights which let C and E get greater relative scores than those of other airports. Are there any relationships between these two approaches? In other words, is there any relationship between Eqs. 5.18 and 5.30?

In order to answer the above question, suppose that Eq. 5.18 is rewritten for $l$ ($l = 1, 2, \ldots, 8$) as follows:

$$\begin{aligned}
&\max\ s_1^- + s_2^- + s_3^- + s_4^- + s_1^+ + s_2^+ + s_3^+,\\
&\text{Subject to}\\
&x_{11}\lambda_1 + x_{21}\lambda_2 + \cdots + x_{81}\lambda_8 + s_1^- = x_{l1},\\
&x_{12}\lambda_1 + x_{22}\lambda_2 + \cdots + x_{82}\lambda_8 + s_2^- = x_{l2},\\
&x_{13}\lambda_1 + x_{23}\lambda_2 + \cdots + x_{83}\lambda_8 + s_3^- = x_{l3},\\
&x_{14}\lambda_1 + x_{24}\lambda_2 + \cdots + x_{84}\lambda_8 + s_4^- = x_{l4},\\
&y_{11}\lambda_1 + y_{21}\lambda_2 + \cdots + y_{81}\lambda_8 - s_1^+ = y_{l1},\\
&y_{12}\lambda_1 + y_{22}\lambda_2 + \cdots + y_{82}\lambda_8 - s_2^+ = y_{l2},\\
&y_{13}\lambda_1 + y_{23}\lambda_2 + \cdots + y_{83}\lambda_8 - s_3^+ = y_{l3},\\
&\lambda_i \geq 0, \quad \text{for} \quad i = 1, 2, \ldots, 8,\\
&s_j^- \geq 0, \quad \text{for} \quad j = 1, 2, 3, 4,\\
&s_k^+ \geq 0, \quad \text{for} \quad k = 1, 2, 3.
\end{aligned} \tag{6.6}$$

From the first seven constraints in Eq. 6.6, the following seven equalities are simply deduced.

$$\begin{aligned}
s_1^- &= \phantom{-}x_{l1} - x_{11}\lambda_1 - x_{21}\lambda_2 - \cdots - x_{81}\lambda_8,\\
s_2^- &= \phantom{-}x_{l2} - x_{12}\lambda_1 - x_{22}\lambda_2 - \cdots - x_{82}\lambda_8,\\
s_3^- &= \phantom{-}x_{l3} - x_{13}\lambda_1 - x_{23}\lambda_2 - \cdots - x_{83}\lambda_8,\\
s_4^- &= \phantom{-}x_{l4} - x_{14}\lambda_1 - x_{24}\lambda_2 - \cdots - x_{84}\lambda_8,\\
s_1^+ &= -y_{l1} + y_{11}\lambda_1 + y_{21}\lambda_2 + \cdots + y_{81}\lambda_8,\\
s_2^+ &= -y_{l2} + y_{12}\lambda_1 + y_{22}\lambda_2 + \cdots + y_{82}\lambda_8,\\
s_3^+ &= -y_{l3} + y_{13}\lambda_1 + y_{23}\lambda_2 + \cdots + y_{83}\lambda_8.
\end{aligned} \tag{6.7}$$

Assume that the first equality in Eq. 6.7 is multiplied by $w_1^-$, a positive real number, and so on for other equations by the positive real numbers $w_2^-$, $w_3^-$, $w_4^-$, $w_1^+$, $w_2^+$, and $w_3^+$, respectively, as Eq. 6.8 illustrates.

$$\begin{aligned}
w_1^- s_1^- &= \phantom{-}x_{l1}w_1^- - x_{11}\lambda_1 w_1^- - x_{21}\lambda_2 w_1^- - \cdots - x_{81}\lambda_8 w_1^-,\\
w_2^- s_2^- &= \phantom{-}x_{l2}w_2^- - x_{12}\lambda_1 w_2^- - x_{22}\lambda_2 w_2^- - \cdots - x_{82}\lambda_8 w_2^-,\\
w_3^- s_3^- &= \phantom{-}x_{l3}w_3^- - x_{13}\lambda_1 w_3^- - x_{23}\lambda_2 w_3^- - \cdots - x_{83}\lambda_8 w_3^-,\\
w_4^- s_4^- &= \phantom{-}x_{l4}w_4^- - x_{14}\lambda_1 w_4^- - x_{24}\lambda_2 w_4^- - \cdots - x_{84}\lambda_8 w_4^-,\\
w_1^+ s_1^+ &= -y_{l1}w_1^+ + y_{11}\lambda_1 w_1^+ + y_{21}\lambda_2 w_1^+ + \cdots + y_{81}\lambda_8 w_1^+,\\
w_2^+ s_2^+ &= -y_{l2}w_2^+ + y_{12}\lambda_1 w_2^+ + y_{22}\lambda_2 w_2^+ + \cdots + y_{82}\lambda_8 w_2^+,\\
w_3^+ s_3^+ &= -y_{l3}w_3^+ + y_{13}\lambda_1 w_3^+ + y_{23}\lambda_2 w_3^+ + \cdots + y_{83}\lambda_8 w_3^+,
\end{aligned} \tag{6.8}$$

From Eq. 6.8, the following equation is yielded.

$$\begin{aligned}\sum_{j=1}^{4} w_j^- s_j^- + \sum_{k=1}^{3} w_k^+ s_k^+ = & \left(\sum_{j=1}^{4} x_{lj} w_j^- - \sum_{k=1}^{3} y_{lk} w_k^+\right) \\ & -\lambda_1\left(\sum_{j=1}^{4} x_{1j} w_j^- - \sum_{k=1}^{3} y_{1k} w_k^+\right) \\ & -\lambda_2\left(\sum_{j=1}^{4} x_{2j} w_j^- - \sum_{k=1}^{3} y_{2k} w_k^+\right) - \ldots \\ & -\lambda_8\left(\sum_{j=1}^{4} x_{8j} w_j^- - \sum_{k=1}^{3} y_{8k} w_k^+\right)\end{aligned}$$

The above equation is less than equal to $\sum_{j=1}^{4} x_{ij} w_j^- - \sum_{k=1}^{3} y_{ik} w_k^+$, where $\lambda_i \geq 0$ and $\sum_{j=1}^{4} x_{ij} w_j^- - \sum_{k=1}^{3} y_{ij} w_j^+ \geq 0$, for $i = 1, 2, \ldots, 8$. Now, if $w_j^- \geq 1$, for $j = 1$, 2, 3, 4 and $w_k^+ \geq 1$, for $k = 1, 2, 3$, hence

$$\begin{aligned}&\left(\sum_{j=1}^{4} s_j^- + \sum_{k=1}^{3} s_k^+\right) \leq \sum_{j=1}^{4} w_j^- s_j^- + \sum_{k=1}^{3} w_k^+ s_k^+ = \\ &\left(\sum_{j=1}^{4} x_{lj} w_j^- - \sum_{k=1}^{3} y_{lk} w_k^+\right) - \lambda_1\left(\sum_{j=1}^{4} x_{1j} w_j^- - \sum_{k=1}^{3} y_{1k} w_k^+\right) \\ &-\lambda_2\left(\sum_{j=1}^{4} x_{2j} w_j^- - \sum_{k=1}^{3} y_{2k} w_k^+\right) - \cdots - -\lambda_8\left(\sum_{j=1}^{4} x_{8j} w_j^- - \sum_{k=1}^{3} y_{8k} w_k^+\right) \\ &\qquad \leq \left(\sum_{j=1}^{4} x_{lj} w_j^- - \sum_{k=1}^{3} y_{lk} w_k^+\right).\end{aligned} \tag{6.9}$$

The left-hand side of Eq. 6.9, $\sum_{j=1}^{4} s_j^- + \sum_{k=1}^{3} s_k^+$, represents the objective of Eq. 6.6 with the upper bound as $\sum_{j=1}^{4} w_j^- s_j^- + \sum_{k=1}^{3} w_k^- s_k^+$. On the other hand, the equation, $\sum_{j=1}^{4} w_j^- s_j^- + \sum_{k=1}^{3} w_k^+ s_k^+$, is the lower bound for the right-hand side of Eq. 6.9, that is, $\sum_{j=1}^{4} x_{lj} w_j^- - \sum_{k=1}^{3} y_{lk} w_k^+$. Since the left-hand side of Eq. 6.9 is maximized, there is a need to find the weights, $w_j^-$ and $w_k^+$, to introduce a very strong upper bound. The upper bound can be found by minimizing the right-hand side of Eq. 6.9, given by the following linear programming.

$$\begin{aligned}&\min\left(\sum_{j=1}^{4} x_{lj} w_j^- - \sum_{k=1}^{3} y_{lk} w_k^+\right), \\ &\text{Subject to} \\ &\sum_{j=1}^{4} x_{lj} w_j^- - \sum_{k=1}^{3} y_{lk} w_k^+ \geq 0, \quad \text{for} \quad i = 1, 2, \ldots, 8, \\ &w_j^- \geq 1, \quad \text{for} \quad j = 1, 2, 3, 4, \\ &w_k^+ \geq 1, \quad \text{for,} \quad k = 1, 2, 3.\end{aligned} \tag{6.10}$$

Since, $\min(\sum_{j=1}^{4} x_{lj} w_j^- - \sum_{k=1}^{3} y_{lk} w_k^+) = -\max(\sum_{k=1}^{3} y_{lk} w_k^+ - \sum_{j=1}^{4} x_{lj} w_j^-)$, then, Eq. 6.10 is equivalent with Eqs. 5.30 and 5.31. If Eq. 6.6 (Eq. 5.18) is called the *primal* linear programming, Eq. 6.10 (Eq. 5.31) is called the *dual* linear programming. An optimal solution for a feasible and bounded primal linear programming is an optimal solution for the feasible and bounded dual linear programming.

For instance, the optimal slacks for airport C by Eq. 5.18 are shown in Eq. 5.20 with seven decimal digits, which yields that, the optimal objective of Eq. 5.18 becomes as 308,138.7537662. The optimal weights for airport C by Eq. 5.30 are also represented in Table 5.21 with seven decimal digits, which yield that the optimal objective of Eq. 5.30 becomes as −0.03081387537662 with fourteen decimal digits.

As a result, the optimal solution for the objective of Eq. 5.18 is equal to −0.0000001 times to the optimal objective of Eq. 5.30, which means these two equations are equivalent.

Section 5.6.4.3 explains that some of the weights in Eq. 5.29 can also select 0 or the restrictions for the weights can be different from one weight to another. In other words, Eq. 6.10 is an especial case of the following equation, where $W_j^- \geq 0$, for $j = 1, 2, 3, 4$ and $W_k^- \geq 0$, for $k = 1, 2, 3$.

$$\begin{aligned}
&\min\left(\sum_{j=1}^{4} x_{lj}w_j^- - \sum_{k=1}^{3} y_{lk}w_k^+\right),\\
&\text{Subject to}\\
&\sum_{j=1}^{4} x_{lj}w_j^- - \sum_{k=1}^{3} y_{lk}w_k^+ \geq 0, \quad \text{for} \quad i = 1, 2, \ldots, 8,\\
&w_j^- \geq W_j^-, \quad \text{for} \quad j = 1, 2, 3, 4,\\
&w_k^+ \geq W_k^- \quad \text{for,} \quad k = 1, 2, 3.
\end{aligned} \tag{6.11}$$

The dual linear programming for Eq. 6.11 is given as:

$$\begin{aligned}
&\min\left(\sum_{j=1}^{4} W_j^- s_j^- + \sum_{k=1}^{3} W_k^+ s_k^+\right),\\
&\text{Subject to}\\
&\sum_{i=1}^{8} x_{ij}\lambda_i + s_j^- = x_{lj}, \quad \text{for} \quad j = 1, 2, 3, 4,\\
&\sum_{i=1}^{8} y_{ik}\lambda_i - s_k^+ = y_{lk}, \quad \text{for} \quad k = 1, 2, 3,\\
&\lambda_i \geq 0, \quad \text{for} \quad i = 1, 2, \ldots, 8,\\
&s_j^- \geq 0, \quad \text{for} \quad j = 1, 2, 3, 4,\\
&s_k^- \geq 0 \quad \text{for,} \quad k = 1, 2, 3.
\end{aligned} \tag{6.12}$$

As can be seen, for instance, while one of the $W_j^- (j = 1, 2, 3, 4)$ is equal to 0 in Eq. 6.11, the corresponded coefficient of the slack $s_j^-$ in the objective of Eq. 6.12 is equal to 0, which means, the optimization is measured based on the strengths or the weaknesses of other factors. Of course, the slack $s_j^-$ may not be 0 when $W_j^- = 0$, and the optimal value of $s_j^-$ is measured from the constraints in Eq. 6.12.

Note that since the purpose of Eq. 6.11 is to estimate the maximum value of the relative scores of an airport in Eq. 5.24 in Form 3, it is not necessarily important whether one of the coefficient of the slacks, (that is, $W_j^-$ for some $j = 1, 2, 3, 4$ or $W_k^+$ for some $k = 1, 2, 3$) is 0, as illustrated in Sects. 5.6.4.1 and 5.6.4.2 as well as Figs. 3.14, 3.18, 4.11 and 4.13 in Chaps. 3 and 4.

On the other hand, Eq. 5.18 (Eq. 6.12) measures whether an airport has done the job right according to the radiate, the convexity and the wholly dominant approaches, as illustrated in Sect. 5.4. Eq. 6.11 is also a dual linear programming for Eq. 6.12, thus, in order to linearly measure the partially dominant concept (to introduce the airports which have done the job well), the combination of these three linear approaches is required. In other words, the combination of the convexity and the wholly dominant approaches (without the radiate approach) is not enough to find the airports which have done the job well, although, this combination finds the airports which have done the job right, according to the approach. It is again noticed that, the concept of doing the job right is not able to rank or benchmark airports. The maximum value of the relative scores of an airport is not also a relative score in comparison with the maximum value of the relative scores of another airport. In addition, how to find the maximum (minimum) values of the relative scores or how to introduce the practical points according to available data, do not necessarily provide the suitable weights to introduce the relative scores of the airports. The concept of doing the job right depends upon how to introduce the practical points by an approach, but the concept of doing the job well depends upon the values by Eq. 5.4 while a set of weights are estimated. None of the introduced models by Eqs. 5.25, 5.26, 6.34. and 6.35 and so on, are proposed to find the airports which have done the job well. These approaches and models are only used to provide a better view to introduce the weights for measuring the relative scores of the airports with lower risks, in order to identify the airports which have done the job well, while there is no any information about the weights.

## 6.4 Examining a Delta Neighborhood

In Chap. 2, while A has the highest rank the agencies Q and K, which are next to A, have the higher ranks in comparison with other agencies as well. While D has the greatest relative score, its neighborhood, P, also has a greater relative score in comparison with almost all of the other agencies. Similarly, the differences between the relative scores of the points in the neighborhood of H and the relative score of H should be negligible. This means while very small errors introduced in the factors of an airport, the suggested relative scores as well as the ranks for these two situations should not have significant differences. How can a measure be introduced to satisfy these statements?

### *6.4.1 Introducing a Delta Neighborhood*

Suppose that a point in the neighborhood of H, which is wholly dominated by H, is selected. This point has seven components which none of its input factor components should be less than the corresponded input factors of H and none of its output factor components should be greater than the corresponded output factors of H. In other words, if the components of airport H is $(x_{81}, x_{82}, x_{83}, x_{84}, y_{81}, y_{82}, y_{83})$, a

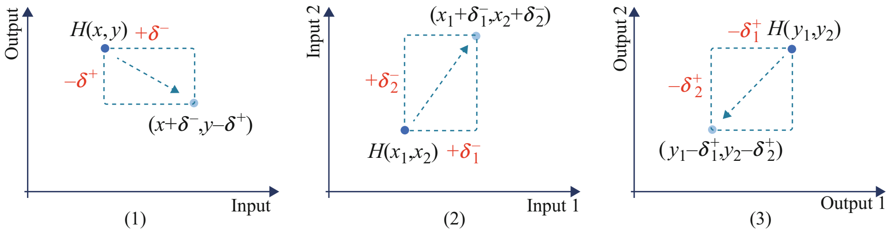



**Fig. 6.3** Neighborhoods of H in two dimensions

neighborhood of H, which is wholly dominated by H, can be introduced as Eq. 6.13, where $\delta_j^-$ and $\delta_k^+$ are non-negative real values, for $j = 1, 2, 3, 4$ and $k = 1, 2, 3$.

$$\left(x_{81} + \delta_1^-, x_{82} + \delta_2^-, x_{83} + \delta_3^-, x_{84} + \delta_4^-, y_{81} - \delta_1^+, y_{82} - \delta_2^+, y_{83} - \delta_3^+\right). \quad (6.13)$$

Since each factor can have different unit of measurements, the values $\delta_j^-$ and $\delta_k^+$ should have the same unit of measurement corresponded to the factors.

One way to introduce the vector $\left(\delta_1^-, \delta_2^-, \delta_3^-, \delta_4^-, \delta_1^+, \delta_2^+, \delta_3^+\right)$ is to use the specified airport measurement approximation. Note that this selection of deltas is later extended for different measurement approximations, and will be optimized in the next chapter.

Suppose that the same proportion of each factor is considered to introduce $\delta_j^-$ and $\delta_k^+$, for $j = 1, 2, 3, 4$ and $k = 1, 2, 3$, by the following equations, for $\delta \in [0, 1)$:

$$\delta_j^- = \delta \times x_{8j} \quad \text{and} \quad \delta_k^+ = \delta \times y_{8k} \quad (6.14)$$

The values $\delta_j^-$, for $j = 1, 2, 3, 4$, and $\delta_k^+$, for $k = 1, 2, 3$ in Eq. 6.14 have the same units of measurement with corresponded factors. From Eq. 6.14, the point in a δ-neighborhood of H (for $\delta \in [0, 1]$) can be introduced as follows:

$$((1 + \delta) \times x_{81}, \ldots, (1 + \delta) \times x_{84}, (1 - \delta) \times y_{81}, \ldots, (1 - \delta) \times y_{83}). \quad (6.15)$$

When $\delta = 0$ in Eq. 6.15, (that is, selecting a 0%-neighborhood), Eq. 6.15 represents H, and when $\delta = 1$, (that is, selecting a 100%-neighborhood of H), the point $(2x_{81}, 2x_{82}, 2x_{83}, 2x_{84}, 0, 0, 0)$ is deduced.

To simplify the illustration, assume that H has only two (three) components; thus, the above technique displays the opposite vertex of a rectangular (rectangle cube) in comparison with H. For example, Fig. 6.3 illustrates a neighborhood of H in two dimensions by three different cases. In case (1), H has one input and one output factors. In case (2), H has two input factors and in case (3), H has two output factors.

### 6.4.2 The Maximum Value of the Relative Scores

By using the same approach from Sect. 5.6, to measure the maximum value of the relative scores of the introduced point in a δ-neighborhood of H in Eq. 6.15, the following equation should be solved for $l = 8$.

$$\max\left\{\frac{\frac{(1-\delta)(y_{l1}w_1^+ + y_{l2}w_2^+ + y_{l3}w_3^+)}{(1+\delta)(x_{l1}w_1^- + x_{l2}w_2^- + x_{l3}w_3^- + x_{l4}w_4^-)}}{\max\left\{\frac{y_{i1}w_1^+ + y_{i2}w_2^+ + y_{i3}w_3^+}{x_{i1}w_1^- + x_{i2}w_2^- + x_{i3}w_3^- + x_{i4}w_4^-} : i = 1, 2, \ldots, 8\right\}} : w_j^- \geq 0, w_k^+ \geq 0\right\}. \quad (6.16)$$

As illustrated in Sect. 5.6, the best benchmarking for H by the fractional programming in Eq. 6.16 might not be found by Forms 1, 2 and 3. Indeed, the strengths or the weaknesses of both input and output factors are not simultaneously measured by Forms 1 and 2. Form 3 is also the same as Form 1 (Form 2), while the same constraints of Form 1 (Form 2) are selected for Form 3. Of course, Form 3 can simultaneously measure the strengths and the weaknesses of both input and output factors. Nonetheless, different restrictions for the weights in Form 3 may yield different outcomes. Form 3 also does not measure the magnitude of the linear combination of the input factors in comparison with the linear combination of the output factors, which is necessary for measuring the relative scores of the airports. Even if Forms 1, 2 and 3 did not have these shortcomings, and Eqs. 5.21 and 6.16 could easily be solved, only the maximum values of the relative scores of the airports are measured which neither represent the desired relative scores for the airports nor provide a measure to examine the differences between the proposed scores for an airport and its delta neighborhood.

In addition, the coefficient, (1 − δ)/(1 + δ), is the only difference between Eqs. 5.21 and 6.16, and does not affect the optimal weights in the both equations. In other words, the optimal weights to measure the maximum value of the relative scores of a δ-neighborhood of H suggest the maximum value of the relative scores for H as well. This phenomenon is due to the way of introducing a delta neighborhood, which has the same proportionate increasing of the values of input factors and decreasing of the values of output factors. From such an approach, we try to find a new measure (different with Eq. 5.21 and Forms 1–3) which lets us calculate Types 1–4 is Sect. 1.2 and Types 5 and 6 in Sect. 4.4.

In contrast, while the delta value is changed, the objective in Form 3 is not equivalent for $\delta$-neighborhoods of H ($l = 8$), as Eq. 6.17 displays. This important difference can lead us to introduce a new measure to bridge between the concepts of doing the job right and doing the job well.

$$\min\left(\sum_{j=1}^{4}(1+\delta)x_{lj}w_j^- - \sum_{k=1}^{3}(1-\delta)y_{lk}w_k^+\right),$$
$$\text{Subject to}$$
$$\sum_{j=1}^{4}x_{ij}w_j^- - \sum_{k=1}^{3}y_{ik}w_k^+ \geq 0 \quad \text{for} \quad i=1,2,\ldots,8, \tag{6.17}$$
$$w_j^- \geq 10^{-7}, \quad \text{for} \quad j=1,2,3,4,$$
$$w_k^+ \geq 10^{-7}, \quad \text{for} \quad k=1,2,3.$$

The objective in Eq. 6.17 depends upon the value of delta and using Form 3 for a δ-neighborhood of H may not suggest the relative score of 1 for H. Nevertheless, if the suggested optimal weights in Eq. 6.17 for a delta neighborhood of H do not yield the relative score equal to 1 for H, another airport in Table 5.1 should have the relative score equal to 1. We use this phenomenon to find a new measure based upon the value of delta.

Since all of the weights in Eq. 6.17 are restricted as greater than equal to 0.0000001, Eq. 6.17 is equivalent with Eqs. 5.30 and 5.31. From the findings in Sect. 6.2, the dual linear programming for Eq. 6.17 is given as:

$$\max 10^{-7}\left(\sum_{j=1}^{4}s_j^- + \sum_{k=1}^{3}s_k^+\right),$$
$$\text{Subject to}$$
$$\sum_{i=1}^{8}x_{ij}\lambda_i + s_j^- = (1+\delta)x_{lj}, \quad \text{for} \quad j=1,2,3,4,$$
$$\sum_{i=1}^{8}y_{ik}\lambda_i - s_k^+ = (1-\delta)y_{lk}, \quad \text{for} \quad k=1,2,3, \tag{6.18}$$
$$\lambda_i \geq 0, \quad \text{for} \quad i=1,2,\ldots,8,$$
$$s_j^- \geq 0, \quad \text{for} \quad j=1,2,3,4,$$
$$s_k^+ \geq 0, \quad \text{for} \quad k=1,2,3.$$

Since the weight/worth/price of each factor is not known, the unit of measurement for each slack is also unknown. Therefore, the objective in Eq. 6.18 may not be valid. In other words, the restrictions for the weights in Eq. 6.17 should not be introduced blindly.

One simple way to relax the units of measurement in the objective of Eq. 6.18 is to divide each slack by the value of the corresponded factor instead of dividing by $10^{+7}$ or another real number, (that is, using the specified airport measurement approximation). In other words, the coefficients of the slacks in Eq. 6.12 can be selected by $W_j^- = 1/x_{lj}$, for $j = 1, 2, 3, 4$, and $W_k^- = 1/y_{lk}$, for $k = 1, 2, 3$. Of course, the results depend on this approach, but in the end, the model is extended to deal with different approaches as well.

From the assumption to relax the units of measurement in the objective of Eq. 6.18, the weights in the dual linear programming (Eq. 6.11), $w_j^+$ and $w_k^-$, are restricted by $W_j^- = 1/x_{lj}$, for $j = 1, 2, 3, 4$, and $W_k^- = 1/y_{lk}$, for $k = 1, 2, 3$, respectively. Hence, Eq. 6.19 is yielded instead of Eq. 6.17, for $l = 1, 2, \ldots, 8$.

$$\begin{aligned}
&\min\left(\sum_{j=1}^{4}(1+\delta)x_{lj}w_j^- - \sum_{k=1}^{3}(1-\delta)y_{lk}w_k^+\right),\\
&\text{Subject to}\\
&\sum_{j=1}^{4}x_{ij}w_j^- - \sum_{k=1}^{3}y_{ik}w_k^+ \geq 0, \quad \text{for,} \quad i=1,2,\ldots,8,\\
&w_j^- \geq 1/x_{lj}, \quad \text{for} \quad j=1,2,3,4,\\
&w_k^+ \geq 1/y_{lk}, \quad \text{for} \quad k=1,2,3.
\end{aligned} \tag{6.19}$$

The optimal values for the weights $w_j^-$ and $w_k^+$ in Eq. 6.19, while $l = 8$ and $\delta = 0$, yield that the relative score of H (measured by Eq. 5.3) becomes as 1, which is the maximum value of the relative scores of H.

If the value of the delta is changed to 0.01, which means a 1% neighborhood of H is selected, the optimal values for the weights, $w_j^-$ and $w_k^+$, in Eq. 6.19 yield the same relative score for H, whereas the relative score for that neighborhood is $0.\overline{9801}$. Table 6.6 illustrates the relative scores of H and its neighborhood in five decimal digits while the value of delta is changed from 0 to 0.01, 0.1, 0.2, 0.5, 0.6, 0.9, and 0.99 in Eq. 6.19.

As $\delta \to 1$, that is, the neighborhood of H is selected farther and farther from H, the optimal values for the weights, $w_j^{-*}$ and $w_k^{+*}$, in Eq. 6.18 suggest the smaller and smaller relative score for that neighborhood, whereas the optimal values suggest the same relative score for H. Are the same outcomes for H in Table 6.6 deduced for other airports A, B, D, F and G as well? (Note that these airports have the maximum value of the relative scores equal to 1, when $\delta = 0$).

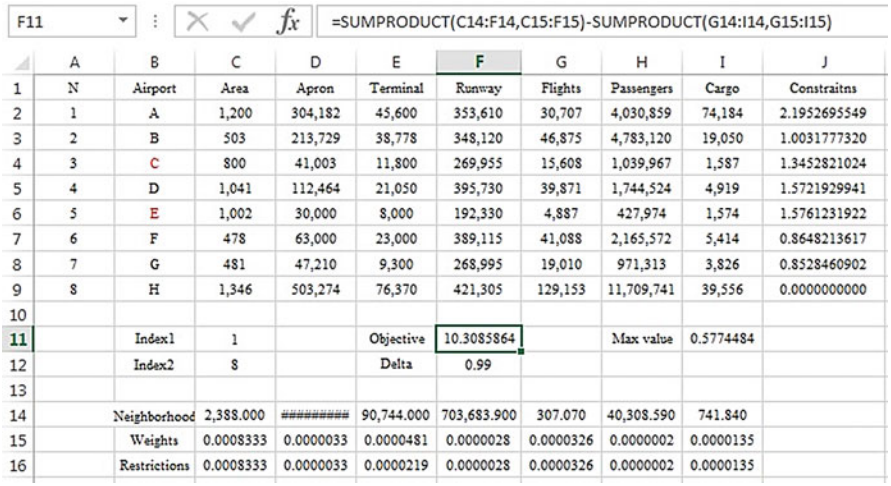
In order to solve Eq. 6.19 for each airport by the Microsoft Excel Solver 2013 software the following steps can be followed:

1. Copy the 9 columns of Table 5.1 on an Excel sheet into cells A1:I9, as Fig. 6.4 depicts.
2. Label B11 as ‘Index1’, B12 as ‘Index2’, B14 as ‘Neighborhood’, B15 as ‘Weights’, B16 as ‘Restrictions’, E11 as ‘Objective’, E12 as ‘Delta’, H11 as ‘Max value’, J1 as ‘Constraints’.
3. Assign the values 0, 0.01, 0.1, 0.2, 0.5, 0.6, 0.9 and 0.99 to cells K1-R1.
4. Assign the number 1 to cells C11 and C12.
5. Assign the command ‘=(1 + $F12)*Index(C2:C9,$C11)’ to C14. Then, copy C14 and paste it to D14, E14 and F14.
6. Assign the command ‘=(1-$F12)*Index(H2:H9,$C11)’ to G14. Then, copy G14 and paste it to H14 and I14.
7. Assign the command ‘=1/Index(C2:C9,$C11)’ to C16. Then, copy C16 and paste it to D16-I16.
8. Assign the following command to F11

   ‘=Sumproduct(C14:F14,C15:F15)-Sumproduct(G14:I14,G15:I15)’.

**Table 6.6** The relative scores of H and its delta neighborhood

| Delta | 0 | 0.01 | 0.1 | 0.2 | 0.5 | 0.6 | 0.9 | 0.99 |
|---|---|---|---|---|---|---|---|---|
| Neighborhood | 1 | 0.980 | 0.818 | 0.667 | 0.333 | 0.25 | 0.053 | 0.005 |
| H | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |



F11 =SUMPRODUCT(C14:F14,C15:F15)-SUMPRODUCT(G14:I14,G15:I15)

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | N | Airport | Area | Apron | Terminal | Runway | Flights | Passengers | Cargo | Constraitns |
| 2 | 1 | A | 1,200 | 304,182 | 45,600 | 353,610 | 30,707 | 4,030,859 | 74,184 | 2.1952695549 |
| 3 | 2 | B | 503 | 213,729 | 38,778 | 348,120 | 46,875 | 4,783,120 | 19,050 | 1.0031777320 |
| 4 | 3 | C | 800 | 41,003 | 11,800 | 269,955 | 15,608 | 1,039,967 | 1,587 | 1.3452821024 |
| 5 | 4 | D | 1,041 | 112,464 | 21,050 | 395,730 | 39,871 | 1,744,524 | 4,919 | 1.5721929941 |
| 6 | 5 | E | 1,002 | 30,000 | 8,000 | 192,330 | 4,887 | 427,974 | 1,574 | 1.5761231922 |
| 7 | 6 | F | 478 | 63,000 | 23,000 | 389,115 | 41,088 | 2,165,572 | 5,414 | 0.8648213617 |
| 8 | 7 | G | 481 | 47,210 | 9,300 | 268,995 | 19,010 | 971,313 | 3,826 | 0.8528460902 |
| 9 | 8 | H | 1,346 | 503,274 | 76,370 | 421,305 | 129,153 | 11,709,741 | 39,556 | 0.0000000000 |
| 10 | | | | | | | | | | |
| 11 | | Index1 | 1 | | Objective | 10.3085864 | | Max value | 0.5774484 | |
| 12 | | Index2 | 8 | | Delta | 0.99 | | | | |
| 13 | | | | | | | | | | |
| 14 | | Neighborhood | 2,388.000 | ######### | 90,744.000 | 703,683.900 | 307.070 | 40,308.590 | 741.840 | |
| 15 | | Weights | 0.0008333 | 0.0000033 | 0.0000481 | 0.0000028 | 0.0000326 | 0.0000002 | 0.0000135 | |
| 16 | | Restrictions | 0.0008333 | 0.0000033 | 0.0000219 | 0.0000028 | 0.0000326 | 0.0000002 | 0.0000135 | |

**Fig. 6.4** Setting Excel to solve Eq. 6.19

9. Assign the following command to I11

   '=Sumproduct(Index(G2:I9,C11,0),G15:I15)/Sumproduct(Index(C2:F9, C11,0),C15:F15)'.

10. Assign the following command into J2.

    '=Sumproduct(C$15:F$15,C2:F2)-Sumproduct(G$15:I$15,G2:I2)' (Fig. 6.5).

11. Copy J2 and paste it to J3-J9
12. Assign the command '=Index(K1:R1,C12)' to F12.
13. Open 'Solver Parameters' window.
14. Assign 'F11' into 'Set Objective' and choose 'Min'.
15. Assign 'C15:I15' into 'By Changing Variable Cells'.
16. Click on 'Add' and assign 'J2:J9' into 'Cell Reference', then select '>=', and assign '0' into 'Constraint'.
17. Click on 'Add' and assign 'C15:I15' into 'Cell Reference', then select '>=', and assign 'C16:I16' into 'Constraint'. Then click on 'OK'.
18. Tick 'Make Unconstrained Variables Non-Negative'.
19. Choose 'Simplex LP' from 'Select a Solving Method'
20. Click on 'Solve'.
21. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.
22. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet.
23. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.
24. From the toolbar menu, click on 'Tools> References...>' and make sure 'Solver' is ticked, and then 'OK' (Fig. 6.6).

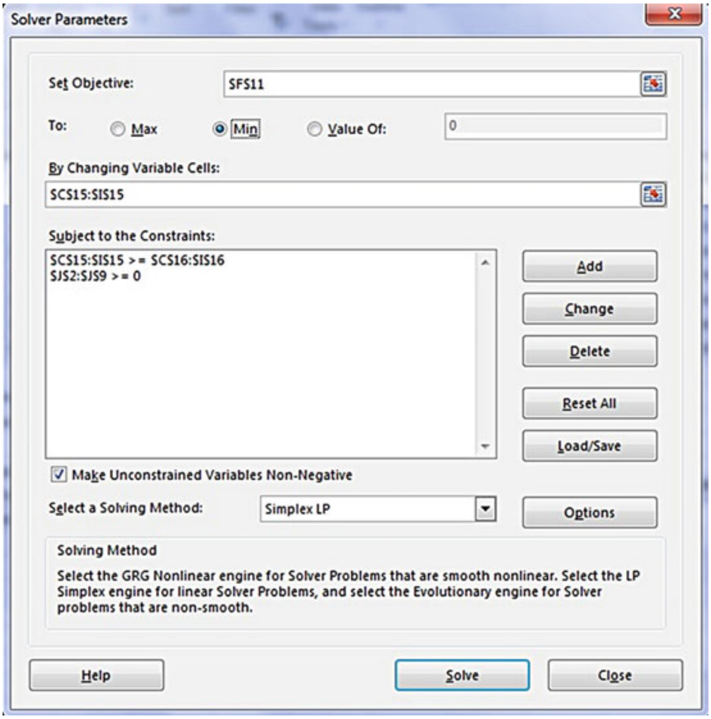



**Fig. 6.5** Setting Solver to solve Eq. 6.19

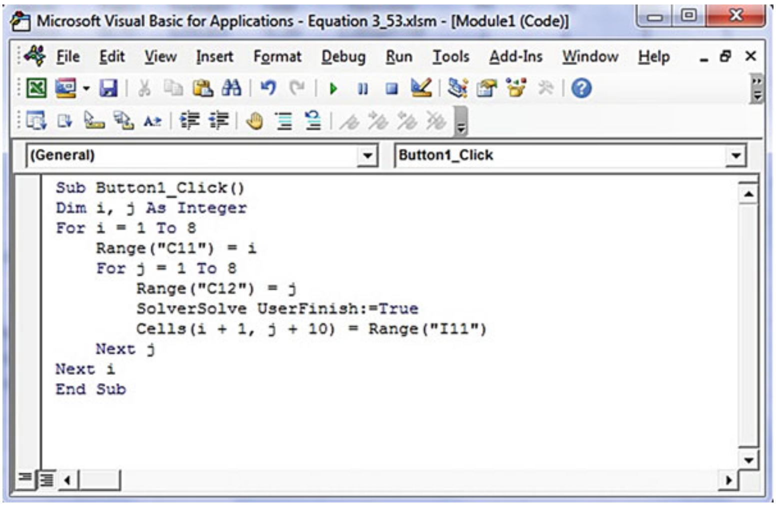



**Fig. 6.6** Setting VBA to solve Eq. 6.19

25. Inside of the 'Microsoft Visual Basic for Applications' window, write the following commands between 'Sub Button1_Click ()' and 'End Sub'.

```
Dim i, j As Integer
For i = 1 To 8
   Range("C11") = i
   For j = 1 To 8
      Range("C12") = j
      SolverSolve UserFinish:=True
      Cells(i + 1, j + 10) = Range("I11")
   Next j
Next i
```

26. Close the 'Microsoft Visual Basic for Applications' window.
27. Click on the small rectangle which was automatically made on the Excel sheet in Step 20.
28. The maximum values of the relative scores of airports by solving Eq. 6.19 are represented into cells K2:R9. Table 6.7 illustrates the transpose of the results in cells K2:K9.

As can be seen in Table 6.7, H is the only airport that even the optimal weights for its 99% neighborhood by Eq. 6.19 yield the relative score of 1 for H.

Similarly, F (and A) is the airport that even the optimal weights for its 60% (50%) neighborhood yield the relative score of 1 for F (A) by Eq. 6.19. In contrast, the optimal weights for 10% neighborhood of G, B and D do not suggest the maximum value of the relative scores of G, B and D.

### *6.4.3 The Dual Linear Programming of Form 3*

From another point of view, let's examine the dual linear programming of Eq. 6.19, as Eq. 6.20 represents, for $l = 1, 2, \ldots, 8$.

$$
\begin{aligned}
&\max \textstyle\sum_{j=1}^{4} (s_j^-/x_{lj}) + \sum_{k=1}^{3} (s_k^+/y_{lk}),\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{8} x_{ij}\lambda_i + s_j^- = x_{lj} + \delta x_{lj}, \quad \text{for} \quad j = 1, 2, 3, 4,\\
&\textstyle\sum_{i=1}^{8} y_{ik}\lambda_i - s_k^+ = y_{lk} - \delta y_{lk}, \quad \text{for} \quad k = 1, 2, 3,\\
&\lambda_i \geq 0, \quad \text{for} \quad i = 1, 2, \ldots, 8,\\
&s_j^- \geq 0, \quad \text{for} \quad j = 1, 2, 3, 4,\\
&s_k^+ \geq 0, \quad \text{for} \quad k = 1, 2, 3.
\end{aligned}
\tag{6.20}
$$

**Table 6.7** The relative scores of airports from Eq. 6.19

| Neighborhood (%) | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 0 | 1.00000 | 1.00000 | 0.63235 | 1.00000 | 0.35629 | 1.00000 | 1.00000 | 1.00000 |
| 1 | 1.00000 | 1.00000 | 0.63235 | 0.90138 | 0.35629 | 1.00000 | 1.00000 | 1.00000 |
| 10 | 1.00000 | 0.84846 | 0.55080 | 0.81641 | 0.35629 | 1.00000 | 0.85288 | 1.00000 |
| 20 | 1.00000 | 0.84846 | 0.55080 | 0.81641 | 0.35629 | 1.00000 | 0.85288 | 1.00000 |
| 50 | 1.00000 | 0.84846 | 0.55080 | 0.60664 | 0.35629 | 1.00000 | 0.74203 | 1.00000 |
| 60 | 0.57745 | 0.84846 | 0.32648 | 0.36776 | 0.35629 | 1.00000 | 0.54036 | 1.00000 |
| 90 | 0.57745 | 0.84846 | 0.32648 | 0.36776 | 0.35629 | 0.57041 | 0.54036 | 1.00000 |
| 99 | 0.57745 | 0.84846 | 0.32648 | 0.36776 | 0.35629 | 0.57041 | 0.54036 | 1.00000 |

**Table 6.8** The optimal value of the objective of Eq. 6.20

| Delta | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.00000 | 0.00000 | 4.18463 | 0.00000 | 5.42004 | 0.00000 | 0.00000 | 0.00000 |
| 0.0001 | 0.00124 | 0.00154 | 4.18649 | 0.04940 | 5.42118 | 0.00114 | 0.00238 | 0.00080 |
| 0.01 | 0.12441 | 0.15438 | 4.37043 | 1.79755 | 5.53424 | 0.11346 | 0.23748 | 0.08000 |
| 0.1 | 1.24413 | 1.34554 | 5.93963 | 3.90847 | 6.56204 | 1.13464 | 2.18542 | 0.80000 |

The units of measurement for the slacks $s_j^-$ and $s_k^+$ are the same as the units of measurement for the $x_{lj}$ and $y_{lk}$, for $j = 1, 2, 3, 4$ and $k = 1, 2, 3$, respectively. Thus, the objective in Eq. 6.20 has the unity scale. While the value of delta is 0, Eq. 6.20 is the same as Eq. 6.21, (where $W_j^- = 1/x_{lj}$, for $j = 1, 2, 3, 4$, and $W_k^- = 1/y_{lk}$, for $k = 1, 2, 3$), and the objective for the airports A, B, D, F, G and H are 0. In addition, when the value of delta in Eq. 6.20 is positive, at least one of the slacks, $s_j^-$ and $s_k^+$, are positive. Hence, the objective of Eq. 6.20 is not zero for $\delta > 0$.

The objective in Eq. 6.20 maximizes the slacks, which display potential decrease of the input factors and potential increase of the output factors for airport number $l$ ($l = 1, 2, \ldots, 8$), where $W_j^- = 1/x_{lj}$, for $j = 1, 2, 3, 4$, and $W_k^- = 1/y_{lk}$, for $k = 1, 2, 3$. Table 6.8 represents the optimal value of the objective of Eq. 6.20 with 5 decimal digits for four different values of delta.

The scores in Table 6.8 depend on the values of delta, and varied from one airport to another, however, the scores are neither between 0 and 1, nor allow measuring the corresponded relative scores for airports in Table 6.7, similar to Eq. 6.19. Therefore, suppose that the optimal solution for Eq. 6.20 is $\left(\lambda_{l1}^*, \lambda_{l2}^*, \lambda_{l3}^*, \lambda_{l4}^*, \lambda_{l5}^*, \lambda_{l6}^*, \lambda_{l7}^*, \lambda_{l8}^*, s_{l1}^{-*}, s_{l2}^{-*}, s_{l3}^{-*}, s_{l4}^{-*}, s_{l1}^{+*}, s_{l2}^{+*}, s_{l3}^{+*}\right)$, for airport number $l\,(l = 1, 2, \ldots, 8)$. Assume that $x_{lj}^*$ and $y_{lk}^*$ are defined as follows:

$$\begin{aligned} x_{lj}^* &= x_{lj} + \delta x_{lj} - s_j^{-*} \quad \text{for} \quad j = 1, 2, 3, 4, \\ y_{lk}^* &= y_{lk} - \delta y_{lk} + s_k^{+*}, \quad \text{for} \quad j = 1, 2, 3. \end{aligned} \tag{6.21}$$

As illustrated in Sects. 5.3 and 5.4, the optimal slacks $s_j^{-*}$ for $j = 1, 2, 3, 4$ and $s_k^{+*}$ for $k = 1, 2, 3$, represent the lack of performance of the following point (Eq. 6.22), which is a neighborhood of airport number $l$.

$$((1 + \delta) \times x_{l1}, \ldots, (1 + \delta) \times x_{l4}, (1 - \delta) \times y_{l1}, \ldots, (1 - \delta) \times y_{l3}). \tag{6.22}$$

Equation 6.21 displays that the point $\left(x_{l1}^*, x_{l2}^*, x_{l3}^*, x_{l4}^*, y_{l1}^*, y_{l2}^*, y_{l3}^*\right)$ is found from Eq. 6.20, by decreasing the excess of the input factors and increasing the shortage of the output factors of the point in Eq. 6.22, (regarding the introduced restrictions for each factor by the inverse value of the corresponded factor). Both the airport number $l$ and the suggested point in Eq. 6.21, $\left(x_{l1}^*, x_{l2}^*, x_{l3}^*, x_{l4}^*, y_{l1}^*, y_{l2}^*, y_{l3}^*\right)$, dominate the point in Eq. 6.22. The scores of these two points may be compared by the following equation, where $W_j^- = 1/x_{lj}$, for $j = 1, 2, 3, 4$, and $W_k^- = 1/y_{lk}$, for $k = 1, 2, 3$.

**Table 6.9** The suggested score by Eq. 6.23

| Delta | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 0 | 1.0000 | 1.0000 | 0.3557 | 1.0000 | 0.2898 | 1.0000 | 1.0000 | 1.0000 |
| 0.0001 | 0.9998 | 0.9998 | 0.3556 | 0.9847 | 0.2898 | 0.9999 | 0.9996 | 1.0000 |
| 0.01 | 0.9806 | 0.9802 | 0.3482 | 0.6244 | 0.2898 | 0.9861 | 0.9554 | 1.0000 |
| 0.1 | 0.8410 | 0.8449 | 0.2984 | 0.4444 | 0.2898 | 0.8751 | 0.6455 | 1.0000 |

$$\frac{\sum_{k=1}^{3} W_k^+ y_{lk} / \sum_{j=1}^{4} W_j^- x_{lj}}{\sum_{k=1}^{3} W_k^+ y_{lk}^* / \sum_{j=1}^{4} W_j^- x_{lj}^*}. \tag{6.23}$$

Equation 6.23 can be less than equal to 1 or greater than 1. If the score of the point $\left(x_{l1}^*, x_{l2}^*, x_{l3}^*, x_{l4}^*, y_{l1}^*, y_{l2}^*, y_{l3}^*\right)$ is greater than or equal to that of the airport number $l$, (that is, the value of Eq. 6.23 is less than or equal to 1), the suggested point from Eq. 6.20 has a better performance in comparison with the airport number $l$, (regarding the introduced restrictions for the weights in Eq. 6.19). Table 6.9 illustrates the value of Eq. 6.23 with 5 decimal digits for each airport while the value of delta is 0, 0.0001, 0.01 and 0.1. The results in Table 6.9 represent the same outcomes as the results in Table 6.8 as well as providing the scores between 0 and 1.

### *6.4.4 Introducing a New Measure*

Since the optimal solution for Eq. 6.20 is not unique, the value of Eq. 6.23 may vary from one optimal solution to another. In addition, Eq. 6.20 represents the outcomes by Form 3 which does not measure the magnitude of the linear combination of the output factors in comparison with the linear combination of the input factors, (and this is a difference between Eqs. 6.16 and 6.17). Therefore, if the smallest value of Eq. 6.23 for an airport is measured, this means, the best point in the generated area by the constraints (the feasible area), $\left(x_{l1}^*, x_{l2}^*, x_{l3}^*, x_{l4}^*, y_{l1}^*, y_{l2}^*, y_{l3}^*\right)$, corresponded to the introduced delta value, is suggested by a new measure, in order to display the shortcomings of the performance of that airport, (regarding the introduced restrictions for the weights in Eq. 6.19). Even if the suggested point in this situation is not unique, the score is optimal regarding the value of delta. In order to find the minimum value of Eq. 6.23, according to the constraints in Eq. 6.20, let's extend Eq. 6.23 by using Eq. 6.21, as follows:

$$\begin{aligned}
&\frac{\sum_{k=1}^{3} W_k^+ y_{lk} / \sum_{j=1}^{4} W_j^- x_{lj}}{\sum_{k=1}^{3} W_k^+ y_{lk}^* / \sum_{j=1}^{4} W_j^- x_{lj}^*} = \\
&= \frac{\sum_{k=1}^{3} W_k^+ y_{lk} / \sum_{j=1}^{4} W_j^- x_{lj}}{\sum_{k=1}^{3} W_k^+ \left(y_{lk} - \delta y_{lk} + s_k^{+*}\right) / \sum_{j=1}^{4} W_j^- \left(x_{lj} + \delta x_{lj} - s_j^{-*}\right)} \\
&= \frac{\sum_{j=1}^{4} W_j^- \left(x_{lj} + \delta x_{lj} - s_j^{-*}\right) / \sum_{j=1}^{4} W_j^- x_{lj}}{\sum_{k=1}^{3} W_k^+ \left(y_{lk} - \delta y_{lk} + s_k^{+*}\right) / \sum_{k=1}^{3} W_k^+ y_{lk}} \\
&= \frac{\sum_{j=1}^{4} W_j^- \left(x_{lj} + \delta x_{lj} - s_j^{-*}\right) / \sum_{j=1}^{4} W_j^- x_{lj}}{\sum_{k=1}^{3} W_k^+ \left(y_{lk} - \delta y_{lk} + s_k^{+*}\right) / \sum_{k=1}^{3} W_k^+ y_{lk}}.
\end{aligned} \tag{6.24}$$

Hence, the objective of Eq. 6.20 can be replaced by Eq. 6.24, given by:

$$\begin{aligned}
&\min \frac{\sum_{j=1}^{4} W_j^- (x_{lj} + \delta x_{lj} - s_j^-) / \sum_{j=1}^{4} W_j^- x_{lj}}{\sum_{k=1}^{3} W_k^+ (y_{lk} - \delta y_{lk} + s_k^+) / \sum_{k=1}^{3} W_k^+ y_{lk}}, \\
&\text{Subject to} \\
&\textstyle\sum_{i=1}^{8} x_{ij}\lambda_i + s_j^- = x_{lj} + \delta x_{lj}, \quad \text{for} \quad j = 1,2,3,4, \\
&\textstyle\sum_{i=1}^{8} y_{ik}\lambda_i - s_k^+ = y_{lk} - \delta y_{lk}, \quad \text{for} \quad k = 1,2,3, \\
&\lambda_i \geq 0, \quad \text{for} \quad i = 1,2,\ldots,8, \\
&s_j^- \geq 0, \quad \text{for} \quad j = 1,2,3,4, \\
&s_k^+ \geq 0, \quad \text{for} \quad k = 1,2,3.
\end{aligned} \tag{6.25}$$

Equation 6.25 is a non-linear programming, and finds the minimum value of Eq. 6.23, according to the constraints in Eq. 6.20. In addition, the terms $\sum_{j=1}^{4} W_j^- x_{lj}$ and $\sum_{k=1}^{3} W_k^+ y_{lk}$ in Eq. 6.25 are constant real numbers for airport number $l$ ($l = 1,2,\ldots,8$). Thus, the model suggests the best point in the feasible area in comparison with airport number $l$, regarding the delta value and $W_j^-$ and $W_k^-$. For instance, when the specified airport measurement approximation is used, that is, $W_j^- = 1/x_{lj}$, for $j = 1, 2, 3, 4$, $W_k^- = 1/y_{lk}$, for $k = 1, 2, 3$, and $\delta = 0$, the following non-linear programming in Eq. 6.26 is deduced.

The term in the numerator of the objective in Eq. 6.26, $(1/4)\sum_{j=1}^{4}\left[1 - \left(s_j^-/x_{lj}\right)\right]$, that is, $(1/4)\sum_{j=1}^{4}\left[\left(x_{lj} - s_j^-\right)/x_{lj}\right]$, measures the mean proportional reduction rate of the input factors, and the term in the denominator of the objective in Eq. 6.26, $(1/3)\sum_{k=1}^{3}\left[1 + \left(s_k^+/y_{lk}\right)\right]$, that is, $(1/3)\sum_{k=1}^{3}\left[\left(y_{lk} + s_k^+\right)/y_{lk}\right]$, measures the mean proportional expansion rate of the output factors.

$$\begin{aligned}
&\min \frac{(1/4)\sum_{j=1}^{4}\left[1-(s_j^-/x_{lj})\right]}{(1/3)\sum_{k=1}^{3}\left[1+(s_k^+/y_{lk})\right]},\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{8} x_{ij}\lambda_i + s_j^- = x_{lj}, \quad \text{for} \quad j=1,2,3,4,\\
&\textstyle\sum_{i=1}^{8} y_{ik}\lambda_i - s_k^+ = y_{lk}, \quad \text{for} \quad k=1,2,3,\\
&\lambda_i \geq 0, \quad \text{for} \quad i=1,2,\ldots,8,\\
&s_j^- \geq 0, \quad \text{for} \quad j=1,2,3,4,\\
&s_k^+ \geq 0, \quad \text{for} \quad k=1,2,3.
\end{aligned} \tag{6.26}$$

In order to introduce a delta linear programming, suppose that the denominator in Eq. 6.25, $\sum_{k=1}^{3} W_k^+\left(y_{lk} - \delta y_{lk} + s_k^+\right)/\sum_{k=1}^{3} W_k^+ y_{lk}$, is equal to $1/t$, for a positive real number $t$. Therefore, the following equation is deduced.

$$\begin{aligned}
&\min\left(\textstyle\sum_{j=1}^{4} W_j^-\left(x_{lj} + \delta x_{lj} - s_j^-\right)/\sum_{j=1}^{4} W_j^- x_{lj}\right)t,\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{3} W_k^+(y_{lk} - \delta y_{lk} + s_k^+)/\sum_{k=1}^{3} W_k^+ y_{lk} = 1/t,\\
&\textstyle\sum_{i=1}^{8} x_{ij}\lambda_i + s_j^- = x_{lj} + \delta x_{lj}, \quad \text{for} \quad j=1,2,3,4,\\
&\textstyle\sum_{i=1}^{8} y_{ik}\lambda_i - s_k^+ = y_{lk} - \delta y_{lk}, \quad \text{for} \quad k=1,2,3,\\
&\lambda_i \geq 0, \quad \text{for} \quad i=1,2,\ldots,8,\\
&s_j^- \geq 0, \quad \text{for} \quad j=1,2,3,4,\\
&s_k^+ \geq 0, \quad \text{for} \quad k=1,2,3,\\
&t > 0.
\end{aligned} \tag{6.27}$$

By multiplying the variable $t$ to each term of the objective in Eq. 6.27, as well as multiplying the variable $t$ to each constraint of the equivalent Eq. 6.28 is yielded.

$$\begin{aligned}
&\min\left(\textstyle\sum_{j=1}^{4} W_j^-(x_{lj}t + \delta x_{lj}t - ts_j^-)/\sum_{j=1}^{4} W_j^- x_{lj}\right),\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{3} W_k^+(y_{lk}t - \delta y_{lk}t + ts_k^+)/\sum_{k=1}^{3} W_k^+ y_{lk} = 1,\\
&\textstyle\sum_{i=1}^{8} x_{ij}t\lambda_i + ts_j^- = x_{lj}t + \delta x_{lj}t, \quad \text{for} \quad j=1,2,3,4,\\
&\textstyle\sum_{t=1}^{8} y_{ik}t\lambda_i - ts_k^+ = y_{lk}t - \delta y_{lk}t, \quad \text{for} \quad k=1,2,3,\\
&t\lambda_i \geq 0, \quad \text{for} \quad i=1,2,\ldots,8,\\
&ts_j^- \geq 0, \quad \text{for} \quad j=1,2,3,4,\\
&ts_k^+ \geq 0, \quad \text{for} \quad k=1,2,3,\\
&t > 0.
\end{aligned} \tag{6.28}$$

Now, the variables $t\lambda_i$, $ts_j^-$ and $ts_k^+$, can be renamed as $\Lambda_i$, $S_j^-$ and $S_k^+$, respectively, for $i = 1, 2, \ldots, 8$, $j = 1, 2, 3, 4$ and $k = 1, 2, 3$.

$$\begin{aligned}
&\min\Big( \textstyle\sum_{j=1}^{4} W_j^{-}(x_{lj}t + \delta x_{lj}t - S_j^{-}) / \sum_{j=1}^{4} W_j^{-} x_{lj}\Big),\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{3} W_k^{+}(y_{lk}t - \delta y_{lk}t + S_k^{+}) / \sum_{k=1}^{3} W_k^{+} y_{lk} = 1,\\
&\textstyle\sum_{i=1}^{8} x_{ij}\Lambda_i + S_j^{-} = x_{lj}t + \delta x_{lj}t \quad \text{for} \quad j = 1,2,3,4,\\
&\textstyle\sum_{i=1}^{8} y_{ik}\Lambda_i - S_k^{+} = y_{lk}t - \delta y_{lk}t, \quad \text{for} \quad k = 1,2,3,\\
&\Lambda_i \geq 0, \quad \text{for} \quad i = 1,2,\ldots,8,\\
&S_j^{-} \geq 0, \quad \text{for} \quad j = 1,2,3,4,\\
&S_k^{+} \geq 0, \quad \text{for} \quad k = 1,2,3,\\
&t > 0.
\end{aligned} \tag{6.29}$$

We can again rename the variables $\Lambda_i, S_j^{-}$ and $S_k^{+}$ by $\lambda_i, s_j^{-}$ and $s_k^{+}$, respectively, for $i = 1, 2, \ldots, 8$, $j = 1, 2, 3, 4$ and $k = 1, 2, 3$, and introduce the following linear programming

$$\begin{aligned}
&\min\Big( \textstyle\sum_{j=1}^{4} W_j^{-}(x_{lj}t + \delta x_{lj}t - s_j^{-}) / \sum_{j=1}^{4} W_j^{-} x_{lj}\Big),\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{3} W_k^{+}(y_{lk}t - \delta y_{lk}t + s_k^{+}) / \sum_{k=1}^{3} W_k^{+} y_{lk} = 1,\\
&\textstyle\sum_{i=1}^{8} x_{ij}\lambda_i + s_j^{-} = x_{lj}t + \delta x_{lj}t, \quad \text{for} \quad j = 1,2,3,4,\\
&\textstyle\sum_{i=1}^{8} y_{ik}\lambda_i - s_k^{+} = y_{lk}t - \delta y_{lk}t, \quad \text{for} \quad k = 1,2,3,\\
&\lambda_i \geq 0, \quad \text{for} \quad i = 1,2,\ldots,8,\\
&s_j^{-} \geq 0, \quad \text{for} \quad j = 1,2,3,4,\\
&s_k^{+} \geq 0, \quad \text{for} \ k = 1,2,3,\\
&t > 0.
\end{aligned} \tag{6.30}$$

Equation 6.30 can still be simplified as follows.

$$\begin{aligned}
&\min\Big( (1+\delta)t - \textstyle\sum_{j=1}^{4} W_j^{-} s_j^{-} / \sum_{j=1}^{4} W_j^{-} x_{lj}\Big),\\
&\text{Subject to}\\
&(1-\delta)t + \textstyle\sum_{k=1}^{3} W_k^{+} s_k^{+} / \sum_{k=1}^{3} W_k^{+} y_{lk} = 1,\\
&\textstyle\sum_{i=1}^{8} x_{ij}\lambda_i + s_j^{-} = t(1+\delta)x_{lj} \quad \text{for} \quad j = 1,2,3,4,\\
&\textstyle\sum_{i=1}^{8} y_{ik}\lambda_i - s_k^{+} = t(1-\delta)y_{lk} \quad \text{for} \quad k = 1,2,3,\\
&\lambda_i \geq 0, \quad \text{for} \quad i = 1,2,\ldots,8,\\
&s_j^{-} \geq 0, \quad \text{for} \quad j = 1,2,3,4,\\
&s_k^{+} \geq 0, \quad \text{for} \quad k = 1,2,3,\\
&t > 0.
\end{aligned} \tag{6.31}$$

In order to solve Eq. 6.31 by the Microsoft Excel Solver 2013 software while $W_j^{-} = 1/x_{lj}$, for $j = 1, 2, 3, 4$, $W_k^{-} = 1/y_{lk}$, for $k = 1, 2, 3$, and $\delta$ is 0, 0.0001, 0.01 and 0.1 the following instructions are introduced.

C17 =SUMPRODUCT(C2:C9,$J2:$J9)+C18

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | N | Airport | Area | Apron | Terminal | Runway | Flights | Passengers | Cargo | Lambdas |
| 2 | 1 | A | 1,200 | 304,182 | 45,600 | 353,610 | 30,707 | 4,030,859 | 74,184 | 1.000000 |
| 3 | 2 | B | 503 | 213,729 | 38,778 | 348,120 | 46,875 | 4,783,120 | 19,050 | 0.000000 |
| 4 | 3 | C | 800 | 41,003 | 11,800 | 269,955 | 15,608 | 1,039,967 | 1,587 | 0.000000 |
| 5 | 4 | D | 1,041 | 112,464 | 21,050 | 395,730 | 39,871 | 1,744,524 | 4,919 | 0.000000 |
| 6 | 5 | E | 1,002 | 30,000 | 8,000 | 192,330 | 4,887 | 427,974 | 1,574 | 0.000000 |
| 7 | 6 | F | 478 | 63,000 | 23,000 | 389,115 | 41,088 | 2,165,572 | 5,414 | 0.000000 |
| 8 | 7 | G | 481 | 47,210 | 9,300 | 268,995 | 19,010 | 971,313 | 3,826 | 0.000000 |
| 9 | 8 | H | 1,346 | 503,274 | 76,370 | 421,305 | 129,153 | 11,709,741 | 39,556 | 0.000000 |
| 10 | | | | | | | | | | |
| 11 | | Index 1 | 1 | | Index 2 | 1 | | | | |
| 12 | | | | | | | | | | |
| 13 | | W | 0.000833 | 0.000003 | 0.000022 | 0.000003 | 0.000033 | 0.000000 | 0.000013 | |
| 14 | | | | | | | | | | |
| 15 | | Constraint | 1 | | t | 1 | | Objective | 1.000000 | |
| 16 | | | | | | | | | | |
| 17 | | Left | 1,200 | 304,182 | 45,600 | 353,610 | 30,707 | 4,030,859 | 74,184 | |
| 18 | | Slacks | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | |
| 19 | | Right | 1200 | 304182 | 45600 | 353610 | 30707 | 4030859 | 74184 | |

**Fig. 6.7** Setting Excel to solve Eq. 6.31

1. Copy the 9 columns of Table 5.1 on an Excel sheet into cells A1:I9, as Fig. 6.7 depicts.
2. Label B11 as ‘Index1’, E11 as ‘Index2’, B13 as ‘Weights’, B15 as ‘Constraint’, B17 as ‘Left side’, B18 as ‘Slacks’, B19 as ‘Right side’, E15 as ‘*t*’, H15 as ‘Objective’, and J1 as ‘Lambdas’.
3. Assign the values 0, 0.0001, 0.01 and 0.1 to cells K1-N1.
4. Assign number 1 to cells C11 and F11.
5. Assign the command ‘=1/Index(C2:C9,$C11)’ to C13. Then, copy C13 and paste it to cells D13-I13.
6. Assign the command ‘=Sumproduct(C2:C9,$J2:$J9) + C18’ to C17. Then, copy C17 and paste it to D17, E17 and F17.
7. Assign the command ‘=Sumproduct(G2:G9,$J2:$J9)-G18’ to G17. Then, copy G17 and paste it to H17 and I17.
8. Assign ‘=$F15*(1 + Index($K1:$N1,$F11))*Index(C2:C9,$C11)’ to C19. Then, copy C19 and paste it to D19, E19 and F19.
9. Assign ‘=$F15*(1-Index($K1:$N1,$F11))*Index(G2:G9,$C11)’ to G19. Then, copy G19 and paste it to H19 and I19 (Fig. 6.8).
10. Assign one of the following commands to C15

‘=F15*(1-Index(K1:N1,F11)) + Sumproduct(G13:I13,G18:I18) / Sumproduct(G13:I13,Index(G2:I9,C11,0))’,

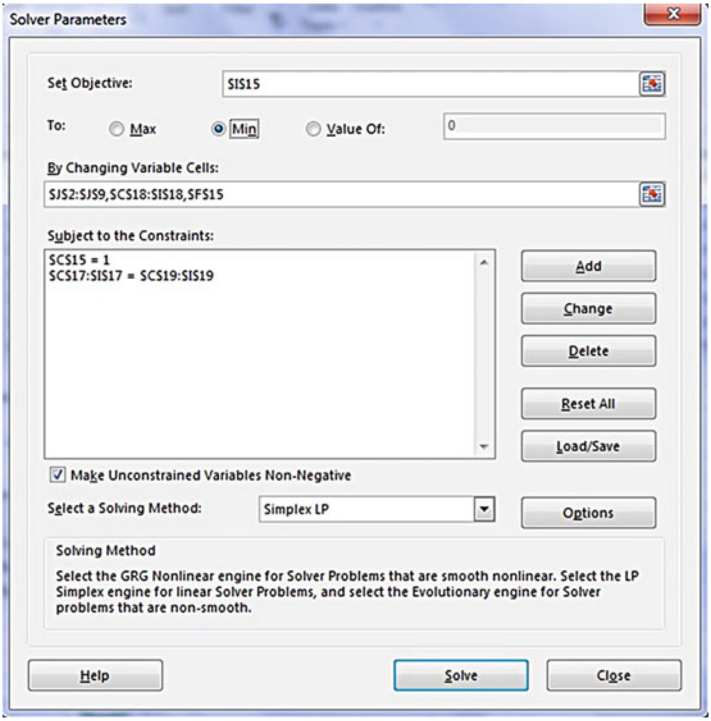



**Fig. 6.8** Setting Solver to solve Eq. 6.31

or
'= Sumproduct(G13:I13,G19:I19 + G18:I18)/Sumproduct(G13:I13,Index(G2:I9,C11,0))'.

11. Assign one of the following commands to I15

'=F15*(1 + Index(K1:N1,F11))-(Sumproduct(C13:F13,C18:F18)/Sumproduct(C13:F13,Index(C2:F9,C11,0)))',
or
'= Sumproduct(C13:F13,C19:F19-C18:F18)/Sumproduct(C13:F13,Index(C2:F9,C11,0))'.

12. Open 'Solver Parameters' window.
13. Assign 'I15' into 'Set Objective' and choose 'Min'.
14. Assign 'J2:J9, C18:I18, F15' into 'By Changing Variable Cells'.
15. Click on 'Add' and assign 'C15' into 'Cell Reference', then select '=', and assign '1' into 'Constraint'.
16. Click on 'Add' and assign 'C17:I17' into 'Cell Reference', then select '=', and assign 'C19:I19' into 'Constraint'. Then click on 'OK'.
17. Tick 'Make Unconstrained Variables Non-Negative'.

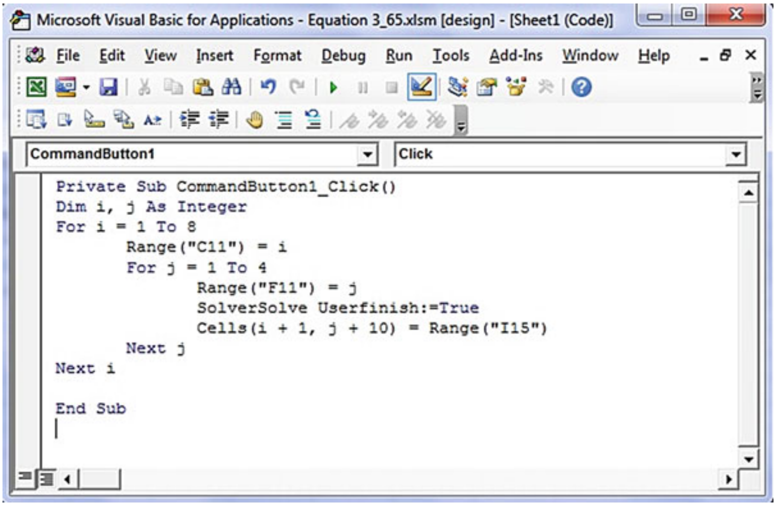



**Fig. 6.9** Setting VBA to solve Eq. 6.31

18. Choose 'Simplex LP' from 'Select a Solving Method' and 'Solve'.
19. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.
20. Click on the first icon, 'Button (Form Control)', and then click on a place in the Excel sheet (Fig. 6.9).
21. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.
22. From the toolbar menu, click on 'Tools> References...>' and make sure 'Solver' is ticked, and then 'OK'.
23. Inside of the 'Microsoft Visual Basic for Applications' window, write the following commands between 'Sub Button1_Click ()' and 'End Sub'.

```
Dim i, j As Integer
For i = 1 To 8
    Range("C11") = i
    For j = 1 To 4
        Range("F11") = j
        SolverSolve   Userfinish:=True
        Cells(i + 1, j + 10) = Range("I15")
    Next j
Next i
```

24. Close the 'Microsoft Visual Basic for Applications' window.
25. Click on the small rectangle which was automatically made in the Excel sheet which was created in Step 21.

**Table 6.10** The suggested score by Eq. 6.31

| Delta | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 0 | 1.00000 | 1.00000 | 0.35565 | 1.00000 | 0.24032 | 1.00000 | 1.00000 | 1.00000 |
| 0.0001 | 0.99980 | 0.99980 | 0.35558 | 0.98468 | 0.24031 | 0.99986 | 0.99955 | 0.99997 |
| 0.01 | 0.98056 | 0.98016 | 0.34820 | 0.62436 | 0.23956 | 0.98607 | 0.95536 | 0.99739 |
| 0.1 | 0.84096 | 0.84421 | 0.29844 | 0.44439 | 0.23327 | 0.87510 | 0.64546 | 0.97558 |

26. The minimum values of Eq. 6.20 are represented into cells K2:N9 for $\delta$ value equal to 0, 0.0001, 0.01 and 0.1.

Table 6.10 illustrates the transpose of the results in cells K2:N9.

The results in Table 6.10 are almost completely the same as the results in Table 6.9, except a few negligible differences. For instance, the minimum value of Eq. 6.23 for B while $\delta = 0.1$ is less than the corresponded score in Table 6.9. In addition, the scores for E and H, while delta is not zero, are different in Tables 6.8 and 6.9.

As shown in Table 6.9, the values of Eq. 6.23 were the same for airport H, when delta was 0, 0.0001, 0.01 and 0.1 in Eq. 6.20, however, Eq. 6.31 discriminates the differences while the value of delta is changed. Since Eq. 6.20 is changed to Eq. 6.31, it is valuable to find the dual linear programming of Eq. 6.31 to see the differences between the models. The dual linear programming for Eq. 6.31 is given by:

$$
\begin{aligned}
&\max\ \tau, \\
&\text{Subject to} \\
&(1-\delta)\tau + \textstyle\sum_{j=1}^{4} (1+\delta)x_{lj}w_j^- - \sum_{k=1}^{3} (1-\delta)y_{lk}w_k^+ = (1+\delta), \\
&\textstyle\sum_{k=1}^{3} y_{ik}w_k^+ - \sum_{j=1}^{4} x_{ij}w_j^- \le 0, \quad \text{for} \quad i = 1, 2, \ldots, 8, \\
&w_j^- \ge W_j^- / \textstyle\sum_{j=1}^{4} W_j^- x_{lj}, \quad \text{for} \quad j = 1, 2, 3, 4, \\
&w_k^+ \ge W_k^+ \tau / \textstyle\sum_{k=1}^{3} W_k^+ y_{lk}, \quad \text{for} \quad k = 1, 2, 3.
\end{aligned}
\tag{6.32}
$$

The same results in Table 6.10 are found for Eq. 6.32 by the Microsoft Excel Solver 2013 software, which shows the results are exactly optimal. In addition, Eq. 6.32 is equivalent with the following linear programming.

$$\begin{aligned} & \min \sum_{j=1}^{4} (1+\delta)x_{lj}w_j^- - \sum_{k=1}^{3} (1-\delta)y_{lk}w_k^+, \\ & \text{Subject to} \\ & \sum_{j=1}^{4} x_{ij}w_j^- - \sum_{k=1}^{3} y_{ik}w_k^+ \geq 0, \quad \text{for} \quad i = 1, 2, \ldots, 8, \\ & w_j^- \geq W_j^- / \sum_{j=1}^{4} W_j^- x_{lj}, \quad \text{for} \quad j = 1, 2, 3, 4, \\ & w_k^+ \geq W_k^+ [1 + \delta + \sum_{k=1}^{3} (1-\delta)y_{lk}w_k^+ - \sum_{j=1}^{4} (1+\delta)x_{lj}w_j^-] / \\ & \qquad \sum_{k=1}^{3} W_k^+ (1-\delta)y_{lk}, \quad \text{for} \quad k = 1, 2, 3. \end{aligned} \tag{6.33}$$

As can be seen, the difference between Eq. 6.19 and 6.33 is the different restrictions for the weights $w_j^-$ and $w_k^+$, for $j = 1, 2, 3, 4$ and $k = 1, 2, 3$. For instance, while $W_j^- = 1/x_{lj}$ and $W_k^+ = 1/y_{lk}$, for $j = 1, 2, 3, 4$ and $k = 1, 2, 3$, and $\delta = 0$, the restrictions for the weights in Eq. 6.33 are $w_j^- \geq 1/(4x_{lj})$ and $w_k^+ \geq [1 + \sum_{k=1}^{3} y_{lk}w_k^+ - \sum_{j=1}^{4} x_{lj}w_j^-]/(3y_{lk})$, for $j = 1, 2, 3, 4$ and $k = 1, 2, 3$. In contrast, the restrictions for the weights in Eq. 6.19 are $w_j^- \geq 1/x_{lj}$ and $w_k^+ \geq 1/y_{lk}$, for $j = 1, 2, 3, 4$ and $k = 1, 2, 3$. Note that this selection of the restrictions for the weights, $w_j^-$ and $w_k^+$ in Eq. 6.32 (that is, $W_j^- = 1/x_{lj}$ and $W_k^+ = 1/y_{lk}$) when $\delta$ is zero, is the same as the specified airport measurement approximation. This means that the weights are different from one airport to another and the results can be used for finding those airports which have done the job right but the scores are not relatively meaningful (see Theorem 4.1). However, when the value of $\delta$ is not zero, Eq. 6.32 (or Eq. 6.31) finds the best points in the feasible area, according to the introduced value of delta and the introduced restrictions for the weights.

As the assigned value of delta is greater and greater, a larger area of the feasible area is introduced (which includes more number of observed airports) to compare airport number $l$ with all the points in that area, and to find the best point which has done the job well in comparison with airport number $l$. At the end of this process, airport number $l$ can find its relative score among other airports, according to the introduced restrictions for the weights. The relative score for airport number $l$, when delta is large enough, is the same as the result in the diameter of Table 5.9 and the score is relatively meaningful, according to the corresponded row for that airport in Table 5.9.

As a result, the shortcomings to measure the magnitude of the linear combinations of the output factors in comparison with the linear combination of the input factors are rectified. Instead of finding the maximum value of the relative scores of airport number $l$, the exact relative score of the airport is measured, according to the restrictions of the weights. If the way of introducing the restrictions for the weights satisfies Theorem 4.1, the exact relative scores of airports are measured. For instance, suppose that Eq. 6.32 is applied while the restrictions of the weights are introduced by $W_1^- = 10$, $W_2^- = 1$, $W_3^- = 100$, $W_4^- = 10$, $W_1^+ = 100$, $W_2^+ = 1$, and $W_3^+ = 10$.

Table 6.11 represents the results by Eq. 6.32 for several values of delta. The scores of airports with five decimal digits are not changed for $\delta \geq 0.6$, and are the

**Table 6.11** The suggested score by Eq. 6.32

| Delta | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 0 | 1.0000 | 1.0000 | 0.4874 | 1.0000 | 0.1668 | 1.0000 | 1.0000 | 1.0000 |
| 0.0001 | 0.9997 | 0.9992 | 0.4873 | 0.9956 | 0.1668 | 0.9999 | 0.9994 | 1.0000 |
| 0.001 | 0.9970 | 0.9956 | 0.4866 | 0.9578 | 0.1668 | 0.9992 | 0.9940 | 1.0000 |
| 0.01 | 0.9709 | 0.9555 | 0.4800 | 0.8312 | 0.1668 | 0.9918 | 0.9407 | 1.0000 |
| 0.1 | 0.7793 | 0.6434 | 0.4101 | 0.5347 | 0.1668 | 0.9125 | 0.5146 | 1.0000 |
| 0.2 | 0.6518 | 0.6302 | 0.3292 | 0.4620 | 0.1668 | 0.8117 | 0.4148 | 1.0000 |
| 0.3 | 0.5688 | 0.6302 | 0.3292 | 0.4620 | 0.1668 | 0.6940 | 0.3918 | 1.0000 |
| 0.4 | 0.5104 | 0.6302 | 0.3292 | 0.4620 | 0.1668 | 0.5550 | 0.3918 | 1.0000 |
| 0.5 | 0.4671 | 0.6302 | 0.3292 | 0.4620 | 0.1668 | 0.4998 | 0.3918 | 1.0000 |
| 0.6 | 0.4608 | 0.6302 | 0.3292 | 0.4620 | 0.1668 | 0.4998 | 0.3918 | 1.0000 |
| 0.7 | 0.4608 | 0.6302 | 0.3292 | 0.4620 | 0.1668 | 0.4998 | 0.3918 | 1.0000 |
| 0.8 | 0.4608 | 0.6302 | 0.3292 | 0.4620 | 0.1668 | 0.4998 | 0.3918 | 1.0000 |
| 0.9 | 0.4608 | 0.6302 | 0.3292 | 0.4620 | 0.1668 | 0.4998 | 0.3918 | 1.0000 |
| 0.99 | 0.4608 | 0.6302 | 0.3292 | 0.4620 | 0.1668 | 0.4998 | 0.3918 | 1.0000 |
| 0.999 | 0.4608 | 0.6302 | 0.3292 | 0.4620 | 0.1668 | 0.4998 | 0.3918 | 1.0000 |
| Rank | 5.0000 | 2.0000 | 7.0000 | 4.0000 | 8.0000 | 3.0000 | 6.0000 | 1.0000 |

same as the results in Table 5.2. The last row in Table 6.11 also illustrates the ranks of airports which are the same as the ranks in Table 5.2.

In fact, Eq. 6.32 provides a relationship between the concepts of 'doing the job right' and 'doing the job well', by changing the values of delta. When the value of delta is zero, Eq. 6.32 represents the airports which have done the job right according to the radiate, the convexity and the wholly dominant approaches, and as the value of delta increases, Eq. 6.32 displays the airports which have done the job well according to the introduced restrictions for the weights. This is due to this fact that the model compares every point in the feasible area (regarding the value of delta) with the evaluated airport, as the objective of Eq. 6.25 displays. When the value of delta is large enough, every factor of the evaluated airport can be regulated to find the point which minimizes the objective in Eq. 6.25.

In spite of all the above valuable findings, in order to discriminate the airports which have done the job well, there is still a need to estimate the restrictions of the weights by expert judgment, while the units of measurement/worth/prices/weights of factors are unknown.

## 6.5 Conclusion

In this chapter, one set of weights models and the delta neighborhood models are introduced to measure the performance of the eight airports, in which each airport has four input factors and three output factors. The robust linear programming, the delta neighborhood model, is gradually proposed to bridge the gap between the concepts of doing the job right and doing the job well, corresponding to a delta parameter. The parameter belongs to the set of non-negative real numbers. When the parameter is equal to zero, the airports which 'have done the job right' are identified, according to the radiate, the convexity, and the wholly dominant approaches. As the parameter increases, the airports which have done the job well are identified according to the introduced restrictions for the weights and the value of the parameter. As discussed repeatedly, the concept of doing the job right depends upon the approaches which are used to create the practical points. In contrast, the concept of doing the job well depends upon the introduced restrictions for the weights. Therefore, while the units of measurement (weights, worth or prices) of the factors are unknown, decision making is not pure, but there are several ways to estimate the results, as discussed in this chapter. In order to have accurate results, we need to know 'how practical points should be generated' and 'how the weights should be restricted' by expert judgment.

## 6.6 Exercises

6.1. Recall Exercise 5.1, by add the point, (856, 164,358, 29,237, 329,895, 40,900, 3,359,134, 18,764), as the 9th airport (labeled I) in Table 3.1. Then,

- 6.1.1. Find the rank and the relative score of airport I by the following equations and describe the results.
  - 6.1.1.1. Eq. 6.2
  - 6.1.1.2. Eq. 6.3
  - 6.1.1.3. Eq. 6.4
  - 6.1.1.4. Eq. 6.5
- 6.1.2. Find the rank and the relative score of airport I using,
  - 6.1.2.1. Eq. 6.19
  - 6.1.2.2. Eqs. 6.20 and 6.23
  - 6.1.2.3. Eq. 6.26
  - 6.1.2.4. Eq. 6.31
  - 6.1.2.5. Eq. 6.32

6.2. Using Form 2, find the minimum score for the relative scores of A while G gets the relative score of 1?

6.3. Show that the dual linear programming of Eq. 5.25 is Eq. 6.34, and solve the model on an Excel sheet. Compare the results with the outcomes in Table 5.13, and at least illustrate two shortcomings of the model.

$$\begin{aligned}&\max\varphi,\\&\text{Subject to}\\&\textstyle\sum_{i=1}^{8}x_{ij}\lambda_i\le x_{lj},\quad\text{for}\quad j=1,2,3,4,\\&\textstyle\sum_{i=1}^{8}y_{ik}\lambda_i\ge y_{lk}\varphi,\quad\text{for}\quad k=1,2,3,\\&\lambda_i\ge 0,\quad\text{for}\quad i=1,2,\ldots,8.\end{aligned}\tag{6.34}$$

6.4. Show that the dual linear programming of Eq. 5.26 is Eq. 6.35, and solve the model on an Excel sheet. Compare the results with the outcomes in Table 5.16, and at least illustrate two shortcomings of the model.

$$\begin{aligned}&\min\theta,\\&\text{Subject to}\\&\textstyle\sum_{i=1}^{8}x_{ij}\lambda_i\le x_{lj}\theta,\quad\text{for}\quad j=1,2,3,4,\\&\textstyle\sum_{i=1}^{8}y_{ik}\lambda_i\ge y_{lk},\quad\text{for}\quad k=1,2,3,\\&\lambda_i\ge 0,\quad\text{for } i=i=1,2,\ldots,8.\end{aligned}\tag{6.35}$$

6.5. Find the relationship between the suggested scores by the models in Exercises 6.3 and 6.4.
6.6. Find the dual linear programming for Eq. 5.17.
6.7. Give an example that the suggested scores by Eqs. 5.31 and 6.23 are greater than that of Eqs. 5.25 and 5.26.
6.8. Find the minimum value of delta in Eq. 6.32, where the scores of the airports are not changed for every greater value of that delta.
6.9. Solve Eq. 6.25 when the restrictions of the weights are introduced as $W_j^- = 1/ave\{x_{ij}, i = 1, 2 \ldots, 8\}$ and $W_k^+ = 1/ave\{y_{ik}, i = 1, 2 \ldots, 8\}$, for $j$ = 1, 2, 3, 4 and $k$ = 1, 2, 3.
6.10. Prove that Eqs. 6.36 and 6.37 are equivalent when the value of delta is large enough, ($\delta > 0$ and $x_j$ is a variable corresponded to $j^{\text{th}}$ input factor). In addition, $x_j^* = x_{lj} - s_j^{-*} + \delta/W_j^-$, for $\delta \gg 0$.

$$\begin{aligned}&\max\textstyle\sum_{j=1}^{4}W_j^- s_j^-,\\&\text{Subject to}\\&\textstyle\sum_{i=1}^{8}x_{ij}\lambda_i+s_j^-=x_{lj}+\delta/W_j^-,\quad\text{for}\quad j=1,2,3,4,\\&\textstyle\sum_{i=1}^{8}y_{ik}\lambda_i-s_k^+=y_{lk},\quad\text{for}\quad k=1,2,3,\\&\lambda_i\ge 0,\quad\text{for}\quad i=1,2,\ldots,8,\\&s_j^-\ge 0,\quad\text{for}\quad j=1,2,3,4,\\&s_k^+\ge 0,\quad\text{for}\quad k=1,2,3.\end{aligned}\tag{6.36}$$

$$\begin{aligned}
&\min \textstyle\sum_{j=1}^{4} W_j^- x_j,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{8} x_{ij}\lambda_i \le x_j, \quad \text{for} \quad j = 1,2,3,4,\\
&\textstyle\sum_{i=1}^{8} y_{ik}\lambda_i \ge y_{lk}, \quad \text{for} \quad k = 1,2,3,\\
&\lambda_i \ge 0, \quad \text{for} \quad i = 1,2,\ldots,8
\end{aligned} \tag{6.37}$$

6.11. Prove that Eqs. 6.38 and 6.39 are equivalent when the value of delta is large enough, where $\delta > 0$ and $y_k$ is a variable corresponded to $k^{\text{th}}$ output factor. In addition, $y_k^* = y_{lk} + s_k^{+*} - \delta/W_k^+$, for $\delta \gg 0$.

$$\begin{aligned}
&\max \textstyle\sum_{k=1}^{3} W_k^+ s_k^+,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{8} x_{ij}\lambda_i + s_j^- = x_{lj}, \quad \text{for} \quad j = 1,2,3,4,\\
&\sum_{i=1}^{8} y_{ik}\lambda_i - s_k^+ = y_{lk} - \delta/W_k^+, \quad \text{for} \quad k = 1,2,3,\\
&\lambda_i \ge 0, \quad \text{for} \ i = 1,2,\ldots,8,\\
&s_j^- \ge 0, \quad \text{for} \quad j = 1,2,3,4,\\
&s_k^+ \ge 0, \quad \text{for} \quad k = 1,2,3
\end{aligned} \tag{6.38}$$

$$\begin{aligned}
&\max \textstyle\sum_{k=1}^{3} W_k^+ y_k,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{8} x_{ij}\lambda_i \le x_{lj}, \quad \text{for} \quad j = 1,2,3,4,\\
&\textstyle\sum_{i=1}^{8} y_{ik}\lambda_i \ge y_k, \quad \text{for} \ k = 1,2,3,\\
&\lambda_i \ge 0, \quad \text{for} \quad i = 1,2,\ldots,8.
\end{aligned} \tag{6.39}$$

6.12. Prove that Eqs. 6.40 and 6.41 are equivalent when the value of delta is large enough.

$$\begin{aligned}
&\max \textstyle\sum_{j=1}^{4} W_j^- s_j^- + \sum_{k=1}^{3} W_k^+ s_k^+,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{8} x_{ij}\lambda_i + s_j^- = x_{lj} + \delta/W_j^-, \quad \text{for} \quad j = 1,2,3,4,\\
&\textstyle\sum_{i=1}^{8} y_{ik}\lambda_i - s_k^+ = y_{lk} - \delta/W_k^+, \quad \text{for} \quad k = 1,2,3,\\
&\lambda_i \ge 0, \quad \text{for} \quad i = 1,2,\ldots 8,\\
&s_j^- \ge 0, \quad \text{for} \quad j = 1,2,3,4,\\
&s_k^+ \ge 0, \quad \text{for} \quad k = 1,2,3.
\end{aligned} \tag{6.40}$$

$$\begin{aligned}
&\max \sum_{k=1}^{3} W_k^+ y_k - \sum_{j=1}^{4} W_j^- x_j.\\
&\text{Subject to}\\
&\sum_{i=1}^{8} x_{ij}\lambda_i \le x_j, \quad \text{for} \quad j = 1,2,3,4,\\
&\sum_{i=1}^{8} y_{ik}\lambda_i \ge y_k, \quad \text{for} \quad k = 1,2,3,\\
&\lambda_i \ge 0, \quad \text{for} \quad i = 1,2,\ldots,8.
\end{aligned} \tag{6.41}$$

6.13. Prove that the results of Eqs. 6.42 can also be calculated by Eq. 6.39.

$$\begin{aligned}
&\max \sum_{k=1}^{3} W_k^+ y_k - \sum_{j=1}^{4} W_j^- x_j\\
&\text{Subject to}\\
&x_j = \sum_{i=1}^{8} x_{ij}\lambda_i \le x_{lj}, \quad \text{for} \quad j = 1,2,3,4,\\
&y_k = \sum_{i=1}^{8} y_{ik}\lambda_i \ge y_{lk}, \quad \text{for} \quad k = 1,2,3,\\
&\lambda_i \ge 0, \quad \text{for} \quad i = 1,2,\ldots,8.
\end{aligned} \tag{6.42}$$

6.14. Solve Eqs. 6.36 and 6.37 when $W_1^- = 10$, $W_2^- = 1$, $W_3^- = 100$, $W_4^- = 10$, for data in Table 3.1 and different values for $\delta > 0$. Describe the results and the weaknesses of these models in comparison with Eq. 6.31.

6.15. Solve Eqs. 6.38 and 6.39 when $W_1^+ = 100$, $W_2^+ = 1$, and $W_3^+ = 10$, for data in Table 3.1 and different values for $\delta > 0$. Describe the results and the weaknesses of these models in comparison with Eq. 6.31.

6.16. Solve Eqs. 6.40 and 6.41 when $W_1^- = 10$, $W_2^- = 1$, $W_3^- = 100$, $W_4^- = 10$, $W_1^+ = 100$, $W_2^+ = 1$, and $W_3^+ = 10$, for data in Table 3.1 and different values for $\delta > 0$. Describe the results and the weaknesses of these models in comparison with Eq. 6.31.

6.17. Find the dual linear programming for Eqs. 6.37, 6.39 and 6.41.

# Chapter 7
# Data Envelopment Analysis

## 7.1 Introduction

The outcomes from the previous chapters provide useful information from the literature of operations research and economics on measuring the performance of a set of homogenous firms with multiple input factors and multiple output factors as well as ranking and benchmarking firms. If firms are not homogenous, the situation is the same as when each factor has a different unit of measurement from one firm to another, and therefore, no meaningful discrimination can be expressed, unless the simple conditions of discrimination, which are represented in Sect. 2.3, are satisfied. In this chapter, the concepts introduced in the previous chapters are adapted with the literature and the philosophical background is discussed.

## 7.2 Reestablishing the Introduced Phrases

In the previous chapters, several phrases are repeatedly used, such as, 'doing the job well', 'doing the well job' and 'doing the useful job'. Since different purposes of discrimination in real-life applications may introduce different meanings for these concepts, each concept should transparently be defined according to the purpose of discrimination to avoid causing any doubts of confusion in research, findings, statements, and so on. In other words, any confusion or misinterpretation about these concepts and phrases which are misleading, cause awkward outcomes, and unfair decision making.

In this book, the purpose of discrimination for a set of homogenous firms (factories, organization, divisions and so on), in which each firm has multiple input factors and multiple output factors, is to find the firms which have lesser values of the input factors and greater values of the output factors. For such an aim,



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the ratio of a linear combination of the output factors to a linear combination of the input factors is introduced as the measure for the purpose of discrimination. Therefore, the weight/price/worth of each factor is required to introduce the linear combination of the factors and allow measuring the performances of the firms, according to Types 1–6. Since the firms are homogenous, the weight/price/worth of each factor should not be varied from one firm to another, unless all differences known, and multiplied to the values of the factors before the evaluation, as illustrated in Sects. 2.3 and 4.3.

The concepts of 'doing the job right', 'doing the job well', 'doing the useful job' and so on, were similarly introduced in the literature of economics and operations research, but differently interpreted with several words and phrases in recent decades, such as, *efficiency*, *technical efficiency*, *price efficiency*, *productive efficiency*, *relative efficiency*, *economic efficiency*, *allocative efficiency*, *overall efficiency*, *productivity*, and so on.

Philosophically, we should avoid using several phrases for a concept, and should cautiously clarify whether the relationship between that concept and what we express is meaningful. For instance, the word 'efficiency' is commonly used instead of the phrases 'technical efficiency' and 'relative efficiency'. If these phrases illustrate the same concept, the word 'efficiency' should be enough to mention that concept and any further terminology is redundant and misleading.

The same criticism can be illustrated for the phrases 'price efficiency', 'overall efficiency', 'productive efficiency' and so on.

According to the Cambridge English dictionary, the word 'efficiency' means "the condition or fact of producing the results you want without waste, or a particular way in which this is done", the phrase 'technical efficiency' means "a situation in which a company or a particular machine produces the largest possible number of goods with the time, materials, labor, etc. that are available", and the word 'relative' means "as judged or measured in comparison with something else". Even from the literal definition, we can see the terms are clearly different in meaning. Therefore, it is vital to review the meaning of these phrases in the literature of economics, engineering and operations research, and reintroduce them with industry-wide accuracy and understanding.

### 7.2.1 The Technical Efficiency Measurement

Suppose that there are several homogenous firms which each firm uses a set of input factors to produce a set of output factors. In the literature of *the production theory*, a *Production Possibility Set* (PPS) is a set of all possible situations which a set of output factors can be produced from a set of input factors. A *production function* (*production frontier*) is also a function that gives the maximum possible values of the output factors from the values of the input factors. The points on the production function are called the *technically efficient* points, and this definition is matched to the literal definition as well. None of the coordinates of the technically efficient points can be improved without worsening another coordinate, that is, none of the values of the input (output) factors can be decreased (increased) without increasing the value of another input factor or decreasing the value of another output factor. A point which is in the PPS, and does not lie on the production function is called *technically inefficient*. When a point is technically inefficient, at least one of its input or output factors can be improved to reach the production function in order to be technically efficient. In other words, a technical inefficient point is dominated by one technical efficient point at least.

Figure 7.1 depicts the production function for a set of six firms, labeled A-F, which each firm has two input factors to produce a single constant output factor, as well as the related PPS and the technically efficient and inefficient firms.

The horizontal axis in Fig. 7.1 represents the values of the first input factor per unit of the output factor and the vertical axis represents the values of the second input factor per unit of the output factor. The curve $SS'$ in Fig. 7.1 is called the production function and the above area of the curve is the related PPS.

The firms A-D which lie on the production function are technically efficient and the firms E and F, which are inside the PPS, are technically inefficient. Firms A-C wholly dominate E; for instance, E and C used the same value of the first input factor, but E used the greater value of the second input factor, and for this reason E is technically inefficient. In other words, the points which are inside the PPS can be

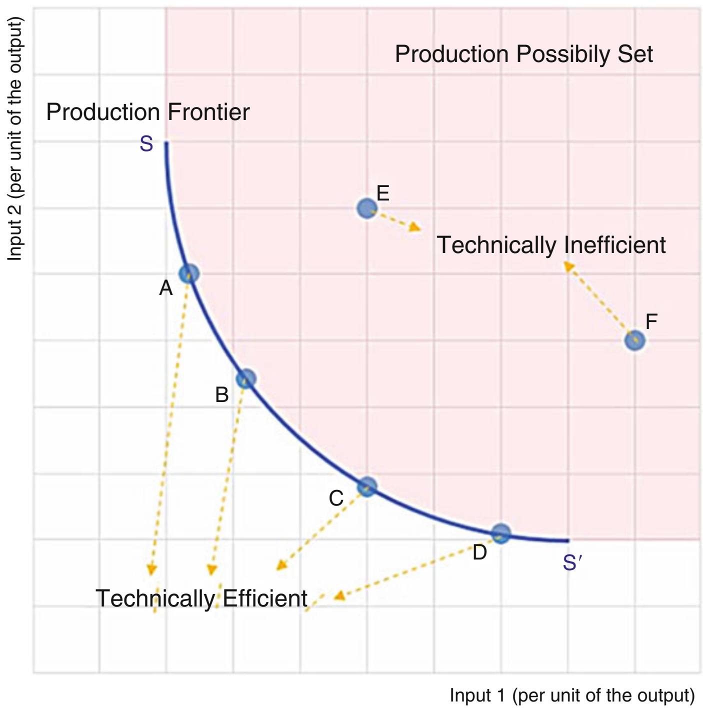


**Fig. 7.1** A PPS of a set of six firms

compared with at least one point which is on the frontier of the PPS in Fig. 7.1. Is this phenomenon enough to discriminate between the firms A-F? Is the performance of D always better than the performance of E? Is there any point inside of the PPS which has a better performance than B?

The above illustrations are similar to the discussions about Figs. 3.5 and 3.6 in Chap. 3. Indeed, the PPS is the practical region; the production function is the frontier of the feasible area, and the technically efficient points are the points which have done the job right. In other words, the concept of doing the job right is the same as the concept of technical efficiency, and the word ‘technical’ refers to the used technology (approach) to introduce the practical points.

As illustrated in Chaps. 1–6, when a firm has done the job right, it means that the firm has produced the maximum possible values of the output factors from a set of the input factors. In addition, the concept of doing the job right depends upon the introduced approach to generate the practical points, and this is the same as the concept of technical efficiency which depends upon the use of technology to define the production function.

It is repeatedly illustrated in Chaps. 1–6 that the concept of doing the job right is only a necessary condition to discriminate between firms, and is not enough to introduce the firms which have done the job well. For instance, both firms A and D have done the job right in Fig. 7.1; however, if it is supposed that they have done the job well at the same time, a paradox is generated according to illustration in Sects. 4.2.5 and 4.3. If a firm lies on the production function, it does not logically say that the firm has done the job well. It is possible that a firm which does not lie on the production function has done the job better than the firm which lies on the production function, as illustrated in Chaps. 1–6. In other words, the discrimination between firms based on the production function only, (even if the production function is exactly available), is not valid. The important pros of technical efficiency are to estimate the production function and find the firms which can be candidates for the concept of doing the job well, (without introducing the firm which has done the job well). A firm which has done the job well in comparison with all other firms is technically efficient and lies on the production function, but the points on the production function, which are technically efficient, have not necessarily done the job well.

### 7.2.2 *Efficiency Measurement*

Let’s suppose that the line $TT'$ has the same slope as the ratio of the prices/weights/worth of the two input factors, as depicted in Fig. 7.2. From the figure, D has done the job well in comparison with all other firms.

Thus, the firms can be arranged from the highest rank to the lowest rank, given by D, C, B, A, F and E, respectively. As illustrated in Sect. 4.4, point P is not practical (according to the PPS), but has the same worth as D’s performance, and allows discrimination between E and D. According to Type 5, E can increase the

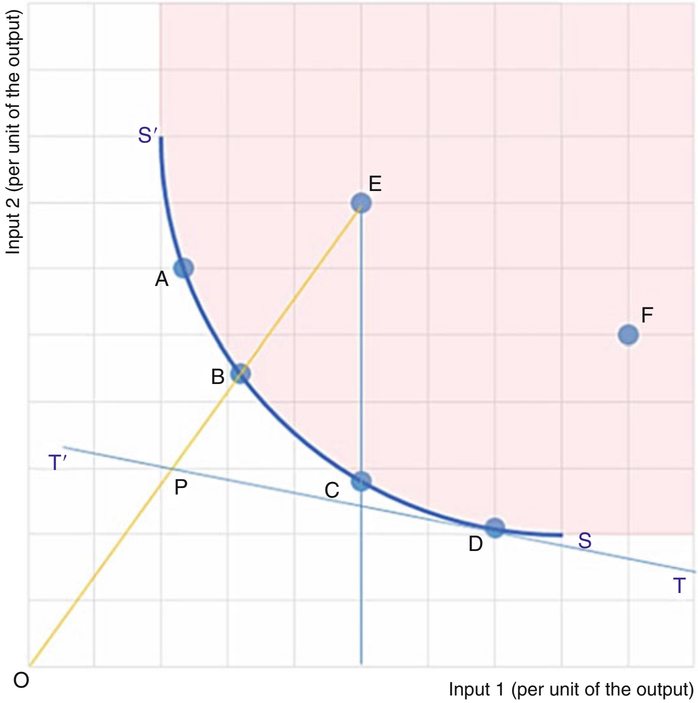


**Fig. 7.2** The price/overall/relative/allocative/economic efficiency of E

value of the first input factor and decrease the value of the second output factor to perform as well as D within the PPS.

From the literature of economics, the ratio of *OB*/*OE* is called *the technical efficiency of E*, the ratio of *OP*/*OB* is called *the price efficiency of B* or *the allocative efficiency* of E, and the ratio of *OP*/*OE* is called the ratio of *the* o*verall efficiency of E* or *the economic efficiency* of E. All of these ratios are less than equal to 1.

As can be seen, the overall efficiency of E has exactly the same meaning as *the price efficiency of E*, which can also be measured by multiplying the technical efficiency of E and the allocative efficiency of E, that is, $(OB/OE) \times (OP/OB) = OP/OE$. Therefore, at least two of phrases, 'price efficiency', 'overall efficiency' or 'economic efficiency', are redundant.

The concept of allocative efficiency also does not provide a ranking tool similar to the concept of technical efficiency and just describes *the non-technical efficiency*; hence, there is no reason to use a new phrase. Indeed, *the price inefficiency of E* can be decomposed by *the technical inefficiency of E* and *the non-technical inefficiency of E*, thus we avoid using extra and superfluous phrases in this book.

From these definitions, the price (overall/economic) efficiency of D is 1 and the price efficiency of the other firms is always less than equal to 1, thus, the meaning of the price (overall/economic) efficiency can also be interpreted as 'the relative efficiency'. As illustrated in Chap. 2, the equation $w_1x_1 + w_2x_2$ has the same value

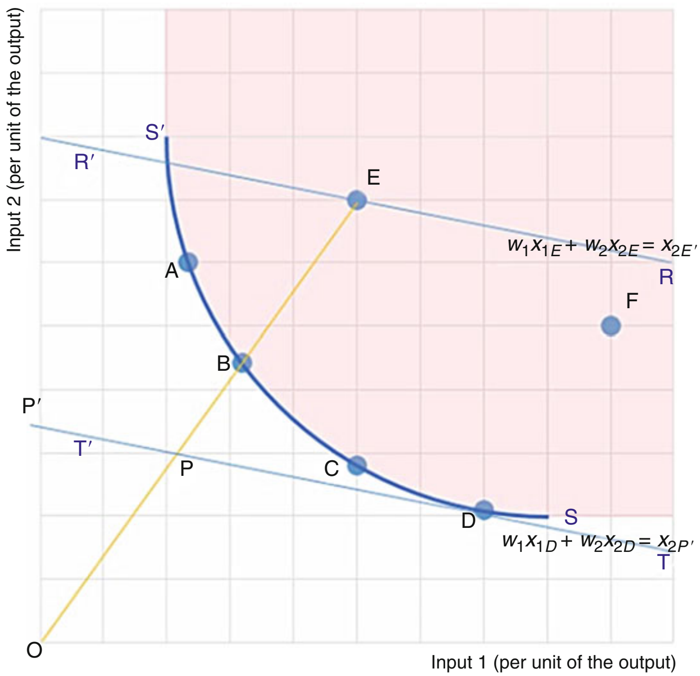


**Fig. 7.3** The concept of doing the job well

for every point $(x_1, x_2)$ on the line $TT'$, and is equal to $w_1x_{1D} + w_2x_{2D}$, where $w_1$ and $w_2$ are the corresponded prices/weights/worth of the first input factor and the second input factor, respectively.

As Fig. 7.3 illustrates, the price (overall/economic) efficiency of E, $OP/OE$, is equal with the ratio of $(w_1x_{1P} + w_2x_{2P})/(w_1x_{1E} + w_2x_{2E})$, which is equal to $(w_1x_{1D} + w_2x_{2D})/(w_1x_{1E} + w_2x_{2E})$. Indeed, the lines $TT'$ and $RR'$ are parallel and the ratio of $OP/OE$ is equal to the ratio of $OP'/OE'$ in the triangle $OEE'$. Hence, the linear combination of the input factors of every point of the PPS is compared with the linear combination of D's input factors. Thus, the provided score is relatively meaningful, and the price (overall/economic) efficiency of E can be introduced as the relative efficiency of E as well. This outcome precisely defines the concept of doing the job well and can be expressed in one word 'efficiency', which is the condition or fact of producing the results that we would want without waste.

Note that, in the literature of operations research, 'relative efficiency' is usually considered as 'technical efficiency', which is incorrect, as illustrated in Chaps. 1–6. In definition of technical efficiency, no suitable discrimination between the points on the production function is introduced. The provided ratio for the technical efficiency, such as, $OB/OE$, is a fake relative score, and does not yield a valid comparison between E and other points in the PPS. In fact, the pros and cons of the technical efficiency are the same as that of doing the job right, and the provided score for the concept of doing the job right is not relatively meaningful.

**Table 7.1** Reestablishing the concepts/phrases

| The concept/approach in Chaps. 1, 2, and 3 | Reestablishing |
|---|---|
| Doing the job right | Technical efficiency |
| Doing the job well | Efficiency |
| Doing the right job | Technical effectiveness |
| Doing the well job | Effectiveness |
| Doing the useful job | Productivity |
| The wholly dominant approach | Free disposal hull |
| The wholly dominant and convexity approaches | Variable returns to scale |
| The wholly dominant, convexity and radiate approaches | Constant returns to scale |
| The wholly dominant, convexity and inner radiate approaches | Decreasing returns to scale |
| The wholly dominant, convexity and outer radiate approaches | Increasing returns to scale |

Since, the concept of doing the job well depends on the weights/prices/worth of the factors, and requires the concepts of the wholly dominant, the convexity and the radiate approaches to discriminate the firms linearly; the efficiency also depends on the weights/prices/worth of the factors, and at least requires the concepts of the wholly dominant, the convexity and the radiate approaches to discriminate the firms linearly.

In the literature of economics and operations research, the wholly dominant approach is called *Free Disposal Hull* (FDH) technology, the combination of the wholly dominant and the convexity approaches is called *Variable Returns to Scale* (VRS) technology, and the combination of the wholly dominant, the convexity and the radiate approaches is called *Constant Returns to Scale* (CRS) technology, as Table 7.1 illustrates. The technical efficiency (doing the job right) depends on the FDH, VRS or CRS technologies and does not provide discrimination between firms, but the efficiency (doing the job well) depends on the relationships between the factors and at least requires *the CRS technical efficiency* to discriminate firms linearly.

When a firm is not efficient, it is *inefficient*. If one desires to decompose the inefficiency of a firm, inefficiency can be decomposed by the CRS technical inefficiency and the non-CRS-technical inefficiency. The CRS technical inefficiency can also be decomposed by VRS technical inefficiency and non-VRS technical inefficiency, and so on. This topic is discussed in the upcoming sections.

In short, the meaning of CRS technical efficiency should not be misinterpreted as efficiency, similar to the concept of doing the job right which should not be misinterpreted with the concept of doing the job well. *As illustrated in Chap. 1, if using \$200 at most yields \$200, and \$220 yields \$700, the point (220, 700) is more efficient than the point (200, 200), and this is our suitable choice, regardless of whether we are applying VRS, FDH or any other approaches to define the production function.* In other words, when our purpose is to rank a set of homogenous firms, at least CRS technical efficiency should be measured. Of course, after finding the best firm and measure the concept of partially dominant, the exact returns to scale is required to estimate the production function.

*Please note, from here forward in this book, instead of the phrase 'CRS technical efficiency' the phrase 'technical efficiency' will be exclusively used.*

### *7.2.3 The Productivity Measurement*

The concept of doing the job well is also introduced as '*productivity*' in the literature of economics. There is no problem if one desires to call 'doing the job well' as productivity and one can use the pair 'technical efficiency and productivity' instead of the pair 'technical efficiency and efficiency'; nonetheless, as illustrated in Sect. 2.3, after measuring the concept of doing the job well, there is still a need to measure whether the outcomes satisfy the goals of firms. Indeed, the concept of 'doing the well job' is different from the concept of 'doing the job well', and requires another meaningful name.

According to Cambridge English dictionary, "the ability to be successful and produce the intended results" is called '*effectiveness*'. Therefore, 'effectiveness' can be used for the concept of 'doing the well job' and 'productivity' can be used for the concept of 'doing the useful job', which is a combination of both efficiency and effectiveness. The word 'productivity' means "the rate at which a person, company or country does useful work", according to Cambridge English dictionary. Therefore, it is suggested that the commonly utilized phrases and concepts be reestablished according to the following table.

To clearly explain the above, *let's suppose that a set of 9 homogenous banks, labeled A-I, are selected. Assume that the aim of discrimination is (1) to find the banks which have used a smaller number of tellers to service a greater number of customers, and (2) to find the banks which have at least serviced $y_1$ number of customers in the period of evaluation. Suppose that the production function is available and the location of each bank in the Cartesian coordinate plane is depicted in* Fig. 7.4.

The blue curve represents the production function, the horizontal axis illustrates the number of tellers and the vertical axis displays the number of customers. The banks A-G lie on the production function, and are VRS-technically efficient, and the banks H and I are VRS-technically inefficient.

Similar to Chap. 1, by considering the ratio of the number of customers to the number of tellers, D is the most efficient bank followed by E, H and F, respectively. H is not technically efficient, but, for instance, H is more efficient than A which is VRS-technically efficient.

The banks C, D, E, F, G and H have at least serviced $y_1$ number of customers, and are *effective*. F is the most effective bank followed by G, E and H. Therefore, the banks C, D, E, F and H are the most productive banks which are most efficient and most effective at the same time in comparison with other banks.

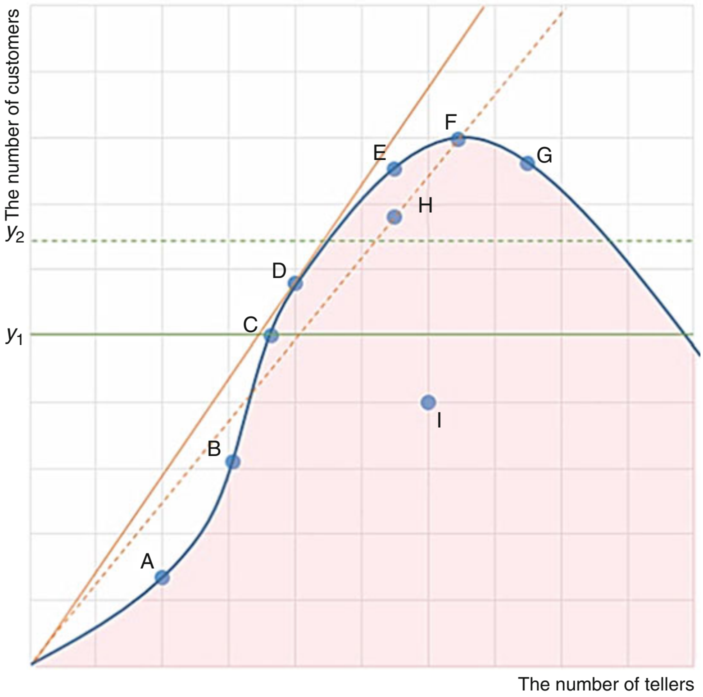


**Fig. 7.4** The production function

The above results can also be seen in Fig. 7.5, where the horizontal axis displays the relative efficiency of each bank and the vertical axis represents the relative effectiveness of each bank. As can be seen, D has the relative score equal to 1 and F has the relative effectiveness equal to 1. The red area illustrates the non-productive banks, which are the banks A, B and I, and have relative effectiveness scores less than that of C and relative efficiency scores less than 0.8.

G is the most effective bank after F, but has the relative efficiency score less than 0.8. The (dark and light) green area represents the productive banks which have relative efficiency scores more than 0.8 and relative effectiveness scores more than that of C.

If the goal of evaluation, which is at least servicing $y_1$ number of customers in period of evaluation, is changed to at least servicing $y_2$ number of customers, as Figs. 7.4 and 7.5 illustrate, even the most efficient banks D and C are not called productive due to the lack of their effectiveness. In this case, the banks E, F and H are the most productive banks among the banks A-I, as the dark green area displays in Fig. 7.5.

The concept of effectiveness is always required in real-life applications, for instance, no bank desires to decrease *consumer satisfaction* and no firm works

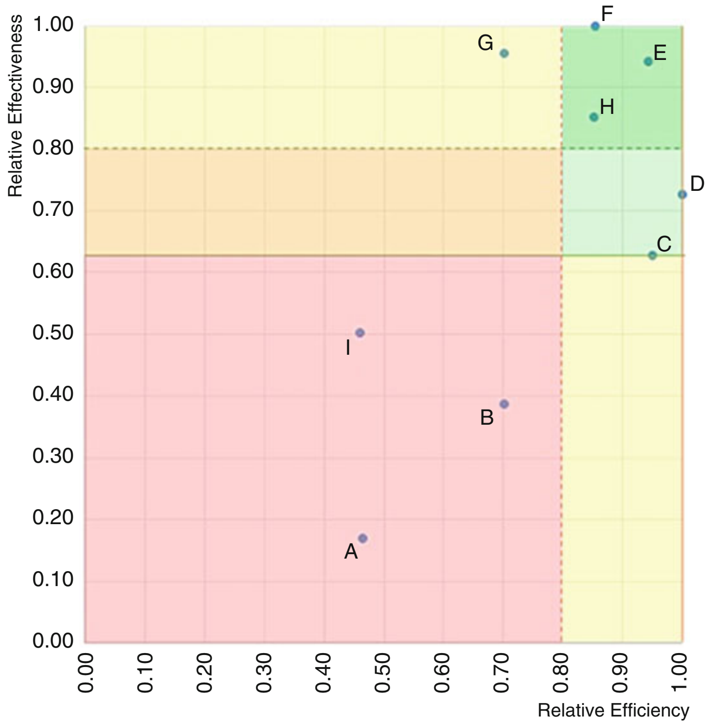


**Fig. 7.5** The productivity measurement

without a plan or some requirements. It is also possible that each bank has different goals for effectiveness index; for instance, a bank may only prefer the production function values according to its set of input factors and in this case, if a different goal for each bank does not affect the homogeneity of the banks, the technical effectiveness index can also be calculated.

In short, there are several concepts which provide the most important indexes to discriminate the performance of homogenous firms. The concept of doing the job right, which is called technical efficiency in this book, is not enough to discriminate between firms. The technical efficiency is usually interpreted as efficiency in the literature, and if one would like to use such a term for the concept of doing the job right, one should be aware that such efficiency is neither enough to discriminate between firms, nor should the corresponded index be used to rank and benchmark the firms. In order to rank and benchmark a set of homogenous firms, the concept of doing the job well is required, and this concept is called efficiency in this book. The efficiency is usually interpreted as productivity in the literature of economics, and similarly if one would like to use the word productivity for the concept of doing the job well, one should know that there is still a need to measure the effectiveness of firms. Therefore, Table 7.1 is provided to reintroduce the concepts of efficiency, effectiveness and productivity with industry-wide accuracy and understanding.

## 7.3 Data Envelopment Analysis

A non-parametric technique to estimate the technical efficiency and the efficiency of a set of homogenous firms was proposed by Farrell (1957). His method estimates the production function non-parametrically, which was similarly suggested by Debreu (1951) and Koopmans (1951). Førsund and Hjalmarsson (1974) illustrated the notion of efficiency in the macro, the industry and the micro levels, and clearly displayed and demonstrated the differences between the production frontier and efficiency. Soon after, Charnes et al. (1978) proposed a mathematical construction and a linear programming model to introduce technically efficient firms with multiple input factors and multiple output factors. They called the mathematical construction '*Data Envelopment Analysis'* (DEA) and the model '*Charnes, Cooper and Rhodes*' (CCR). CCR generates a PPS based upon a set of available homogenous firms, and non-parametrically and linearly estimates the production function; thus, the firms which lie on the frontier of that PPS are called technically efficient. CCR is the same as Forms 1 and 2 (Eqs. 5.25 and 5.26) in Chap. 5 which only introduce technically efficient firms, and the provided scores by CCR (Forms 1 and 2) are neither relatively meaningful, nor can be used to rank and benchmark the firms, as explained in Chap. 5.

In addition, Forms 1 and 2 (CCR) even fail to measure the technical inefficiency completely, and only calculate the output view of technical efficiency (that is, increasing the values of output factors without measuring the excess of input factors) or calculate only the input view of technical efficiency (that is, decreasing the values of input factors without measuring the shortage of output factors). For this reason, Färe and Lovell (1978) noted on Farrell's measurement of technical efficiency and CCR, and proposed a *Russell measure*, to simultaneously deal with both input and output views of technical efficiency. Their proposed model is a non-linear programming, and difficult to solve; thus, Pastor et al. (1999) proposed *Enhanced Russell Measure* (ERM) (see Exercise 7.3) to measure technical efficiency of firms, and avoid computational and interpretative difficulties with the Russell measure.

On the other hand, Charnes et al. (1985) proposed an *Additive model* (ADD), which is the same as Form 3 (Eq. 5.31) in Chap. 5, to remove the shortcomings of CCR (Forms 1 and 2) to measure the technical inefficiency of firms. Nonetheless, ADD (Form 3) is also not a perfect model to measure the technical inefficiency, as illustrated in Chap. 5. Therefore, Tone (2001) proposed a *Slack-Based Measure* (SBM) model (Eq. 6.26) to measure technical inefficiency of firms. He proved that (1) ERM and SBM are equivalent in that the lambda's values that are optimal for one are also optimal for the other, (2) the SBM measurement corresponds to the mean proportional rate of input factors' reduction and the mean proportional rate of output factors' expansion, (3) the SBM measurement is monotone, decreasing in each input and output slack, and (4) it is invariant with respect to the unit of measurement of each input and output item.

As explained in Chap. 5, all proposed models, such as CCR (Forms 1 and 2), ADD (Form 3) and SBM (Eq. 6.26) are provided to measure the technical inefficiency which neither provides a ranking and benchmarking tool, nor introduces the relative efficiency scores for firms. Note that there are a lot of proposed models based on CCR (Forms 1 and 2) in the literature of operations research since 1978 which have the same (or more) mentioned shortcomings to focus on technical efficiency (doing the job right) instead of efficiency (doing the job well), in order to discriminate a set of homogenous firms. While these studies not be further upon here, but readers can examine several of these studies, and review their discrimination and ranking tools as simple exercises, see for example the topic in Exercise 7.4.

Sexton et al. (1986) wisely noted the shortcomings of CCR (Forms 1 and 2) and stated that DEA cannot be used to analyze or comment on a firm's (price) efficiency and a firm can be technically efficient, but (price) inefficient. Thus, they proposed a *cross efficiency* model, which was supposed to measure the score that a particular firm receives when it is rated by another firm. Nonetheless, the cross efficiency score for a firm is also not a relative score for that firm; it is an average value of the relative scores of that firm, according to some specified sets of weights, which neither should be used to rank firms, nor is relatively meaningful, similar to discussions in Sects. 2.2.1, 3.5.3.1., and 4.2. The average values of the relative scores of firms similar to the maximum (minimum, first quartile, and so on) values of the relative scores, are not relatively meaningful and should not be suggested as the relative scores of firms, as logically illustrated in Sect. 4.2 as well.

In order to decrease the above shortcomings, Khezrimotlagh et al. (2013) proposed δ-Kourosh and Arash Method (δ-KAM) to bridge between technical efficiency and efficiency. Their proposed model is the same as Eq. 6.20, which is improved to Eq. 6.25 in this book, and will be improved again to cover several new topics.

The score of KAM is different from the scores of other models in the literature, and can be used as a fair judgment tool for ranking and benchmarking firms. The words 'Kourosh' and 'Arash' are also symbolic and referred to 'justice' and 'border' in ancient linguistic history of Persia.

From illustration in Sect. 6.3, while the value of delta is 0, the results of 0-KAM identify the firms which are technically efficient and technically inefficient, and should not be used to rank or benchmark firms. As the value of delta increases, the results of δ-KAM can be used to rank and benchmark firms, according to the value of delta and the introduced assumptions for weights/prices of input and output factors.

As is explained in Chaps. 1–6 the technical efficiency (as well as production function) depends upon the way of introducing the practical points and the efficiency depends upon the weights/worth/prices of the factors, and at least require the combination of the radiate, the convexity and the wholly dominant approaches to linearly estimate the efficiency scores of a set of homogenous firms. In the next sections, KAM is improved to measure the efficiency of firms with the least requirements to the radiate, the convexity and the wholly dominant approaches.

### 7.3.1 DEA Axioms

DEA construction can be used as a non-parametric tool for estimating the production function of a set of homogenous firms. In order to explain DEA, suppose that there are $n$ firms, labeled $F_i\,(i = 1, 2, \ldots, n)$, and each firm has $m$ input factors with the values $x_{ij}\,(j = 1, 2, \ldots, m)$ and $p$ output factors with the values $y_{ik}\,(k = 1, 2, \ldots, p)$. Let's suppose that $X_i$ is the vector of the input factors for $i$th firm, that is, $X_i = (x_{i1}, x_{i2}, \ldots, x_{im})$, and $Y_i$ is the vector of the output factors for $i$th firm, that is, $Y_i = (y_{i1}, y_{i2}, \ldots, y_{ip})$, where $i = 1, 2, \ldots, n$. The corresponded PPS to these firms are generated by the following axioms, where $X = (x_1, x_2, \ldots, x_m)$ and $Y = (y_1, y_2, \ldots, y_p)$:

Axiom 1. Each observed firm should be belonged to the PPS, that is, $\forall i\ (X_i, Y_i) \in$ PPS.

Axiom 2. If $(X, Y) \in$ PPS, then $(\lambda X, \lambda Y) \in$ PPS, for $\lambda \geq 0$, that is, for every point $(X, Y)$ in the PPS, the same proportionate increase (decrease) in input factors results the same proportionate increase (decrease) in the output factors.

Axiom 3. Every point $(X, Y)$ of the line-segment which connects each two points of the PPS should belong to the PPS, that is, if $(X', Y') \in$ PPS and $(X'', Y'') \in$ PPS, then $(X, Y) = [\lambda(X', Y') + (1 - \lambda)(X'', Y'')] \in$ PPS, for $\lambda \in [0, 1]$.

Axiom 4. Every point $(X', Y')$ which has greater or equal values of the input factors with the same values of the output factors in comparison with at least one of the points in the PPS should belong to the PPS, that is, if $X' \geq X$ and $(X, Y) \in PPS$, then $(X', Y) \in$ PPS, (where $X \geq X' \equiv \forall j : x_j \geq x'_j$).

Axiom 5. Every point $(X', Y')$ which has lesser or equal values of the output factors with the same values of the input factors in comparison with at least one of the points in the PPS should belonged the PPS, that is, if $Y' \leq Y$ and $(X, Y) \in$ PPS, then $(X, Y') \in$ PPS, (where $Y \leq Y' \equiv \forall k : y'_k \leq y_k$).

Axiom 6. The PPS is the intersection of all PPSs which have the above properties.

The above axioms are provided from Charnes et al. (1978). As illustrated in Chap. 1, the second axiom is the radiate approach, the third axiom is the convexity approach, and the fourth and fifth axioms are the wholly dominant approach. Therefore, the generated PPS is a CRS-PPS. In order to generate the CRS-PPS linearly, there is a need to apply the convexity approach for the observations without upper bound for $\lambda$, and after that apply the wholly dominant approach, as illustrated in Chap. 1. In other words, the CRS-PPS is a set of points, $(X', Y')$, with $m + p$ dimensions, that is, $\left(x'_1, x'_2, \ldots, x'_m, y'_1, y'_2, \ldots, y'_p\right)$, which have the following conditions $\sum_{i=1}^{n} \lambda_i x_{ij} \leq x'_j$, for $j = 1, 2, \ldots, m$, and $y'_k \leq \sum_{i=1}^{n} \lambda_i y_{ik}$, for $k = 1, 2, \ldots, p$, where $\lambda_i \geq 0$ for $i = 1, 2, .., n$.

The matrix view of $m$ inequalities, $\sum_{i=1}^{n} \lambda_i x_{ij} \leq x'_j$, for $j = 1, 2, \ldots, m$, can be represented as follows:

$$[\lambda_1\ \lambda_2\ \ldots\ \lambda_n]_{1\times n}\times\begin{bmatrix}x_{11} & x_{12} & \ldots & x_{1m}\\ x_{21} & x_{22} & \ldots & x_{2m}\\ \vdots & \vdots & \vdots & \vdots\\ x_{n1} & x_{n2} & \ldots & x_{nm}\end{bmatrix}_{n\times m}\leq[x'_1\ \ x'_2\ \ \ldots\ \ x'_m]_{1\times m}. \tag{7.1}$$

Let's suppose that $X^j=[x_{1j}\quad x_{2j}\quad\ldots\quad x_{nj}]^t{}_{1\times n}$, (that is, the transpose of $j^{\text{th}}$ column in the above matrix of the input factors), which displays $j^{\text{th}}$ values of the input factor of each firm, where $j = 1, 2, \ldots, m$. In addition, assume that $\Lambda=[\lambda_1\quad\lambda_2\quad\ldots\quad\lambda_n]_{1\times n}$, thus, the above equation can be rewritten as follows:

$$\Lambda\cdot X^j\leq x'_j,\quad \text{for}\quad j=1,2,\ldots,m. \tag{7.2}$$

The notation '·' in Eq. 7.2 represents *the inner product*, that is, $\Lambda\cdot X^j=\lambda_1\times x_{1j}+\lambda_2\times x_{2j}+\ldots+\lambda_n\times x_{nj}=\sum_{i=1}^{n}\lambda_i x_{ij}$.

In addition, the matrix view of $p$ inequalities, $y'_k\leq\sum_{i=1}^{n}\lambda_i y_{ik}$, for $k = 1, 2, \ldots, p$, can be represented as follows:

$$[\lambda_1\quad\lambda_2\quad\ldots\quad\lambda_n]_{1\times n}\times\begin{bmatrix}y_{11} & y_{12} & \cdots & y_{1p}\\ y_{21} & y_{22} & \cdots & y_{2p}\\ \vdots & \vdots & \vdots & \vdots\\ y_{n1} & y_{n2} & \cdots & y_{np}\end{bmatrix}_{n\times p}\geq[y'_1\quad y'_2\quad\cdots\quad y'_p]_{1\times p}.$$

Let's suppose that $Y^k=[y_{1k}\quad y_{2k}\quad\ldots\quad y_{nk}]^t{}_{1\times n}$, (that is, the transpose of $k$th column in the above matrix of the output factors), which displays $k$th values of the input factor of each firm, where $k = 1, 2, \ldots, p$. Therefore, the above equation can be rewritten as follows:

$$\Lambda\cdot Y^k\geq y'_k\quad \text{for}\quad k=1,2,\ldots,p. \tag{7.3}$$

From Eqs. 7.2 and 7.3, the CRS-PPS, which is denoted by $T_C$, is given by ($i = 1, 2, \ldots, n, j = 1, 2, \ldots, m,$and $k = 1, 2, \ldots, p$):

$$T_C=\left\{(X',Y')\in\mathbb{R}^{m+p}:\ \ \Lambda\cdot X^j\leq x'_j,\ \ \Lambda\cdot Y^k\geq y'_k,\quad \lambda_i\geq 0,\quad \text{for } i,j,k\right\} \tag{7.4}$$

The frontier of the CRS-PPS is defined as a production function by CRS technology for the firms $F_i(i=1,2,\ldots,n)$, and the firms on the frontier are introduced as technically efficient firms and the other firms are called technically inefficient.

Similar to Chap. 1, if the second axiom (the radiate approach) is removed from Axioms 1–6, the PPS is the VRS-PPS, that is, a PPS which is generated by the combination of the wholly dominant and the convexity approaches. This VRS-PPS, which is also denoted by $T_V$, was proposed by Banker et al. (1984), as follows ($i = 1, 2, \ldots, n, j = 1, 2, \ldots, m,k = 1, 2, \ldots, p$):

$$\left\{(X', Y') \in \mathbb{R}^{m+p} : \Lambda \cdot X^j \leq x'_j, \Lambda \cdot Y^k \geq y'_k, \sum_{\mathrm{i=1}}^{\mathrm{n}} \lambda_i = 1, \lambda_i \geq 0, \text{for } i, j, k\right\}. \quad (7.5)$$

The frontier of the VRS-PPS is defined as a production function by VRS technology for the firms $F_i(i = 1, 2, \ldots, n)$, and the firms on the frontier are introduced as VRS-technically efficient firms and the other firms are called VRS-technically inefficient.

When instead of the radiate approach, the outer radiate approach is used, the lower bound for $\lambda_i$ is greater than 1, that is, $\lambda_i > 1$. The outer radiate approach can be used while the same proportionate increase in the input factors results the greater proportionate increase in the output factors. From Table 7.1, using the outer radiate approach instead of the radiate approach is called Increasing Returns to Scale (IRS) technology. If $\lambda_i \geq 1$, the technology is called Non-Decreasing Returns to Scale (NDRS). The frontier of the NDRS-PPS is defined as a production function by NDRS technology for the firms $F_i(i = 1, 2, \ldots, n)$, and the firms on the frontier are introduced as NDRS-technically efficient firms and the other firms are called NDRS-technically inefficient (See, for instance, Zhu 2014).

If $\lambda_i < 1$, that is, using the inner radiate approach instead of the radiate approach, the same proportionate increase in the input factors results the lesser proportionate increase in the output factors. The inner radiate approach is used while the same proportionate increase in the input factors results in the greater proportionate increase in the output factors. As Table 7.1 displays, the combination of the wholly dominant, the convexity and the inner radiate approaches is called Decreasing Returns to Scale (DRS) technology, and if $\lambda_i \leq 1$, the technology is called Non-Increasing Returns to Scale (NIRS). The frontier of the NIRS-PPS is defined as a production function by NIRS technology for the firms $F_i(i = 1, 2, \ldots, n)$, and the firms on the frontier are introduced as NIRS-technically efficient firms and the other firms are called NIRS-technically inefficient.

An example for IRS (NDRS) technology is the tax income on electricity usage, and for DRS (NIRS) technology is the revenue of publishing a newspaper according to circulation. In addition, the study about *Returns to Scale* (RS) is also important to merge small firms or divide large firms (See, for instance, Coelli et al. 2005).

It is quite possible that a real-life application could have several factors which each factor pursues one of the NDRS, NIRS, and CRS technologies. Nonetheless, the VRS technology is the intersection of all these technologies, and includes all the observations, which certainly provides an estimation of production function to measure VRS technical inefficiencies. The ratio of the technical efficiency over the ratio of VRS-technical efficiency is also called *scale efficiency* in economics (see Cooper et al. 2011).

In short, DEA axioms can be used to estimate the production function which only represents the concept of doing the job right (technical efficiency) with different approaches (technologies); but the provided axioms do not represent the concept of doing the job well (efficiency) and the concepts of allocative models are required (see Førsund et al. 1980). However, as it will be explained, KAM improves DEA to handle both the concepts of doing the job right (technical efficiency) and doing the job well (efficiency) by a single linear programming.

## 7.3.2 DEA Models

In Chap. 5, several linear programming models are introduced to measure the technical inefficiencies. The measurement which considers the possible decreasing of input factors without measuring the shortages of output factors is called *Input Orientation* (IO), and the measurement which considers the possible increasing of output factors without measuring the excesses of input factors is called *Output Orientation* (OO) in DEA (see also Toloo 2014). In other words, IO lets us measure the input technical inefficiencies without measuring the output technical inefficiencies, (that is, Form 2 by Eq. 5.26), OO lets us measure the output technical inefficiencies without measuring the input technical inefficiencies, (that is, Form 1 by Eq. 5.25), and none of these measurements provide a fair discrimination nor they calculate the technical efficiency completely. We also avoid using these measurements to benchmark firms, before calculating efficiency or ranking firms according to the concept of doing the job well.

The measurement which considers both possible decreasing of input factors and possible increasing of output factors is called *Non-Orientation* (NO) in DEA, similar to Form 3 by Eq. 5.28. In order to measure technical inefficiency and solve Eq. 5.28, we use 0-KAM (Eq. 6.31). If the VRS-technical inefficiency is necessary to calculate, we use 0-KAM (Eq. 6.25) when $\sum_{i=1}^{n} \lambda_i = 1$, and decompose technical inefficiency to VRS-technical inefficiency and non-VRS-technical inefficiency. As illustrated, inefficiency can be decomposed to technical inefficiency and non-technical inefficiency (allocative inefficiency). In other words, efficiency can be introduced by multiplying VRS-technical efficiency, non-VRS-technical efficiency (scale efficiency) and non-technical efficiency (allocative efficiency). The results of efficiency can be used for ranking and benchmarking firms, however, none of the decomposed parts of efficiency should independently be used to rank or benchmark firms.

As explained, the provided scores by Form 3 also cannot be used to discriminate firms or describe efficiency. In order to measure efficiency, similar to Forms 1–3, some linear programming proposed by Debreu (1951), Farrell (1957), Färe et al. (1985) and Tone (2002), called *Cost Efficiency* (CE) (Eq. 6.36), *Revenue Efficiency* (RE) (Eq. 6.38) and *Profit Efficiency* (PE) (Eqs. 6.40 and 6.41). Khezrimotlagh (2014) proved that the $\delta$-KAM (Eq. 6.25) is equivalent with these models, (see Exercise 6.10–6.13), while the value of delta is large enough and the coefficients of the slacks are introduced as the available prices. Nonetheless, none of the CE, RE and PE models measure the efficiency of a set of homogenous firms completely, as the next section illustrates. In other words, a suitable model to measure the efficiency of firms should at least satisfy the introduced Types 1–6.

## 7.4 Conclusion

The meaning of technical efficiency, efficiency, effectiveness and productivity are discussed in this chapter. Productivity is a combination of effectiveness and efficiency. In the effectiveness measurement, the factors of each firm are compared with the desired goals of the firms, and in efficiency measurement, the firms' performances are compared to each other. The provided technical efficiency scores should not be used to rank firms; they have unfortunately been wrongly used in the literature of operations research for the last four decades. The discrimination between homogenous firms requires expert judgment either to introduce a set of weights for factors or to specify a measurement approximation to estimate the efficiency scores which are relatively meaningful.

## 7.5 Exercises

7.1. Explain the differences between efficiency and

- 7.1.1. Technical efficiency.
- 7.1.2. Effectiveness.
- 7.1.3. Productivity.

7.2. Describe the following phrases:

- 7.2.1. Allocative efficiency
- 7.2.2. Scale efficiency

7.3. Prove that ERM and SBM are equivalent in that the lambda's values that are optimal for one are also optimal for the other. The ERM is given by the following equation.

$$\begin{aligned}
&\min \frac{(1/m)\sum_{j=1}^{m}\theta_j}{(1/p)\sum_{k=1}^{p}\varphi_k},\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n}\lambda_i x_{ij} \le \theta_j x_{lj}, \quad \text{for} \quad j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n}\lambda_i y_{ik} \ge \varphi_k y_{lk}, \quad \text{for} \quad k = 1, 2, \ldots, p,\\
&\lambda_i \ge 0, \quad \text{for} \quad i = 1, 2, \ldots, n,\\
&0 \le \theta_j \le 1, \quad \text{for} \quad j = 1, 2, \ldots, m,\\
&\varphi_k \ge 0, \quad \text{for} \quad k = 1, 2, \ldots, p.
\end{aligned} \tag{7.6}$$

7.4. The following model is called super-efficiency model (Andersen and Petersen 1993).

$$\begin{aligned}
&\min \theta,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1, i\neq l}^{n}\lambda_i x_{ij} \le \theta_j x_{lj} \quad \text{for} \quad j = 1, 2, \ldots, m\\
&\textstyle\sum_{i=1, i\neq l}^{n}\lambda_i y_{ik} \ge y_{lk}, \quad \text{for} \quad k = 1, 2, \ldots, p,\\
&\lambda_i \ge 0, \quad \text{for} \quad i = 1, 2, \ldots, n,\\
&\theta \ge 0.
\end{aligned} \tag{7.7}$$

7.4.1. Describe Eq. 7.7.
7.4.2. Why the provided ranks for firms $F_i$'s ($i = 1, 2, \ldots, n$) by Eq. 7.7 are not logically and relatively meaningful?
7.4.3. Give a counter example that Eq. 7.7 should not be used to identify outliers as well.
7.4.4. Run a simulation to show the results of BCC Super-efficiency is not even stronger as the result of BCC to rank DMUs.

# Chapter 8
# The Ratio of Output to Input Factors

## 8.1 Introduction

In the previous chapters, we provided transparent steps to learn the foundation of DEA to estimate the performance of a set of homogenous firms. In this chapter, we demonstrate mathematical properties to describe the natural relationships between the DEA frontier and the ratio of output to input factors.

## 8.2 Charnes, Cooper and Rhodes Model

Let's extend Eq. 3.24 in general for $n$ DMUs $A_i$, for $i = 1, 2, \ldots, n$, in which each DMU has $m$ positive input factors $x_{ij}$, for $j = 1, 2, \ldots, m$, and $p$ positive output factors $y_{ik}$, for $k = 1, 2, \ldots, p$. Assume that $A_l$ is evaluated, for $l = 1, 2, \ldots, n$, as Eq. 8.1 displays.

$$\begin{aligned} &\max \frac{\sum_{k=1}^{p} y_{lk} w_k^+}{\sum_{j=1}^{m} x_{lj} w_j^-}, \\ &\text{Subject to} \\ &\frac{\sum_{k=1}^{p} y_{ik} w_k^+}{\sum_{j=1}^{m} x_{ij} w_j^-} \leq 1, \text{for } i = 1, 2, \ldots, n, \\ &w_j^+ \geq 0, \quad \text{for} \quad j = 1, 2, \ldots, m, \\ &w_k^- \geq 0, \quad \text{for} \quad k = 1, 2, \ldots, p. \end{aligned} \tag{8.1}$$

Charnes et al. (1978) proposed Eq. 8.1, and called it Charnes, Cooper and Rhodes' (CCR) model. The optimal value of CCR is at most 1 and we have the following definition:



D. Khezrimotlagh, Y. Chen, *Decision Making and Performance Evaluation Using Data Envelopment Analysis*, International Series in Operations Research & Management Science 269, https://doi.org/10.1007/978-3-319-76345-3_8

**Definition 8.1** A DMU is CCR-efficient if and only if the optimal value of CCR is equal to 1, for some positive optimal solutions $w_j^{-*}$ and $w_k^{+*}$, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

As discussed in the previous chapters, if a DMU is CCR-efficient, the DMU has done the job right, and is technically efficient. Readers should note that if a DMU is CCR-efficient, this does not mean that the DMU has done the job well or the DMU is efficient.

Charnes et al. (1978) transformed CCR to equivalent linear programming models (Forms 1 and 2), using Charnes and Cooper's transformation (1962). They called these two forms as Output-Oriented (OO) CCR and Input Oriented (IO) CCR, respectively.

Equations 8.2 and 8.3 represent IO-CCR and OO-CCR in multiplier forms. Here the word "multiplier" refers to the weights (multipliers) $w_j^+$'s and $w_k^-$'s. As discussed in Chap. 3, IO-CCR only measures the possible decrease in the input factors radially, whereas OO-CCR only measures the possible increase in the output factors radially. In other words, all input factors are decreased with the same proportion, $\theta$, by IO-CCR, where $\theta^*$ is the possible minimum value of $\theta$ and $\theta^* \in (0, 1]$. Similarly, all output factors are increased with the same proportion, $\varphi$, by OO-CCR, where $\varphi^*$ is the possible maximum value of $\varphi$ and $\varphi^* \in [1, +\infty)$

$$
\begin{aligned}
&\theta_l^* = \max \textstyle\sum_{k=1}^{p} y_{lk} w_k^+,\\
&\text{Subject to}\\
&\textstyle\sum_{j=1}^{m} x_{lj} w_j^- = 1,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \le 0, \text{for } i = 1, 2, \ldots, n,\\
&w_j^+ \ge 0, \quad \text{for} \quad j = 1, 2, \ldots, m,\\
&w_k^- \ge 0, \quad \text{for} \quad k = 1, 2, \ldots, p.
\end{aligned}
\tag{8.2}
$$

$$
\begin{aligned}
&\varphi_l^* = \min \textstyle\sum_{j=1}^{m} x_{lj} w_k^-,\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{p} y_{lk} w_k^+ = 1,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \le 0, \text{for } i = 1, 2, \ldots, n,\\
&w_j^+ \ge 0, \quad \text{for} \quad j = 1, 2, \ldots, m,\\
&w_k^- \ge 0, \quad \text{for} \quad k = 1, 2, \ldots, p.
\end{aligned}
\tag{8.3}
$$

**Definition 8.2** A DMU is CCR-efficient if and only if $\theta^* = 1$, for some positive optimal solutions $w_j^{-*}$ and $w_k^{+*}$, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

**Definition 8.3** A DMU is CCR-efficient if and only if $\varphi^* = 1$, for some positive optimal solutions $w_j^{-*}$ and $w_k^{+*}$, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

**Theorem 8.1** *For every DMU,* $\theta^* \varphi^* = 1$.

*Proof* See Exercise 8.3. □

The dual linear programming models for IO-CCR and OO-CCR (Envelopment Forms) are as follows, respectively:

$$\begin{aligned} &\theta_l^* = \min\ \theta_l, \\ &\text{Subject to} \\ &\textstyle\sum_{i=1}^n x_{ij}\lambda_i \le x_{lj}\theta_l, \text{for } j = 1, 2, \ldots, m, \\ &\textstyle\sum_{i=1}^n y_{ik}\lambda_i \ge y_{lk}, \text{for } k = 1, 2, \ldots, p, \\ &\lambda_i \ge 0, \quad \text{for} \quad i = 1, 2, \ldots, n. \end{aligned} \tag{8.4}$$

$$\begin{aligned} &\varphi_l^* = \max\ \varphi_l \\ &\text{Subject to} \\ &\textstyle\sum_{i=1}^n x_{ij}\lambda_i \le x_{lj}, \text{for } j = 1, 2, \ldots, m, \\ &\textstyle\sum_{i=1}^n y_{ik}\lambda_i \ge y_{lk}\varphi_l, \text{for k} = 1, 2, \ldots, p, \\ &\lambda_i \ge 0, \quad \text{for} \quad i = 1, 2, \ldots, n. \end{aligned} \tag{8.5}$$

CCR can easily be solved using Microsoft Excel Solver software. For example, let's consider a data set used in Ali et al. (1995), as Table 8.1 represents. There are 11 DMUs in which each DMU consumes two input factors and produces two output factors.

In order to apply IO-CCR (Eq. 8.2) for data in Table 8.1, by the Microsoft Excel Solver software, the following steps can be followed.

1. Copy the 5 columns of Table 8.1 on an Excel sheet into cells A1:E12.
2. Label A14 as 'Index', and enter number 1 to B14.
3. Label A16 as "Weights" (reserve B16-E16 for changing cells).
4. Label F1 as "Constraints".

**Table 8.1** Example of 11 DMUs with 4 factors

| DMU | Input 1 | Input 2 | Output 1 | Output 2 |
|---|---|---|---|---|
| A01 | 40 | 30 | 160 | 100 |
| A02 | 30 | 60 | 180 | 70 |
| A03 | 93 | 40 | 170 | 60 |
| A04 | 50 | 70 | 190 | 130 |
| A05 | 80 | 30 | 180 | 120 |
| A06 | 35 | 45 | 140 | 82 |
| A07 | 105 | 75 | 120 | 90 |
| A08 | 97 | 67 | 100 | 82 |
| A09 | 100 | 50 | 140 | 40 |
| A10 | 90 | 60 | 140 | 105 |
| A11 | 98 | 65 | 140 | 50 |

5. Assign the following command into F2 ‘=Sumproduct(D2:E2,D$16:E$16)-Sumproduct(B2:C2,B$16:C$16)’.

   Note: the single quotation marks do not belong to the commands.

6. Copy F2 and then paste it into F3-F12.
7. Label E14 as “Constraint” and assign the following command into F14,

   ‘=Sumproduct(Index(B2:C12,B14,0),B16:C16)’

8. Label E18 as “Constraint” and assign the following command into F14,

   ‘=Sumproduct(Index(D2:E12,B14,0),D16:E16)’

9. Open ‘Solver Parameters’ window.
10. Assign ‘F18’ into ‘Set Objective’ and choose ‘Max’.
11. Assign ‘B16:E16’ into ‘By Changing Variable Cells’.
12. Click on ‘Add’ and assign ‘F2:F12’ into ‘Cell Reference’, then select ‘<=’, and assign ‘0’ into ‘Constraint’.
13. Click on ‘Add’ and assign ‘F14’ into ‘Cell Reference’, then select ‘=’, and assign ‘1’ into ‘Constraint’. Then click on OK.
14. Tick ‘Make Unconstrained Variables Non-Negative’.
15. Choose ‘Simplex LP’ from ‘Select a Solving Method’.
16. Click on ‘Solve’.
17. Save or ‘Save As’ your excel file with this type ‘Excel macro-Enabled Workbook (*.xlsm)’.
18. Label H1 as ‘Optimal Theta’.
19. Label I1-L1 as ‘Optimal Weights’ for I1, I2, O1 and O2, respectively.
20. From ‘Developer’ in the toolbar menu, click on the ‘Insert’ icon to open the ‘Form Control’ window.
21. Click on the first icon, ‘Button (Form Control)’, and then click on a place on the Excel sheet.
22. In the opened window with the title ‘Assign Macro’, click on ‘New’. So, the ‘Microsoft Visual Basic for Applications’ window is opened.
23. From the toolbar menu, click on ‘Tools> References...>’ and make sure ‘Solver’ is ticked, and then ‘OK’.
24. Inside of the ‘Microsoft Visual Basic for Applications’ window, write the following commands between ‘Sub Button1_Click ()’ and ‘End Sub’ as Fig. 2.21 depicts.

```
Dim i As Integer
For i = 1 To 11
   Range("B14") = i
   SolverSolve Userfinish:=True
   Range("H" & i + 1) = Round(Range("F18"), 4)
   Range("B16:E16").Copy
   Range("I" & i + 1).Select
   Selection.PasteSpecial Paste:=xlPasteValues
Next i
```

**Table 8.2** The results of IO-CCR for data in Table 8.1

| DMU | $\theta^*$ | $w_1^{-*}$ | $w_2^{-*}$ | $w_1^{+*}$ | $w_2^{+*}$ |
|---|---|---|---|---|---|
| A01 | 1.0000 | 0.0238 | 0.0015 | 0.0000 | 0.0100 |
| A02 | 1.0000 | 0.0156 | 0.0089 | 0.0056 | 0.0000 |
| A03 | 0.7291 | 0.0021 | 0.0200 | 0.0043 | 0.0000 |
| A04 | 1.0000 | 0.0183 | 0.0012 | 0.0000 | 0.0077 |
| A05 | 1.0000 | 0.0042 | 0.0222 | 0.0000 | 0.0083 |
| A06 | 0.9320 | 0.0273 | 0.0010 | 0.0014 | 0.0090 |
| A07 | 0.3564 | 0.0020 | 0.0106 | 0.0000 | 0.0040 |
| A08 | 0.3610 | 0.0022 | 0.0117 | 0.0000 | 0.0044 |
| A09 | 0.4941 | 0.0018 | 0.0165 | 0.0035 | 0.0000 |
| A10 | 0.5122 | 0.0024 | 0.0130 | 0.0000 | 0.0049 |
| A11 | 0.3974 | 0.0014 | 0.0132 | 0.0028 | 0.0000 |

25. Close the 'Microsoft Visual Basic for Applications' window.
26. Click on the small rectangle which was automatically made on the Excel sheet by step 21.
27. The results are represented into cells H2:L12. Column H represents the CCR-scores for DMUs with four decimal digits, and columns I-L illustrate the optimal solutions for the weights, $w_1^{-*}$, $w_2^{-*}$, $w_1^{+*}$, and $w_2^{+*}$.

Table 8.2 illustrates the results of IO-CCR for data in Table 8.1. There are four DMUs which are CCR-efficient, and the rest of DMU are CCR-inefficient.

Note that, the scores in the first column of Table 8.2 are not relatively meaningful, as discussed in Sects. 2.3 and 4.3. As can be seen, the second column in Table 8.3 represents the relative scores of DMUs, according to the set of optimal weights for A01, and the third column in Table 8.3 illustrates the relative scores of DMUs according to the set of optimal weights for A02, and so on. The first row in Table 8.3 also shows that the CCR score for A01 is 1 for every set of weights in Table 8.2. The CCR scores are bold on the diameter of Table 8.3, to emphasize that the scores of CCR in Table 8.2 are not relatively meaningful. For instance, the relative scores for technically efficient DMUs A02 and A04 according to the optimal weights for A11 are 0.610 and 0.540, respectively.

In addition, the instructions to solve the envelopment form of IO-CCR for data in Table 8.1, by the Microsoft Excel Solver software, are as follows:

1. Copy the 5 columns of Table 8.1 on an Excel sheet into cells A1:E12.
2. Label A14 as 'Index', and enter number 1 to B14.
3. Label F1 as "Lambdas" (reserve F2-F12 for changing cells).
4. Label D14 as "Theta" (reserve E14 for the changing cell).
5. Label A16 as "Constraints".
6. Assign the following command into B16

   '=Sumproduct(B2:B12,$F2:$F12)'.

7. Copy B16 and then paste it into C16, D16 and E16.

**Table 8.3** The relative scores of DMUs in Table 8.1

| DMU | A01 | A02 | A03 | A04 | A05 | A06 | A07 | A08 | A09 | A10 | A11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A01 | **1.000** | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| A02 | 0.867 | **1.000** | 0.610 | 0.867 | 0.400 | 1.000 | 0.400 | 0.400 | 0.610 | 0.400 | 0.610 |
| A03 | 0.263 | 0.524 | **0.729** | 0.263 | 0.392 | 0.300 | 0.392 | 0.392 | 0.729 | 0.392 | 0.729 |
| A04 | 1.000 | 0.754 | 0.540 | **1.000** | 0.614 | 1.000 | 0.614 | 0.614 | 0.540 | 0.614 | 0.540 |
| A05 | 0.614 | 0.662 | 1.000 | 0.614 | **1.000** | 0.601 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| A06 | 0.907 | 0.824 | 0.615 | 0.907 | 0.596 | **0.932** | 0.596 | 0.596 | 0.615 | 0.596 | 0.615 |
| A07 | 0.344 | 0.290 | 0.298 | 0.344 | 0.356 | 0.332 | **0.356** | 0.356 | 0.298 | 0.356 | 0.298 |
| A08 | 0.339 | 0.264 | 0.277 | 0.339 | 0.361 | 0.323 | 0.361 | **0.361** | 0.277 | 0.361 | 0.277 |
| A09 | 0.163 | 0.389 | 0.494 | 0.163 | 0.218 | 0.199 | 0.218 | 0.218 | **0.494** | 0.218 | 0.494 |
| A10 | 0.469 | 0.402 | 0.431 | 0.469 | 0.512 | 0.453 | 0.512 | 0.512 | 0.431 | **0.512** | 0.431 |
| A11 | 0.205 | 0.370 | 0.397 | 0.205 | 0.225 | 0.234 | 0.225 | 0.225 | 0.397 | 0.225 | **0.397** |

8. Assign the following command into B17.

   '=Index(B2:B12,$B14)*$E14'

9. Copy B17 and paste it into C17.
10. Assign the following command into D17.

    '=Index(D2:D12,$B14)'

11. Copy D17 and paste it into E17.
12. Open 'Solver Parameters' window.
13. Assign 'E14' into 'Set Objective' and choose 'Min'.
14. Assign 'E14, F2:F12' into 'By Changing Variable Cells'.
15. Click on 'Add' and assign 'B16:C16' into 'Cell Reference', then select '<=', and assign 'B17:C17' into 'Constraint'.
16. Click on 'Add' and assign 'D16:E16' into 'Cell Reference', then select '>=', and assign 'D17:E17' into 'Constraint'. Then click on OK.
17. Tick 'Make Unconstrained Variables Non-Negative'.
18. Choose 'Simplex LP' from 'Select a Solving Method'.
19. Click on 'Solve'.
20. Save or 'Save As' your excel file with this type 'Excel macro-Enabled Workbook (*.xlsm)'.
21. Label H1 as 'Optimal Theta'.
22. Label I1-S1 as 'Optimal Lambdas' for A01-A11, respectively.
23. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.
24. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet.
25. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.
26. From the toolbar menu, click on 'Tools> References...>' and make sure 'Solver' is ticked, and then 'OK'.
27. Inside of the 'Microsoft Visual Basic for Applications' window, write the following commands between 'Sub Button1_Click ()' and 'End Sub' as Fig. 2.21 depicts.

```
Dim i As Integer
For i = 1 To 11
   Range("B14") = i
   SolverSolve Userfinish:=True
   Range("H" & i + 1) = Round(Range("E14"), 4)
   Range("F2:F12").Copy
   Range("I" & i + 1).Select
   Selection.PasteSpecial Transpose:=True
Next i
```

28. Close the 'Microsoft Visual Basic for Applications' window.
29. Click on the small rectangle which was automatically made on the Excel sheet by step 21.

**Table 8.4** The optimal lambdas for data in Table 8.1

| DMU | $\theta^*$ | $\lambda^*_{A01}$ | $\lambda^*_{A02}$ | $\lambda^*_{A04}$ | $\lambda^*_{A05}$ |
|---|---|---|---|---|---|
| A01 | 1.0000 | 1.0000 | | | |
| A02 | 1.0000 | | 1.0000 | | |
| A03 | 0.7291 | 0.2491 | | | 0.7230 |
| A04 | 1.0000 | | | 1.0000 | |
| A05 | 1.0000 | | | | 1.0000 |
| A06 | 0.9320 | 0.2875 | 0.2082 | 0.2975 | |
| A07 | 0.3564 | 0.8465 | | | 0.0446 |
| A08 | 0.3610 | 0.7370 | | | 0.0692 |
| A09 | 0.4941 | 0.4118 | | | 0.4118 |
| A10 | 0.5122 | 0.8963 | | | 0.1280 |
| A11 | 0.3974 | 0.7483 | | | 0.1126 |

30. The results are represented into cells H2:S12. Column H represents the CCR-scores for DMUs with four decimal digits, and columns I-S illustrate the optimal solutions for the lambdas, $\lambda^*_{A01} - \lambda^*_{A11}$, as shown in Table 8.4. Note that positive lambdas are only shown in Table 8.4.

When a DMU lies on the frontier, that is, the DMU is CCR-efficient, the corresponded lambda for that DMU is 1 and the corresponded lambdas for the other DMUs are 0, as displayed in Table 8.4. When a DMU is CCR-inefficient, that is, the DMU does not lie on the frontier, CCR radially projects the DMU on the frontier. In this case, the corresponded positive values of $\lambda^*$ represent the reference sets for that DMU. For instance, the reference sets for A03 are A01 and A05, because $\lambda^*_{A01} = 0.2491 > 0$ and $\lambda^*_{A05} = 0.7230 > 0$. In other words, CCR projects A03 on the line which connects A01 and A05. Similarly, the reference sets for A06 are A01, A02 and A04, because $\lambda^*_{A01} = 0.2875 > 0$, $\lambda^*_{A02} = 0.2082 > 0$ and $\lambda^*_{A04} = 0.2975 > 0$. In this case, CCR projects A06 on the plane which passes A01, A02 and A04.

## 8.3 Banker, Charnes and Cooper Model

Since CCR uses CRS technology, Banker et al. (1984) proposed CCR with VRS technology, and called the model Banker, Charnes and Cooper (BCC). Eqs. 8.6 and 8.7 are IO-BCC and OO-BCC, respectively.

$$
\begin{aligned}
&\theta_l^* = \max\left(\textstyle\sum_{k=1}^{p} y_{lk} w_k^+ + w\right),\\
&\text{Subject to}\\
&\textstyle\sum_{j=1}^{m} x_{lj} w_j^- = 1,\\
&\left(\textstyle\sum_{k=1}^{p} y_{ik} w_k^+ + w\right) - \textstyle\sum_{j=1}^{m} x_{ij} w_j^- \le 0, \text{for } i = 1, 2, \ldots, n,\\
&w_j^+ \ge 0, \quad \text{for} \quad j = 1, 2, \ldots, m,\\
&w_k^- \ge 0, \quad \text{for} \quad k = 1, 2, \ldots, p.
\end{aligned}
\tag{8.6}
$$

$$
\begin{aligned}
&\varphi_l^* = \min(\textstyle\sum_{j=1}^{m} x_{lj} w_j^- + w),\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{p} y_{lk} w_k^+ = 1,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^+ - (\sum_{j=1}^{m} x_{ij} w_j^- + w) \le 0, \text{for } i = 1, 2, \ldots, n,\\
&w_j^+ \ge 0, \quad \text{for} \quad j = 1, 2, \ldots, m,\\
&w_k^- \ge 0 \quad \text{for} \quad k = 1, 2, \ldots, p.
\end{aligned} \tag{8.7}
$$

Similar to CCR, the optimal value of BCC is at most 1, and we have the following definitions:

**Definition 8.4** A DMU is BCC-efficient if and only if $\theta^* = 1$ in Eq. 8.6, for some positive optimal solutions $w_j^{-*}$ and $w_k^{+*}$, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

**Definition 8.5** A DMU is BCC-efficient if and only if $\varphi^* = 1$ in Eq. 8.7, for some positive optimal solutions $w_j^{-*}$ and $w_k^{+*}$, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

The dual linear programming models for IO-BCC and OO-BCC (Envelopment Forms) are as follows, respectively:

$$
\begin{aligned}
&\theta_l^* = \min\theta_l\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i \le x_{lj}\theta_l, \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i \ge y_{lk}, \text{for } k = 1, 2, \ldots, p,\\
&\textstyle\sum_{i=1}^{n} \lambda_i = 1,\\
&\lambda_i \ge 0, \quad \text{for} \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{8.8}
$$

$$
\begin{aligned}
&\varphi_l^* = \min\varphi_l\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i \le x_{lj}, \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i \ge y_{lk}\varphi_l, \text{for } k = 1, 2, \ldots, p,\\
&\textstyle\sum_{i=1}^{n} \lambda_i = 1,\\
&\lambda_i \ge 0, \quad \text{for} \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{8.9}
$$

Since the generated PPS with VRS technology is a subset of the generated PPS with CRS, the BCC score for a DMU is not less than the CCR score for that DMU. In other words, A DMU can be BCC-efficient, but CCR-inefficient.

If the constraint $w \ge 0$ is added to the constraints of the multiplier form of IO-BCC (Eq. 8.6), we have the IO-CCR with the NDRS technology, and if the constraint $w \le 0$ is added to the constraints of the multiplier form of IO-BCC, we have the IO-CCR with the NIRS technology. Similarly, if the constraint $w \ge 0$ is added to the constraints of the multiplier form of OO-BCC (Eq. 8.7), we have the

OO-CCR with the NIRS technology, and if the constraint $w \leq 0$ is added to the constraints of the multiplier form of OO-BCC, we have the OO-CCR with the NDRS technology.

In addition, if the constraint $\sum_{i=1}^{n} \lambda_i \leq 1$ is added to the constraints of the envelopment form of IO-BCC (Eq. 8.8), we have the IO-CCR with the NIRS technology, and if the constraint $\sum_{i=1}^{n} \lambda_i \geq 1$ is added to the constraints of the envelopment form of IO-BCC, we have the IO-CCR with the NDRS technology. Similarly, if the constraint $\sum_{i=1}^{n} \lambda_i \leq 1$ is added to the constraints of the envelopment form of OO-BCC (Eq. 8.9), we have the OO-CCR with the NIRS technology, and if the constraint $\sum_{i=1}^{n} \lambda_i \geq 1$ is added to the constraints of the envelopment form of OO-BCC, we have the OO-CCR with the NDRS technology.

Now, suppose that $s_j^- = x_{lj}\theta - \sum_{i=1}^{n} x_{ij}\lambda_i$, for $j = 1, 2, \ldots, m$, is the excess of the $j$th input factor, and $s_k^+ = y_{lk} - \sum_{i=1}^{n} y_{ik}\lambda_i$, for $k = 1, 2, \ldots, p$, is the shortfall of the $k$th output factor.

**Definition 8.5** (IO-CCR Phase II). A DMU is CCR-efficient if and only if $\theta^* = 1$ and $s_j^{-*} = 0$ and $s_k^{+*} = 0$, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

**Definition 8.6** (OO-CCR Phase II). A DMU is CCR-efficient if and only if $\varphi^* = 1$ and $s_j^{-*} = 0$ and $s_k^{+*} = 0$, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

The same definitions can be illustrated for IO-BCC Phase II and OO-BCC Phase II (see exercise 8.8).

## 8.4 CRS Output-Input Ratio Analysis

As discussed in the previous sections, a DMU labelled $l\,(l = 1, 2, .., n)$ is located on the CRS frontier, that is, DMU$_l$ is technically efficient or DMU$_l$ has 'done the job right', if the optimal value of $\theta$ for DMU$_l$ is 1. This means that

$$\max \frac{\dfrac{\sum_{k=1}^{p} y_{lk} w_k^+}{\sum_{j=1}^{m} x_{lj} w_j^-}}{\max\left\{\dfrac{\sum_{k=1}^{p} y_{ik} w_k^+}{\sum_{j=1}^{m} x_{ij} w_j^-} : i = 1, 2, \ldots, n\right\}} = 1. \tag{8.10}$$

That is, there exist $\tilde{w}_j^- \geq 0$ and $\tilde{w}_k^+ \geq 0$ such that

$$\frac{\sum_{k=1}^{p} \tilde{y}_{lk} w_k^+}{\sum_{j=1}^{m} \tilde{x}_{lj} w_j^-} = \max\left\{\frac{\sum_{k=1}^{p} y_{ik} \tilde{w}_k^+}{\sum_{j=1}^{m} x_{ij} \tilde{w}_j^-} : i = 1, 2, \ldots, n\right\}. \tag{8.11}$$

Similarly, if there are exists $\tilde{w}_j^- \geq 0$ and $\tilde{w}_k^+ \geq 0$ such that Eq. 8.11 holds, DMU$_l$ is located on the CRS frontier, and we have the following theorem.

**Theorem 8.2** *If there exist* $\tilde{w}_j^- \geq 0$, *for* $j = 1, 2, \ldots, m$, *and* $\tilde{w}_k^+ \geq 0$, *for* $k = 1, 2, \ldots, p$, *such that Eq.* 8.11 *holds,* $DMU_l$ *is located on the CRS frontier.*

*Proof* See Exercise 8.7. □

Theorem 8.2 essentially allows a characterization of DMUs that lie on the CRS frontier as having the highest ratio of aggregated outputs to aggregated inputs using some selection of weights (Chen and Ali 2002). The theorem suggests the possibility of finding 'frontier DMUs' by selecting different combinations of input and output factors to determine a specific selection of weights.

Suppose that $J$ is a subset of the indexes of input factors, that is, $J \subseteq \{1, 2, \ldots, m\}$, and $\tilde{w}_j^- = 1$, for $j \in J$, and $\tilde{w}_j^- = 0$, for $j \notin J$. Also assume that $K$ is a subset of the indexes of output factors, that is, $K \subseteq \{1, 2, \ldots, p\}$, and $\tilde{w}_k^+ = 1$, for $k \in K$, and $\tilde{w}_k^+ = 0$, for $k \notin K$. Similar to Theorem 8.2, we have the following theorems.

**Theorem 8.3** *$DMU_l$ is located on the CRS frontier, if the following equation holds*

$$\frac{\sum_{k \in K} y_{lk}}{\sum_{j \in J} x_{lj}} = \max\left\{\frac{\sum_{k \in K} y_{ik}}{\sum_{j \in J} x_{ij}} : i = 1, 2, \ldots, n\right\}. \tag{8.12}$$

*Proof* The proof is simply concluded from Theorem 8.2. □

**Theorem 8.4** *$DMU_l$ is located on the CRS frontier, if the following equation holds*

$$\frac{\sum_{k=1}^{p} y_{lk}}{\sum_{j=1}^{m} x_{lj}} = \max\left\{\frac{\sum_{k=1}^{p} y_{ik}}{\sum_{j=1}^{m} x_{ij}} : i = 1, 2, \ldots, n\right\}. \tag{8.13}$$

*Proof* The proof is simply concluded from Theorem 8.2. □

The outcomes of Theorems 8.2, 8.3, and 8.4 can be examined for data in Table 8.1. As Table 8.5 displays, let's calculate different combinations of input and output factors such as $y_1/x_1$, $y_2/x_1$, $y_1/x_2$, $y_2/x_2$, $(y_1 + y_2)/x_1$, $(y_1 + y_2)/x_2$, $y_1/(x_1 + x_2)$, $y_2/(x_1 + x_2)$ and $(y_1 + y_2)/(x_1 + x_2)$, where $x_1$, $x_2$, $y_1$ and $y_2$ are input 1, input 2, output 1 and output 2, respectively.

Each column in Tables 8.5 and 8.6 represent different combination of input and output factors. The maximum value for each combination is bolded. As can be seen, only DMUs A01, A02, A04 and A06 are located on the CRS frontier, as shown in Tables 8.2-8.4.

In this example, all DMUs on the CRS frontier are found by output–input ratios. However, in contrast with DEA, a performance measure based upon only the ratio of one single output to one single input fails to detect the entirety of performance regarding a set of input and output factors.

**Table 8.5** Applying Theorems 8.2, 8.3 and 8.4 for data in Table 8.1

| DMU | $\frac{y_1}{x_1}$ | $\frac{y_2}{x_1}$ | $\frac{y_1}{x_2}$ | $\frac{y_2}{x_2}$ | $\frac{y_1}{x_1+x_2}$ | $\frac{y_2}{x_1+x_2}$ | $\frac{y_1+y_2}{x_1}$ | $\frac{y_1+y_2}{x_2}$ | $\frac{y_1+y_2}{x_1+x_2}$ |
|---|---|---|---|---|---|---|---|---|---|
| **A01** | 4.00 | 5.33 | 2.50 | 3.33 | **2.29** | **1.43** | 6.50 | 8.67 | **3.71** |
| **A02** | **6.00** | 3.00 | 2.33 | 1.17 | 2.00 | 0.78 | **8.33** | 4.17 | 2.78 |
| A03 | 1.83 | 4.25 | 0.65 | 1.50 | 1.28 | 0.45 | 2.47 | 5.75 | 1.73 |
| **A04** | 3.80 | 2.71 | **2.60** | 1.86 | 1.58 | 1.08 | 6.40 | 4.57 | 2.67 |
| **A05** | 2.25 | **6.00** | 1.50 | **4.00** | 1.64 | 1.09 | 3.75 | **10.00** | 2.73 |
| A06 | 4.00 | 3.11 | 2.34 | 1.82 | 1.75 | 1.03 | 6.34 | 4.93 | 2.78 |
| A07 | 1.14 | 1.60 | 0.86 | 1.20 | 0.67 | 0.50 | 2.00 | 2.80 | 1.17 |
| A08 | 1.03 | 1.49 | 0.85 | 1.22 | 0.61 | 0.50 | 1.88 | 2.72 | 1.11 |
| A09 | 1.40 | 2.80 | 0.40 | 0.80 | 0.93 | 0.27 | 1.80 | 3.60 | 1.20 |
| A10 | 1.56 | 2.33 | 1.17 | 1.75 | 0.93 | 0.70 | 2.72 | 4.08 | 1.63 |
| A11 | 1.43 | 2.15 | 0.51 | 0.77 | 0.86 | 0.31 | 1.94 | 2.92 | 1.17 |

**Table 8.6** More combinations for data in Table 8.1

| DMU | $\frac{y_1}{10x_1+x_2}$ | $\frac{y_1}{x_1+10x_2}$ | $\frac{10y_1+y_2}{x_1}$ | $\frac{y_1+10y_2}{x_1}$ | $\frac{10y_1+y_2}{x_1+x_2}$ |
|---|---|---|---|---|---|
| **A01** | 0.37 | 0.47 | 42.50 | 29.00 | **24.29** |
| **A02** | **0.50** | 0.29 | **62.33** | 29.33 | 20.78 |
| A03 | 0.18 | 0.34 | 18.92 | 8.28 | 13.23 |
| **A04** | 0.33 | 0.25 | 40.60 | **29.80** | 16.92 |
| **A05** | 0.22 | **0.47** | 24.00 | 17.25 | 17.45 |
| A06 | 0.35 | 0.29 | 42.34 | 27.43 | 18.53 |
| A07 | 0.11 | 0.14 | 12.29 | 9.71 | 7.17 |
| A08 | 0.10 | 0.13 | 11.15 | 9.48 | 6.60 |
| A09 | 0.13 | 0.23 | 14.40 | 5.40 | 9.60 |
| A10 | 0.15 | 0.20 | 16.72 | 13.22 | 10.03 |
| A11 | 0.13 | 0.19 | 14.80 | 6.53 | 8.90 |

## 8.5 VRS Output-Input Ratio Analysis

Theorem 8.2 can be extended for VRS technology as well. If a DMU lies on the CRS frontier, the DMU lies on VRS frontier, but not vice versa. Consequently, the following theorems hold.

**Theorem 8.4** If there are exists $\tilde{w}_j^- \geq 0$, for $j = 1, 2, \ldots, m$, and $\tilde{w}_k^+ \geq 0$, for $k = 1, 2, \ldots, p$, and $w \in \mathbb{R}$, such that Eq. 8.14 holds, $\text{DMU}_l$ is located on the VRS frontier.

$$\frac{\sum_{k=1}^{p} y_{lk}\tilde{w}_k^+ + w}{\sum_{j=1}^{m} x_{lj}\tilde{w}_j^-} = \max\left\{\frac{\sum_{k=1}^{p} y_{ik}\tilde{w}_k^+ + w}{\sum_{j=1}^{m} x_{ij}\tilde{w}_j^-} : i = 1, 2, \ldots, n\right\}. \tag{8.14}$$

*Proof* The proof is simply concluded from Theorem 8.2. □

**Theorem 8.5** *If there are exists* $\tilde{w}_j^- \geq 0$, *for* $j = 1, 2, \ldots, m$, *and* $\tilde{w}_k^+ \geq 0$, *for* $k = 1, 2, \ldots, p$, *and* $w \in \mathbb{R}$, *such that Eq.* 8.14 *holds,* $DMU_l$ *is located on the VRS frontier.*

$$\frac{\sum_{j=1}^{m} x_{lj}\tilde{w}_j^- + w}{\sum_{k=1}^{p} y_{lk}\tilde{w}_k^+} = \min\left\{\frac{\sum_{j=1}^{m} x_{ij}\tilde{w}_j^- + w}{\sum_{k=1}^{p} y_{ik}\tilde{w}_k^+} : i = 1, 2, \ldots, n\right\}. \tag{8.15}$$

*Proof* The proof is simply concluded from Theorem 8.2. □

Now, suppose that $\tilde{w}_j^- = 1$, for $j \in J$, $\tilde{w}_k^+ = 0$, for all $k$, and $w = 1$. Thus we have.

**Theorem 8.6** $DMU_l$ *is located on the VRS frontier, if the following equation holds*

$$\sum_{j \in J} x_{lj} = \min\left\{\sum_{j \in J} x_{lj} =: i = 1, 2, \ldots, n\right\}. \tag{8.16}$$

*Proof* The proof is simply concluded from Theorem 8.2. □

Similarly, assume that $\tilde{w}_j^+ = 1$, for $k \in K$, $\tilde{w}_j^- = 0$, for all $j$, and $w = 1$. Thus we have.

**Theorem 8.7** $DMU_l$ *is located on the VRS frontier, if the following equation holds*

$$\sum_{k \in K} y_{lk} = \max\left\{\sum_{k \in K} y_{ik} : i = 1, 2, \ldots, n\right\}. \tag{8.17}$$

*Proof* The proof is simply concluded from Theorem 8.2. □

BCC can easily be solved by adding the command '=Sum(F2:F12)' into cell F13. After that, open 'Solver Parameters' window, and click on 'Add'. Assign 'F13' into 'Cell Reference', then select '=', and assign '1' into 'Constraint'. The outcomes are illustrated in Table 8.7.

Five DMUs A01, A02, A04, A05, A06 are BCC-efficient. A06 is the only BCC-efficient DMU which is not CCR-efficient.

Table 8.8 shows different combinations of input and output factors such as, $(x_1 - 1)/y_2$, $(x_2 - 1)/y_2$ and $(2/100)x_1 + (1/150)x_2$, to exemplify Theorems 8.5 and 8.6. The minimum values in each column are bolded.

As can be seen, all BCC-efficient DMUs can be found by different combination of output–input ratios.

## 8.6 Conclusion

The relationship between ratio analysis and DEA efficiency is revealed in this section. DEA includes the basis of ratio analyses, that is, a unit with the highest rank, in respect to the ratio of a single output to a single input, dominates other units. It is also shown a deficiency of ratio analysis is that it fails to identify all types of dominating units, unlike DEA. In other words, a performance measure based on the

**Table 8.7** The envelopment IO-BCC results for data in Table 8.1

| DMU | $\theta^*$ | $\lambda^*_{A01}$ | $\lambda^*_{A02}$ | $\lambda^*_{A04}$ | $\lambda^*_{A05}$ | $\lambda^*_{A06}$ |
|---|---|---|---|---|---|---|
| A01 | 1.0000 | 1.0000 | | | | |
| A02 | 1.0000 | | 1.0000 | | | |
| A03 | 0.7500 | 0.5000 | | | 0.5000 | |
| A04 | 1.0000 | | | 1.0000 | | |
| A05 | 1.0000 | | | | 1.0000 | |
| A06 | 1.0000 | | | | | 1.0000 |
| A07 | 0.4000 | 1.0000 | | | | |
| A08 | 0.4478 | 0.9142 | | | 0.0858 | |
| A09 | 0.6000 | 1.0000 | | | | |
| A10 | 0.5303 | 0.7727 | | 0.0455 | 0.1818 | |
| A11 | 0.4615 | 1.0000 | | | | |

**Table 8.8** Applying Theorems 8.5 and 8.6 for data in Table 8.1

| DMU | $(x_1 - 1)/y_2$ | $(x_2 - 1)/y_1$ | $0.02x_1 + 0.00\bar{6}x_2$ |
|---|---|---|---|
| **A01** | 0.3900 | 0.1813 | **1.0000** |
| **A02** | 0.4143 | 0.3278 | **1.0000** |
| A03 | 1.5333 | 0.2294 | 2.1267 |
| **A04** | **0.3769** | 0.3632 | 1.4667 |
| **A05** | 0.6583 | **0.1611** | 1.8000 |
| **A06** | 0.4146 | 0.3143 | **1.0000** |
| A07 | 1.1556 | 0.6167 | 2.6000 |
| A08 | 1.1707 | 0.6600 | 2.3867 |
| A09 | 2.4750 | 0.3500 | 2.3333 |
| A10 | 0.8476 | 0.4214 | 2.2000 |
| A11 | 1.9400 | 0.4571 | 2.3933 |

ratio of a single output to a single input fails to capture the entirety of performance with respect to a set of outputs and inputs.

## 8.7 Exercises

8.1. By a counterexample show that Theorem 8.1 is not necessary satisfied for BCC.
8.2. Prove that there exists a positive parameter $t$, in which the following model is equivalent with CCR.

$$
\begin{aligned}
&\max \textstyle\sum_{k=1}^{p} y_{lk} w_k^{+},\\
&\text{Subject to}\\
&\textstyle\sum_{j=1}^{m} x_{lj} w_j^{-} = t,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^{+} - \sum_{j=1}^{m} x_{ij} w_j^{-} \le 0, \text{for } i = 1,2,\ldots,n,\\
&0 \le w_j^{+} \le 1, \quad \text{for} \quad j = 1,2,\ldots,m,\\
&0 \le w_k^{-} \le 1, \quad \text{for} \quad k = 1,2,\ldots,p.
\end{aligned}
\tag{8.18}
$$

8.3. Prove that the following model is equivalent with CCR.

$$
\begin{aligned}
&\max \textstyle\sum_{k=1}^{p} y_{lk} w_k^{+},\\
&\text{Subject to}\\
&\textstyle\sum_{j=1}^{m} x_{lj} w_j^{-} = 1,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^{+} - \sum_{j=1}^{m} x_{ij} w_j^{-} \le 0, \text{for } i = 1,2,\ldots,n,\\
&0 \le w_j^{+} \le 1, \quad \text{for} \quad j = 1,2,\ldots,m,\\
&0 \le w_k^{-} \le 1, \quad \text{for} \quad k = 1,2,\ldots,p.
\end{aligned}
\tag{8.19}
$$

8.4. Prove that the optimal solution of the following model is $w_j^{-*}/\theta^*$, for $j = 1$, 2, ..., $m$, and $w_k^{+*}/\theta^*$, for $k = 1$, 2, ..., $p$.

$$
\begin{aligned}
&\max \textstyle\sum_{k=1}^{p} y_{lk} w_k^{+},\\
&\text{Subject to}\\
&\textstyle\sum_{j=1}^{m} \theta^* x_{lj} w_j^{-} = 1,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^{+} - \sum_{j=1}^{m} \theta^* x_{ij} w_j^{-} \le 0, \text{for } i = 1,2,\ldots,n,\\
&w_j^{+} \ge 0, \quad \text{for} \quad j = 1,2,\ldots,m,\\
&w_k^{-} \ge 0, \quad \text{for} \quad k = 1,2,\ldots,p.
\end{aligned}
\tag{8.20}
$$

8.5. Prove that the optimal solution of the following model is $w_j^{-*}/\varphi^*$, for $j = 1$, 2, ..., $m$, and $w_k^{+*}/\varphi^*$, for $k = 1$, 2, ..., $p$.

$$
\begin{aligned}
&\max \textstyle\sum_{j=1}^{m} x_{lj} w_k^{-},\\
&\text{Subject to}\\
&\textstyle\sum_{k=1}^{p} \varphi^* y_{lk} w_k^{+} = 1,\\
&\textstyle\sum_{k=1}^{p} \varphi^* y_{ik} w_k^{+} - \sum_{j=1}^{m} x_{ij} w_j^{-} \le 0, \text{for } i = 1,2,\ldots,n,\\
&w_k^{+} \ge 0, \quad \text{for} \quad k = 1,2,\ldots,m,\\
&w_k^{-} \ge 1, \quad \text{for} \quad k = 1,2,\ldots,p.
\end{aligned}
\tag{8.21}
$$

8.6. Prove Theorem 8.1.
8.7. Prove Theorem 8.2.
8.8. Solve the following IO-BCC Phase II for data in Table 8.1, where $\theta_l^*$ can be found from Table 8.7.

$$\begin{aligned}
&\max \textstyle\sum_{j=1}^{m} s_j^- + \sum_{k=1}^{p} s_k^+,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = \theta_l^* x_{lj}, \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{lk}, \text{for } k = 1, 2, \ldots, p,\\
&\textstyle\sum_{i=1}^{n} \lambda_i = 1\\
&\lambda_i \geq 0, \quad \text{for} \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{8.22}$$

# Chapter 9
# Production Planning Problem

## 9.1 Introduction

Production in large firms with a centralized decision-making unit usually involves the contribution of several individual units. Supermarket chains or organizations with several workshops include more than one unit in which each unit contributes to a part of the entire production. For example, a sales plan is decided by a bank's cooperate management for its divisions regarding the number of credit cards to be issued and the number of loans to be processed. The central division defines a plan to optimize the average or overall production performance in the entire organization after planning. In this chapter, a production-planning problem which is regularly faced by the central decision-making units is illustrated. Two planning ideas, proposed by Du et al. (2010), are discussed to arrange new input and output plans for all specific units when demand deviations can be predicted.

## 9.2 Twenty Fast-Food restaurants

Assume a set of 20 fast food restaurants that are located in the city of Hefei, Anhui Province, China. These fast food restaurants belong to the same chain, with a central decision-making team of several members who supervise all branches' operations and make future sales plans for them, as well.

Table 9.1 illustrates the given data, where the restaurants are labeled R01–R20. These data are collected for a month, and the central decision-making team attempts to arrange input and sales (output) plans for all 20 fast food restaurants under the same chain for the next month in business.

There are two input factors for each restaurant, man-hour ($10^3$ h) and shop size ($10^2$ $m^2$), which are labeled I1 and I2, respectively. Here, man-hour means the labor force used within a certain period, and shop size means the total rental floor space of



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the restaurant that can be used for the purpose of serving customers. There are also five output factors, such as, the sales of meat dish, vegetable dish, soup, noodles and beverage, all in $10^3$ serving unit, which are labeled O1–O5.

The demand changes for meat dish, vegetable dish, soup, noodles and beverage are forecasted as $\tilde{D}_1 = 3$, $\tilde{D}_2 = 2.4$, $\tilde{D}_3 = -3$, $\tilde{D}_4 = 6$, $\tilde{D}_5 = 3$ ($10^3$ serving) in the next business month, where a positive change represents an increase in demand while a negative change represents a decrease in demand. How should these 20 fast food restaurants be benchmarked to optimize the overall production performance of all restaurants after planning?

### 9.2.1 CCR Efficient Restaurants

The CCR model (See Eq. 8.2 and Form 1), for the 20 restaurants in Table 9.1 is consisted by Eq. 9.1.

**Table 9.1** Example of 20 fast food restaurants

| DMUs | I1 | I2 | O1 | O2 | O3 | O4 | O5 |
|---|---|---|---|---|---|---|---|
| R01 | 3.2 | 2.0 | 2.24 | 2.46 | 1.22 | 3.12 | 0.96 |
| R02 | 3.4 | 2.1 | 2.12 | 2.52 | 1.34 | 3.08 | 0.88 |
| R03 | 3.1 | 1.8 | 2.08 | 2.25 | 1.05 | 2.85 | 0.74 |
| R04 | 3.8 | 2.2 | 2.45 | 2.10 | 1.3 | 2.96 | 0.79 |
| R05 | 4.2 | 2.6 | 2.80 | 2.78 | 1.42 | 3.48 | 1.05 |
| R06 | 4.1 | 2.5 | 2.65 | 2.95 | 1.38 | 3.25 | 0.98 |
| R07 | 3.8 | 2.3 | 2.60 | 2.24 | 1.15 | 3.18 | 0.95 |
| R08 | 3.8 | 2.2 | 2.50 | 2.15 | 1.10 | 3.20 | 0.82 |
| R09 | 2.9 | 1.6 | 2.10 | 2.04 | 0.98 | 2.88 | 0.72 |
| R10 | 4.2 | 2.8 | 2.90 | 2.85 | 1.52 | 3.36 | 1.12 |
| R11 | 3.4 | 2.1 | 2.60 | 2.45 | 1.36 | 3.32 | 0.82 |
| R12 | 4.0 | 2.4 | 2.78 | 2.66 | 1.18 | 3.15 | 0.98 |
| R13 | 3.8 | 2.6 | 2.84 | 2.38 | 1.25 | 3.29 | 0.85 |
| R14 | 3.4 | 1.9 | 2.33 | 2.20 | 1.06 | 2.99 | 0.82 |
| R15 | 2.8 | 1.6 | 2.00 | 2.18 | 1.96 | 2.84 | 0.71 |
| R16 | 3.5 | 2.2 | 2.40 | 2.25 | 1.26 | 2.93 | 0.74 |
| R17 | 4.2 | 2.5 | 2.68 | 2.50 | 1.46 | 3.22 | 0.92 |
| R18 | 3.3 | 1.8 | 2.05 | 2.20 | 1.12 | 3.02 | 0.78 |
| R19 | 3.6 | 1.9 | 2.00 | 2.16 | 1.02 | 2.89 | 0.74 |
| R20 | 3.1 | 1.7 | 2.05 | 2.12 | 0.94 | 2.90 | 0.68 |

$$
\begin{aligned}
&\theta_l^* = \max(y_{l1}w_1^+ + y_{l2}w_2^+ + y_{l3}w_3^+ + y_{l4}w_4^+ + y_{l5}w_5^+),\\
&\text{Subject to}\\
&x_{l1}w_1^- + x_{l2}w_2^- = 1,\\
&\textstyle\sum_{k=1}^{5} y_{ik}w_k^+ - \sum_{j=1}^{2} x_{ij}w_j^- \le 0, \text{for } i = 1, 2, \ldots, 20,\\
&w_j^+ \ge 0, \text{for } j = 1, 2,\\
&w_k^- \ge 0, \text{for} k \;= 1, 2, \ldots, 5.
\end{aligned}
\tag{9.1}
$$

The following instructions solve Eq. 9.1 for data in Table 9.1.

1. Copy the 8 columns of Table 9.1 on an Excel sheet into cells A1:H21, as Fig. 9.1 depicts.
2. Label A23 as ‘Weights’, A25 as ‘Index’, D25 as ‘Constraint’, G25 as ‘Objective’, I1 as ‘Constraints’, J1 as ‘CCR Score’ and K1 as ‘Rank’.
3. Assign number 1 to B25.

H25 =SUMPRODUCT(INDEX(D2:H21,B25,0),D23:H23)

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | DMUs | I1 | I2 | O1 | O2 | O3 | O4 | O5 | Constraints | CCR |
| 2 | R01 | 3.2 | 2.0 | 2.24 | 2.46 | 1.22 | 3.12 | 0.96 | 0.000 | 1.0000 |
| 3 | R02 | 3.4 | 2.1 | 2.12 | 2.52 | 1.34 | 3.08 | 0.88 | -0.113 | 0.9564 |
| 4 | R03 | 3.1 | 1.8 | 2.08 | 2.25 | 1.05 | 2.85 | 0.74 | -0.088 | 0.9365 |
| 5 | R04 | 3.8 | 2.2 | 2.45 | 2.10 | 1.30 | 2.96 | 0.79 | -0.241 | 0.8724 |
| 6 | R05 | 4.2 | 2.6 | 2.80 | 2.78 | 1.42 | 3.48 | 1.05 | -0.200 | 0.9154 |
| 7 | R06 | 4.1 | 2.5 | 2.65 | 2.95 | 1.38 | 3.25 | 0.98 | -0.223 | 0.9253 |
| 8 | R07 | 3.8 | 2.3 | 2.60 | 2.24 | 1.15 | 3.18 | 0.95 | -0.152 | 0.9382 |
| 9 | R08 | 3.8 | 2.2 | 2.50 | 2.15 | 1.10 | 3.20 | 0.82 | -0.196 | 0.8880 |
| 10 | R09 | 2.9 | 1.6 | 2.10 | 2.04 | 0.98 | 2.88 | 0.72 | 0.000 | 1.0000 |
| 11 | R10 | 4.2 | 2.8 | 2.90 | 2.85 | 1.52 | 3.36 | 1.12 | -0.259 | 0.9552 |
| 12 | R11 | 3.4 | 2.1 | 2.60 | 2.45 | 1.36 | 3.32 | 0.82 | -0.135 | 1.0000 |
| 13 | R12 | 4.0 | 2.4 | 2.78 | 2.66 | 1.18 | 3.15 | 0.98 | -0.182 | 0.9454 |
| 14 | R13 | 3.8 | 2.6 | 2.84 | 2.38 | 1.25 | 3.29 | 0.85 | -0.366 | 0.9773 |
| 15 | R14 | 3.4 | 1.9 | 2.33 | 2.20 | 1.06 | 2.99 | 0.82 | -0.066 | 0.9513 |
| 16 | R15 | 2.8 | 1.6 | 2.00 | 2.18 | 1.96 | 2.84 | 0.71 | -0.011 | 1.0000 |
| 17 | R16 | 3.5 | 2.2 | 2.40 | 2.25 | 1.26 | 2.93 | 0.74 | -0.281 | 0.8970 |
| 18 | R17 | 4.2 | 2.5 | 2.68 | 2.50 | 1.46 | 3.22 | 0.92 | -0.270 | 0.8642 |
| 19 | R18 | 3.3 | 1.8 | 2.05 | 2.20 | 1.12 | 3.02 | 0.78 | -0.043 | 0.9546 |
| 20 | R19 | 3.6 | 1.9 | 2.00 | 2.16 | 1.02 | 2.89 | 0.74 | -0.134 | 0.8648 |
| 21 | R20 | 3.1 | 1.7 | 2.05 | 2.12 | 0.94 | 2.90 | 0.68 | -0.078 | 0.9528 |
| 22 | | | | | | | | | | |
| 23 | Weights | 0.0000 | 0.5000 | 0.0000 | 0.0000 | 0.0000 | 0.0926 | 0.7407 | | |
| 24 | | | | | | | | | | |
| 25 | Index | 1 | | Constraint | 1.0000 | | Objective | 1.0000 | | |
| 26 | | | | | | | | | | |

**Fig. 9.1** Setting Excel sheet to solve Eq. 9.1

4. Assign the following command (without quotations mark) into I2,

   ‘=Sumproduct(D2:H2,D$23:H$23)-Sumproduct(B2:C2,B$23:C$23)’.

5. Copy I2 (by Ctrl + C), and paste it (by Ctrl + V) to I3–I21.
6. Assign the command ‘=Rank(J2,J$2:J$21,0)’ into K2.
7. Copy K2 and then paste it to K3–K21.
8. Assign the command ‘=Sumproduct(Index(B2:C21,B25,0),B23:C23)’ into E25.
9. Assign the command ‘=Sumproduct(Index(D2:H21,B25,0),D23:H23)’ into H25.
10. Open ‘Solver Parameters’ window, from ‘DATA’ in toolbar menu, as Fig. 9.2 illustrates.
11. Assign ‘H25’ into ‘Set Objective’ and choose ‘Max’.
12. Assign ‘B23:H23’ into ‘By Changing Variable Cells’.
13. Click on ‘Add’ and assign ‘I2:I21’ into ‘Cell Reference’, then select ‘<=’, and assign ‘0’ into ‘Constraint’.
14. Click on ‘Add’ and assign ‘E25’ into ‘Cell Reference’, then select ‘=’ and assign ‘1’ into ‘Constraint’. Then click on ‘OK’.
15. Tick ‘Make Unconstrained Variables Non-Negative’.

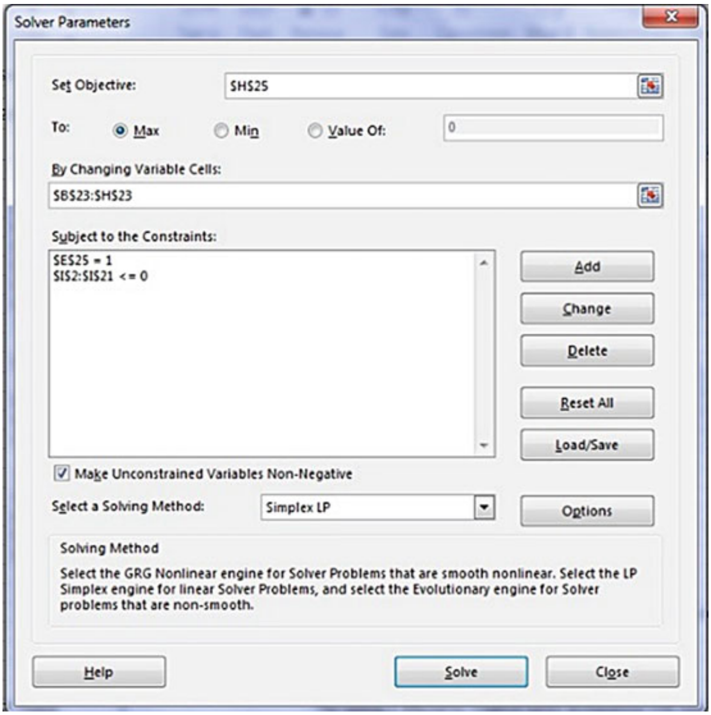



**Fig. 9.2** Setting Solver to solve Eq. 9.1

16. Choose ‘Simplex LP’ from ‘Select a Solving Method’.
17. Click on ‘Solve’ (Fig. 9.3).
18. From ‘Developer’ in the toolbar menu, click on the ‘Insert’ icon to open the ‘Form Control’ window (Fig. 9.4).

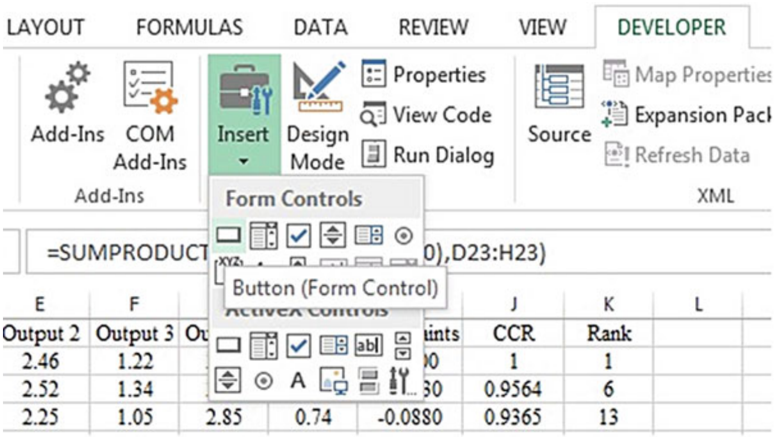



**Fig. 9.3** The Form control menu

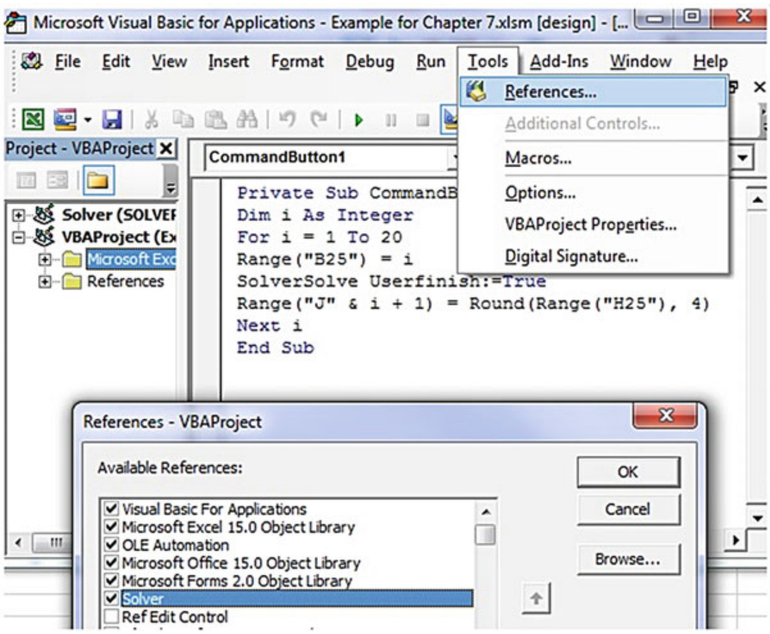



**Fig. 9.4** Setting VBA to solve Eq. 9.1

19. Click on the first icon, ‘Button (Form Control)’, and then click on a place on the Excel sheet.
20. In the opened window with the title ‘Assign Macro’, click on ‘New’. So, the ‘Microsoft Visual Basic for Applications’ window is opened.
21. Inside of the ‘Microsoft Visual Basic for Applications’ window, write the following commands between ‘Sub Button1_Click ()’ and ‘End Sub’.

```
Dim i As Integer
For i = 1 To 20
 Range("B25") = i
 SolverSolve Userfinish:=True
 Range("J" & i + 1) = Round(Range("H25"), 4)
Next i
```

22. Close the ‘Microsoft Visual Basic for Applications’ window.
23. Click on the small rectangle which was automatically made on the Excel sheet and created by step 19. The results are represented to cells J2:J21.

Table 9.2 illustrates the CCR-inefficient and CCR-efficient restaurants. There were four CCR-efficient restaurants, which are bolded in Table 9.2, and the rest of the restaurants were CCR-inefficient.

In the next sections, we construct two ideas to optimize overall performance of these 20 restaurants.

**Table 9.2** The results of solving Eq. 9.1

| DMUs | CCR Scores |
|---|---|
| **1** | **1.0000** |
| 2 | 0.9564 |
| 3 | 0.9365 |
| 4 | 0.8724 |
| 5 | 0.9154 |
| 6 | 0.9253 |
| 7 | 0.9382 |
| 8 | 0.8880 |
| **9** | **1.0000** |
| 10 | 0.9552 |
| **11** | **1.0000** |
| 12 | 0.9454 |
| 13 | 0.9773 |
| 14 | 0.9513 |
| **15** | **1.0000** |
| 16 | 0.8970 |
| 17 | 0.8642 |
| 18 | 0.9546 |
| 19 | 0.8648 |
| 20 | 0.9528 |

## 9.3 Idea 1: Overall Production Performance

In this section, we theoretically construct Idea 1. First we assume that demands for all output factors are positive, that is, increase in demands.

Part 1 Suppose that a set of $n$ DMUs, DMU$_i$, ($i = 1, 2, \ldots, n$) is given in which each DMU has $m$ input factors, $x_{ij}$, ($j = 1, 2, \ldots, m$) and $p$ output factors, $y_{ik}$, ($k = 1, 2, \ldots, p$). Assume that the upper demand change for output $k$ ($k = 1, 2, \ldots, p$) in the next production season can be forecasted as $\tilde{D}_k$, where $\tilde{D}_k > 0$. As a result, with the same values of output factors for the production plan, we should have $\sum_{i=1}^{n} y_{ik} \le \tilde{D}_k$, for $k = 1, 2, \ldots, p$. In order to meet the demand changes, the most preferred input-output plans should be determined for all DMUs.

For Idea 1, we choose a planning principle to maximize the CCR-efficiency of average input and output levels of the entire DMUs. This ensures that the average (overall) production capability for all DMUs to earn their highest potential after planning.

Similar to Eq. 5.24, we have the following model where DMU$_l$ is under evaluation.

$$\begin{aligned}
&\max \frac{y_{l1}w_1^+ + y_{l2}w_2^+ + \cdots + y_{lp}w_p^+}{x_{l1}w_1^- + x_{l2}w_2^- + \cdots + x_{l\mathrm{m}}w_m^-},\\
&\text{Subject to}\\
&\frac{y_{i1}w_1^+ + y_{i2}w_2^+ + \cdots + y_{ip}w_p^+}{x_{i1}w_1^- + x_{i2}w_2^- + \cdots + x_{i\mathrm{m}}w_m^-} \le 1, \text{for}, i = 1, 2, \ldots, n,\\
&w_j^+ \ge 0, \text{for} j = 1, 2, \ldots, m,\\
&w_k^- \ge 0, \text{for} k = 1, 2, \ldots, p.
\end{aligned} \tag{9.2}$$

As illustrated in the previous chapters, CCR uses the CRS technology, so we suppose that for DMU$_i$, $i = 1, 2, \ldots, n$ the input and output factors are changed by the same proportion denoted by $\Delta_i$, $i = 1, 2, \ldots, n$, where $\Delta_i \ge 0$.

Therefore, we assume that the new production plan for input and output factors of DMU$_i$, $i = 1, 2, \ldots, n$ is a DMU with the following factors $(x_{i1} + \Delta_i x_{i1})$, $(x_{i2} + \Delta_i x_{i2})$, $\ldots$, $(x_{im} + \Delta_i x_{im})$, $(y_{i1} + \Delta_i y_{i1})$, $(y_{i2} + \Delta_i y_{i2})$, $\ldots$, $(y_{ip} + \Delta_i y_{ip})$, where $\sum_{i=1}^{n} y_{ik}\Delta_i \le \tilde{D}_k$, for $k = 1, 2, \ldots, p$.

Note that, $\tilde{D}_k$ is the upper demand change for output $k$ ($k = 1, 2, \ldots, p$) in the next production season, and $(x_{ij} + \Delta_i x_{ij}) = (1 + \Delta_i)x_{ij}$, for $j = 1, 2, \ldots, m$, and $(y_{ik} + \Delta_i y_{ik}) = (1 + \Delta_i)y_{ik}$, for $k = 1, 2, \ldots, p$.

Equation 9.2 is not changed if we use the new input and output values for DMU$_i$, $i = 1, 2, \ldots, n$, because

$$\frac{(1+\Delta_i)y_{i1}w_1^+ + (1+\Delta_i)y_{i2}w_2^+ + \cdots + (1+\Delta_i)y_{ip}w_p^+}{(1+\Delta_i)x_{i1}w_1^- + (1+\Delta_i)x_{i2}w_2^- + \cdots + (1+\Delta_i)x_{im}w_m^-} = \frac{y_{i1}w_1^+ + y_{i2}w_2^+ + \cdots + y_{ip}w_p^+}{x_{i1}w_1^- + x_{i2}w_2^- + \cdots + x_{im}w_m^-}. \tag{9.3}$$

Nonetheless, adding the constraints $\sum_{i=1}^{n} y_{ik}\Delta_i \leq \tilde{D}_k$, for $k = 1, 2, \ldots, p$, may change the optimal solutions, as the optimal solutions may not be unique.

The average input and output factors of all DMUs, denoted by $\bar{x}_j$ and $\bar{y}_k$, respectively, and are given by:

$$\begin{aligned} \bar{x}_j &= \frac{(1+\Delta_1)x_{1j} + (1+\Delta_2)x_{2j} + \ldots + (1+\Delta_n)x_{nj}}{n}, \text{for } j = 1, 2, \ldots, m\& \\ \bar{y}_k &= \frac{(1+\Delta_1)y_{1k} + (1+\Delta_2)y_{2k} + \ldots + (1+\Delta_n)y_{nk}}{n}, \text{for } k = 1, 2, \ldots, p, \end{aligned} \tag{9.4}$$

Therefore, we have $n + 1$ DMUs, that is, $n$ DMUs ($\text{DMU}_i$, $i = 1, 2, \ldots, n$) plus the virtual DMU with the average input and output values. Equation 9.2 for this virtual DMU with the average input and output values, where the constraints $\sum_{i=1}^{n} y_{ik}\Delta_i \leq \tilde{D}_k$, for $k = 1, 2, \ldots, p$ are also added, is as follows:

$$\begin{aligned}
&\max \frac{\bar{y}_1 w_1^+ + \bar{y}_2 w_2^+ + \cdots + \bar{y}_\text{p} w_p^+}{x_1 w_1^- + x_2 w_2^- + \cdots + x_\text{m} w_m^-}, \\
&\text{Subject to} \\
&\frac{y_{i1}(1+\Delta_i)w_1^+ + y_{i2}(1+\Delta_i)w_2^+ + \cdots + y_{ip}(1+\Delta_i)w_p^+}{x_{i1}(1+\Delta_i)w_1^- + x_{i2}(1+\Delta_i)w_2^- + \cdots + x_{im}(1+\Delta_i)w_m^-} \leq 1, \text{for } i = 1, 2, \ldots, n, \\
&\frac{\bar{y}_1 w_1^+ + \bar{y}_2 w_2^+ + \cdots + \bar{y}_\text{p} w_p^+}{\bar{x}_1 w_1^- + \bar{x}_2 w_2^- + \cdots + \bar{x}_\text{m} w_m^-} \leq 1, \\
&\textstyle\sum_{i=1}^{n} y_{ik}\Delta_i \leq \tilde{D}_k, \text{for } k = 1, 2, \ldots, p, \\
&w_j^+ \geq 0, \text{for } j = 1, 2, \ldots, m, \\
&w_k^- \geq 0, \text{for } k = 1, 2, \ldots, p, \\
&\Delta_i \geq 0, \text{for } i = 1, 2, \ldots, n.
\end{aligned} \tag{9.5}$$

The above equation yields that,

$$
\begin{aligned}
&\max \frac{\sum_{k=1}^{p} \bar{y}_k w_k^+}{\sum_{j=1}^{m} \bar{x}_j w_j^-}, \\
&\text{Subject to} \\
&\frac{\sum_{k=1}^{p} y_{ik} w_k^+}{\sum_{j=1}^{m} x_{ij} w_j^-} \le 1, \text{for } i = 1, 2, \ldots, n, \\
&\frac{\sum_{k=1}^{p} \bar{y}_k w_k^+}{\sum_{j=1}^{m} \bar{x}_j w_j^-} \le 1, \\
&\sum_{i=1}^{n} y_{ik} \Delta_i \le \tilde{D}_k, \text{for} k = 1, 2, \ldots, p, \\
&w_j^+ \ge 0, \text{for } j = 1, 2, \ldots, m, \\
&w_k^- \ge 0, \text{for } k = 1, 2, \ldots, p, \\
&\Delta_i \ge 0, \text{for } i = 1, 2, \ldots, n.
\end{aligned}
\tag{9.6}
$$

From Eq. 9.4 we have

$$
\frac{\sum_{k=1}^{p} \bar{y}_k w_k^+}{\sum_{j=1}^{m} \bar{x}_j w_j^-} = \frac{\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1 + \Delta_i)}{\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1 + \Delta_i)}. \tag{9.7}
$$

Thus, Eq. 9.6 is equal with the following equation.

$$
\begin{aligned}
&\max \frac{\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1 + \Delta_i)}{\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1 + \Delta_i)}, \\
&\text{Subject to} \\
&\frac{\sum_{k=1}^{p} y_{ik} w_k^+}{\sum_{j=1}^{m} x_{ij} w_j^-} \le 1, \text{for } i = 1, 2, \ldots, n, \\
&\frac{\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1 + \Delta_i)}{\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1 + \Delta_i)} \le 1, \\
&\sum_{i=1}^{n} y_{ik} \Delta_i \le \tilde{D}_k, \text{for } k = 1, 2, \ldots, p, \\
&w_j^+ \ge 0, \text{for } j = 1, 2, \ldots, m, \\
&w_k^- \ge 0, \text{for } k = 1, 2, \ldots, p, \\
&\Delta_i \ge 0, \text{for } i = 1, 2, \ldots, n.
\end{aligned}
\tag{9.8}
$$

Let's assume that $\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1 + \Delta_i) = 1$, thus we have the following converted model.

$$
\begin{aligned}
&\max \sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i),\\
&\text{Subject to}\\
&\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1+\Delta_i) = 1,\\
&\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i) \le 1,\\
&\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \le 1, \text{for } i = 1, 2, \ldots, n,\\
&\sum_{i=1}^{n} y_{ik}\Delta_i \le \tilde{D}_k, \text{for } k = 1, 2, \ldots, p,\\
&w_j^+ \ge 0, \text{for } j = 1, 2, \ldots, m,\\
&w_k^- \ge 0, \text{for } k = 1, 2, \ldots, p,\\
&\Delta_i \ge 0, \text{for } i = 1, 2, \ldots, n.
\end{aligned} \tag{9.9}
$$

As discussed in Sect. 6.2, we can define lower bound for the weights $w_j^+$ and $w_k^-$, that is, $w_j^+ \ge \varepsilon$ and $w_k^- \ge \varepsilon$, for $j = 1, 2, \ldots, m$ and for $k = 1, 2, \ldots, p$, respectively, where $\varepsilon > 0$. In this case, we have

$$
\begin{aligned}
&\max \sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i),\\
&\text{Subject to}\\
&\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1+\Delta_i) = 1,\\
&\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i) \le 1,\\
&\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \le 1, \text{for } i = 1, 2, \ldots, n,\\
&\sum_{i=1}^{n} y_{ik}\Delta_i \le \tilde{D}_k, \text{for } k = 1, 2, \ldots, p,\\
&w_j^+ \ge \varepsilon, \text{for } j = 1, 2, \ldots, m,\\
&w_k^- \ge \varepsilon, \text{for } k = 1, 2, \ldots, p,\\
&\Delta_i \ge 0, \text{for } i = 1, 2, \ldots, n.
\end{aligned} \tag{9.10}
$$

We can also set some restrictions on the changing proportions $\Delta_i$, for $i = 1, 2, \ldots, n$ in Eq. 9.10, such as: $\Delta_i \ge \delta_l^{(i)} \Delta_l$, where $i \ne l$, $i = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, n$.

$$
\begin{aligned}
&\max \sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i),\\
&\text{Subject to}\\
&\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1+\Delta_i) = 1,\\
&\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i) \le 1,\\
&\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \le 1, \quad \text{for} i = 1, 2, \ldots, n,\\
&\sum_{i=1}^{n} y_{ik}\Delta_i \le \tilde{D}_k, \quad \text{for } k = 1, 2, \ldots, p,\\
&w_j^+ \ge 0, \quad \text{for } j = 1, 2, \ldots, m,\\
&w_k^- \ge 0, \quad \text{for } k = 1, 2, \ldots, p,\\
&\Delta_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n,\\
&\Delta_i \ge \delta_l^{(i)} \Delta_l, \quad \text{for } i = 1, 2, \ldots, n \text{ and for } l = 1, 2, \ldots, n \;\text{ where } i \ne l.
\end{aligned} \tag{9.11}
$$

The advantages of the restrictions $\Delta_i \geq \delta_l^{(i)} \Delta_l$ are to reveal the preferences of the central unit and to avoid the possibility that changes in the production plan might happen to a few DMUs only due to the nature of optimization.

Note that, Eqs. 9.10 and 9.11 are nonlinear programming. We later develop the models such that none of the local and global solutions from the non-linear models are critical. In addition, Eq. 9.11 is the same as Eq. 9.10 if $\delta_l^{(i)} = 0$, for $i = 1, 2, \ldots, n$ and for $l = 1, 2, \ldots, n$.

Now, suppose that the data in Table 9.3 are given. There are six DMUs in which three DMUs are CCR-efficient and the other three DMUs are CCR-inefficient.

Assume that the demand changes for Output 1 and Output 2 are predicated as $\tilde{D}_1 = 4$ and $\tilde{D}_2 = 3$ in the next production season, that is, the situation of demand increases in the two outputs.

In order to apply Eq. 9.11 for data in Table 9.3, where $\delta_l^{(i)} = 0.2$, for $i = 1, 2, \ldots, n$ and for $l = 1, 2, \ldots, n$, where $i \neq l$, we have the following instructions.

1. Copy the 5 columns of Table 9.3 on an Excel sheet into cells A1:E7, as Fig. 9.5 depicts.
2. Label A9 as 'Constraint', A11 as 'Weights', A12 as 'SumDelta', D9 as 'Objective', C14 as 'Constraints', C15 as 'Dtilda', F1 as 'Constraints', G1 as 'Deltas', H1 as 'Constraints', G9 as 'delta($i$,$l$)', J1 as 'Target Input1', K1 as 'Target Input2', L1 as 'Target Output1' and M1 as 'Target Output2'.
3. Assign 0.2 into H9.
4. Assign the following command (without quotations mark) into B9,
   '=Sumproduct(B11:C11,B12:C12)'.
5. Assign the following command (without quotations mark) into E9,

   '=Sumproduct(D11:E11,D12:E12)'.

6. Assign the following command (without quotations mark) into B12,

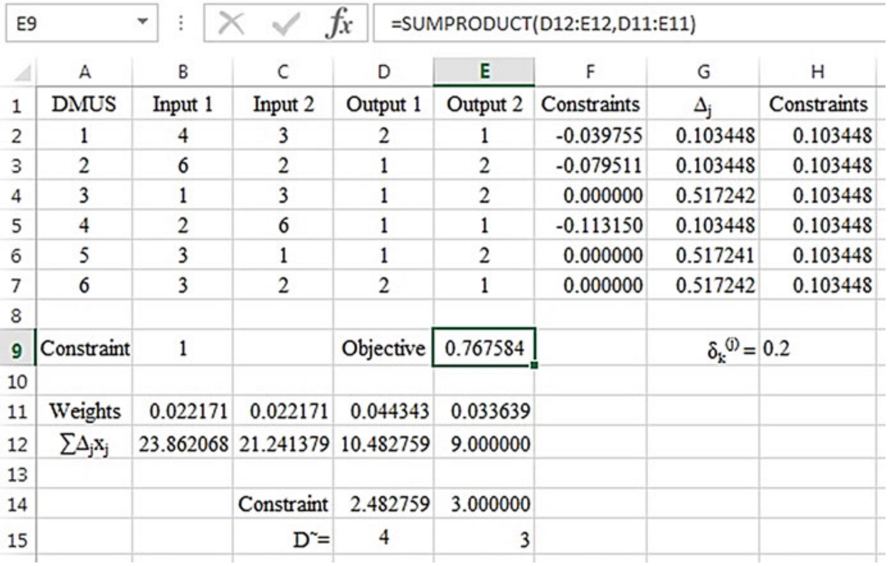
   '=Sumproduct(B2:B7,(1+$G2:$G7))'.

7. Copy B12 (by Ctrl + C), and paste it (by Ctrl + V) to C12, D12 and E12.
8. Assign the following command into D14,

   '=Sumproduct(D2:D7,$G2:$G7)'.

**Table 9.3** Data of six DMUs with four factors

| DMUS | Input 1 | Input 2 | Output 1 | Output 2 | CCR-Score |
|---|---|---|---|---|---|
| 1 | 4 | 3 | 2 | 1 | 0.7368 |
| 2 | 6 | 2 | 1 | 2 | 0.5000 |
| 3 | 1 | 3 | 1 | 2 | 1.0000 |
| 4 | 2 | 6 | 1 | 1 | 0.5000 |
| 5 | 3 | 1 | 1 | 2 | 1.0000 |
| 6 | 3 | 2 | 2 | 1 | 1.0000 |



E9 =SUMPRODUCT(D12:E12,D11:E11)

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | DMUS | Input 1 | Input 2 | Output 1 | Output 2 | Constraints | $\Delta_j$ | Constraints |
| 2 | 1 | 4 | 3 | 2 | 1 | -0.039755 | 0.103448 | 0.103448 |
| 3 | 2 | 6 | 2 | 1 | 2 | -0.079511 | 0.103448 | 0.103448 |
| 4 | 3 | 1 | 3 | 1 | 2 | 0.000000 | 0.517242 | 0.103448 |
| 5 | 4 | 2 | 6 | 1 | 1 | -0.113150 | 0.103448 | 0.103448 |
| 6 | 5 | 3 | 1 | 1 | 2 | 0.000000 | 0.517241 | 0.103448 |
| 7 | 6 | 3 | 2 | 2 | 1 | 0.000000 | 0.517242 | 0.103448 |
| 8 | | | | | | | | |
| 9 | Constraint | 1 | | Objective | 0.767584 | | $\delta_k^{(i)}=$ | 0.2 |
| 10 | | | | | | | | |
| 11 | Weights | 0.022171 | 0.022171 | 0.044343 | 0.033639 | | | |
| 12 | $\sum \Delta_j x_j$ | 23.862068 | 21.241379 | 10.482759 | 9.000000 | | | |
| 13 | | | | | | | | |
| 14 | | | Constraint | 2.482759 | 3.000000 | | | |
| 15 | | | $D^{\sim}=$ | 4 | 3 | | | |

**Fig. 9.5** Setting Excel sheet to solve Eq. 9.11

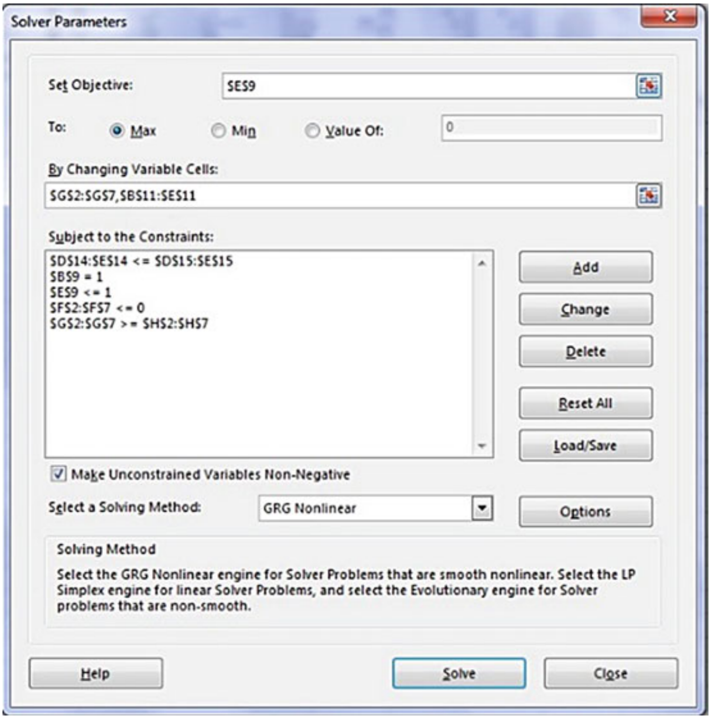



**Fig. 9.6** Setting Solver to solve Eq. 9.11

9. Copy D14 and then paste it to E14.
10. Assign the following command into F2,

    '=Sumproduct(D2:E2,D$11:E$11)- Sumproduct(B2:C2,B$11:C$11)'.

11. Copy F2 and then paste it to F3-F7.
12. Assign the following command into H2,

    '=If(G2=Max(G$2:G$7),H$9*Small(G$2:G$7,5),H$9*Max(G$2:G$7))'.

13. Copy H2 and then paste it to H3-H7.
14. Assign the following command into J2,

    '=(1+$G2)*B2'.

15. Copy J2 and then paste it to J2-M7.
16. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 9.2 illustrates.
17. Assign 'E9' into 'Set Objective' and choose 'Max'.
18. Assign 'G2:G7, B11:E11' into 'By Changing Variable Cells'.
19. Click on 'Add' and assign 'D14:E14' into 'Cell Reference', then select '<=', and assign 'D15:E15' into 'Constraint'.
20. Click on 'Add' and assign 'B9' into 'Cell Reference', then select '=' and assign '1' into 'Constraint'. Then click on 'OK'.
21. Click on 'Add' and assign 'E9' into 'Cell Reference', then select '<=', and assign '1' into 'Constraint'.
22. Click on 'Add' and assign 'F2:F7' into 'Cell Reference', then select '<=', and assign '0' into 'Constraint'.
23. Click on 'Add' and assign 'G2:G7' into 'Cell Reference', then select '>=', and assign 'H2:H7' into 'Constraint'.
24. Tick 'Make Unconstrained Variables Non-Negative'.
25. Choose 'GRG Nonlinear' from 'Select a Solving Method'.
26. Click on 'Solve'.

Table 9.4 illustrates the CCR-inefficient and CCR-efficient restaurants. There were four CCR-efficient restaurants, which are bolded in Table 9.4, and the rest of the restaurants were CCR-inefficient.

**Table 9.4** The results of solving Eq. 9.11

| DMUs | $\Delta_i^*$ | $(1+\Delta_i^*)x_{i1}$ | $(1+\Delta_i^*)x_{i2}$ | $(1+\Delta_i^*)y_{i1}$ | $(1+\Delta_i^*)y_{i2}$ | CCR-Score |
|---|---|---|---|---|---|---|
| 1 | 0.103447 | 4.413790 | 3.310342 | 2.206895 | 1.103447 | 0.736842 |
| 2 | 0.103447 | 6.620685 | 2.206895 | 1.103447 | 2.206895 | 0.500000 |
| 3 | 0.517242 | 1.517242 | 4.551725 | 1.517242 | 3.034483 | 1.000000 |
| 4 | 0.103447 | 2.206895 | 6.620685 | 1.103447 | 1.103447 | 0.499999 |
| 5 | 0.517242 | 4.551726 | 1.517242 | 1.517242 | 3.034484 | 1.000000 |
| 6 | 0.517242 | 4.551727 | 3.034485 | 3.034485 | 1.517242 | 1.000000 |

**Table 9.5** The results of solving Eq. 9.10

| DMUs | $\Delta_i^*$ | $(1+\Delta_i^*)x_{i1}$ | $(1+\Delta_i^*)x_{i2}$ | $(1+\Delta_i^*)y_{i1}$ | $(1+\Delta_i^*)y_{i2}$ | CCR-Score |
|---|---|---|---|---|---|---|
| 1 | 0.000 | 4.000 | 3.000 | 2.000 | 1.000 | 0.737 |
| 2 | 0.000 | 6.000 | 2.000 | 1.000 | 2.000 | 0.500 |
| 3 | 0.667 | 1.667 | 5.000 | 1.667 | 3.333 | 1.000 |
| 4 | 0.000 | 2.000 | 6.000 | 1.000 | 1.000 | 0.500 |
| 5 | 0.000 | 3.000 | 1.000 | 1.000 | 2.000 | 1.000 |
| 6 | 1.667 | 8.000 | 5.333 | 5.333 | 2.667 | 1.000 |

To apply Eq. 9.10, we only need to assume that $\delta_l^{(i)} = 0$, for $i = 1, 2, \ldots, n$ and for $l = 1, 2, \ldots, n$, that is, assign 0 into H9 and run Solver.

Part 2 Now, assume that demands for all output factors are negative, that is, a decrease in demands.

Since the demand decreases and can be 0 at least, thus $\Delta_i$ can be between 0 and 1, that is, $-1 \le \Delta_i \le 0$, which yields $0 \le y_{ik} + \Delta_i y_{ik} \le y_{ik}$. As a result, Eq. 9.11 can be as follows:

$$
\begin{aligned}
&\max \textstyle\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i),\\
&\text{Subject to}\\
&\textstyle\sum_{j=1}^{m} w_j^- \sum_{i=1}^{n} x_{ij}(1+\Delta_i) = 1,\\
&\textstyle\sum_{k=1}^{p} w_k^+ \sum_{i=1}^{n} y_{ik}(1+\Delta_i) \le 1,\\
&\textstyle\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \le 1, \quad \text{for} i = 1, 2, \ldots, n,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\Delta_i \le \tilde{D}_k, \quad \text{and } k = 1, 2, \ldots, p,\\
&w_j^+ \ge \varepsilon, \quad \text{for } j = 1, 2, \ldots, m,\\
&w_k^- \ge \varepsilon, \quad \text{for } k = 1, 2, \ldots, p,\\
&\Delta_i \ge -1, \quad \text{for } i = 1, 2, \ldots, n,\\
&\Delta_i \le -\varepsilon', \quad \text{for } i = 1, 2, \ldots, n,\\
&\Delta_i \ge \delta_l^{(i)}\Delta_l, \quad \text{for } i = 1, 2, \ldots, n \text{ and for } l = 1, 2, \ldots, n \text{ where } i \ne l.
\end{aligned}
\tag{9.12}
$$

Assume that the demand changes for Output 1 and Output 2 in Table 9.3 are predicated as $\tilde{D}_1 = -4, \tilde{D}_2 = -3$ and $\delta_l^{(i)} = 2$, in the next production season, that is, the situation of demand decreases in the two outputs. The results of applying Eq. 9.12, where $\varepsilon = 0$ and $\varepsilon' = 0.0001$ are illustrated in Table 9.6.

Note that, when $\Delta_i \le 0$, for $i = 1, 2, \ldots, n$, then $\Delta_l \le \delta_l^{(i)}\Delta_l \le 0$, where $\delta_l^{(i)} \le 1$, for $i = 1, 2, \ldots, n$ and for $l = 1, 2, \ldots, n$. Thus, the constraints $\Delta_i \ge \delta_l^{(i)}\Delta_l$ yield that $\delta_l^{(i)} = 1$ and $\Delta_1 = \Delta_2 = \ldots = \Delta_n$. To avoid this, we assume that $\delta_l^{(i)} > 1$, for

**Table 9.6** The results of solving Eq. 9.12

| DMUs | $\Delta_i^*$ | $(1+\Delta_i^*)x_{i1}$ | $(1+\Delta_i^*)x_{i2}$ | $(1+\Delta_i^*)y_{i1}$ | $(1+\Delta_i^*)y_{i2}$ | CCR-Score |
|---|---|---|---|---|---|---|
| 1 | −0.4000 | 1.3333 | 1.0000 | 0.6667 | 0.3333 | 0.736842 |
| 2 | −0.8000 | 2.0000 | 0.6667 | 0.3333 | 0.6667 | 0.500000 |
| 3 | −0.4000 | 0.6667 | 2.0000 | 0.6667 | 1.3333 | 1.000000 |
| 4 | −0.8000 | 0.6667 | 2.0000 | 0.3333 | 0.3333 | 0.499999 |
| 5 | −0.4000 | 2.0000 | 0.6667 | 0.6667 | 1.3333 | 1.000000 |
| 6 | −0.4000 | 2.0000 | 1.3333 | 1.3333 | 0.6667 | 1.000000 |

$i = 1, 2, \ldots, n$ and for $l = 1, 2, \ldots, n$ where $i \neq l$, where $\tilde{D}_k < 0$, for $k = 1, 2, \ldots, p$. In addition, if $\Delta_l = 0$, for some $l = 1, 2, \ldots, n$, then $\Delta_i = 0$ for $i = 1, 2, \ldots, n$. Therefore, we consider a very small positive value $\varepsilon'$ such as $\Delta_i \leq -\varepsilon'$, for $i = 1, 2, \ldots, n$.

Part 3 In real life applications, different directions can be planned for demand production changes. In other words, demand changes can be positive changes for some outputs, negative changes for some other outputs, and zero changes for the rest of outputs.

In order to deal with these multiple demand changes, we split all outputs into three groups, based upon their particular demand changing directions.

The three categories are denoted by $O_p$, that is, the sets of outputs with positive demand changes, $O_n$, that is, the sets of outputs with negative demand changes, and $O_z$, that is, the sets of outputs with zero demand changes. Thus, we have $O_p \cup O_n \cup O_z = \{1, 2, \ldots, p\}$. In addition, $\tilde{D}_k$ is positive for $k \in O_p$, $\tilde{D}_k$ is negative for $k \in O_n$ and $\tilde{D}_k$ is 0 for $k \in O_z$.

We now plan the upcoming production for all DMUs to fulfill the demand changes in two steps given by:

Step 1 *Considering the output changes in group* $O_p$*only*. Similar to Part 1, we suppose that the same positive proportional change, $\Delta_i^{(p)} \geq 0$, for $i = 1, 2, \ldots, n$, are defined in all input factors and output factors in $O_p$. After that, Eq. 9.11 is applied and the targets for the input and the output factors are considered as the data for the next step.

The related model for Step 1 is as follows:

$$
\begin{aligned}
&\max \frac{\sum_{k\in O_p} w_k^+ \sum_{i=1}^n y_{ik}(1+\Delta_i^{(p)}) + \sum_{k\notin O_p} w_k^+ \sum_{i=1}^n y_{ik}}{\sum_{j=1}^m w_j^- \sum_{i=1}^n x_{ij}(1+\Delta_i^{(p)})},\\
&\text{Subject to}\\
&\frac{\sum_{k\in O_p} y_{ik}(1+\Delta_i^{(p)})w_k^+ + \sum_{k\notin O_p} y_{ik} w_k^+}{\sum_{j=1}^m x_{ij}(1+\Delta_i^{(p)})w_j^-} \le 1, \quad \text{and } i=1,2,\ldots,n,\\
&\frac{\sum_{k\in O_p} w_k^+ \sum_{i=1}^n y_{ik}(1+\Delta_i^{(p)}) + \sum_{k\notin O_p} w_k^+ \sum_{i=1}^n y_{ik}}{\sum_{j=1}^m w_j^- \sum_{i=1}^n x_{ij}(1+\Delta_i^{(p)})} \le 1,\\
&\sum_{i=1}^n y_{ik}\Delta_i^{(p)} \le \tilde{D}_k, \text{for} k\in O_p,\\
&w_j^+ \ge \varepsilon, \text{for} j=1,2,\ldots,m,\\
&w_k^- \ge \varepsilon, \quad \text{for} k=1,2,\ldots,p,\\
&\Delta_i^{(p)} \ge 0, \quad \text{for} i=1,2,\ldots,n.
\end{aligned}
\tag{9.13}
$$

Of course, we can define $\sum_{j=1}^m w_j^- \sum_{i=1}^n \left(1+\Delta_i^{(p)}\right)x_{ij} = 1$, and convert above fractional programming to the following non-linear programming.

$$
\begin{aligned}
&\max \sum_{k\in O_p} w_k^+ \sum_{i=1}^n y_{ik}(1+\Delta_i^{(p)}) + \sum_{k\notin O_p} w_k^+ \sum_{i=1}^n y_{ik},\\
&\text{Subject to}\\
&\sum_{j=1}^m w_j^- \sum_{i=1}^n x_{ij}(1+\Delta_i^{(p)}) = 1,\\
&\sum_{k\in O_p} w_k^+ \sum_{i=1}^n y_{ik}(1+\Delta_i^{(p)}) + \sum_{k\notin O_p} w_k^+ \sum_{i=1}^n y_{ik} \le 1,\\
&\sum_{k\in O_p} w_k^+ y_{ik}(1+\Delta_i^{(p)}) + \sum_{k\notin O_p} w_k^+ y_{ik} \le \sum_{j=1}^m w_j^- x_{ij}(1+\Delta_i^{(p)}), \text{ for } i=1,2,\ldots,n,\\
&\sum_{i=1}^n y_{ik}\Delta_i^{(p)} \le \tilde{D}_k, \quad \text{for} k\in O_p,\\
&w_j^+ \ge \varepsilon, \quad \text{for} j=1,2,\ldots,m,\\
&w_k^- \ge \varepsilon, \quad \text{for} k=1,2,\ldots,p,\\
&\Delta_i^{(p)} \ge 0, \text{for} i=1,2,\ldots,n.
\end{aligned}
\tag{9.14}
$$

The targets for input and output factors of $\text{DMU}_l$ ($l = 1, 2, \ldots, n$) by Eq. 9.14 are given by:

$$
\begin{aligned}
x_{lj}^{*(1)} &= x_{lj}(1+\Delta_l^{(p)*}), \quad \text{for} j=1,2,\ldots,m,\\
y_{lk}^{*(1)} &= y_{lk}(1+\Delta_l^{(p)*}) \quad \text{for} k\in O_p,\\
y_{lk}^{*(1)} &= y_{lk} \quad \text{for} k\in\{1,2,\ldots,p\} - O_p.
\end{aligned}
\tag{9.15}
$$

Step 2 *Considering the output changes in group $O_n$only*. Similar to Part 2, we suppose that the same negative proportional change, $-1 \leq \Delta_i^{(n)} \leq 0$, for $i = 1, 2, \ldots, n$, are defined in all input factors and output factors in $O_n$ which are measured in Step 1. After that, Eq. 9.12 is applied and the new targets for the input and the output factors are proposed as the new production plans for all DMUs.

$$
\begin{aligned}
&\max \sum_{k \notin O_n} w_k^+ \sum_{i=1}^n y_{ik}^{*(1)} + \sum_{k \in O_n} w_k^+ \sum_{i=1}^n y_{ik}^{*(1)} (1 + \Delta_i^{(n)}),\\
&\text{Subject to} \sum_{j=1}^m w_j^- \sum_{i=1}^n x_{ij}^{*(1)} (1 + \Delta_i^{(n)}) = 1,\\
&\sum_{k \notin O_n} w_k^+ \sum_{i=1}^n y_{ik}^{*(1)} + \sum_{k \in O_n} w_k^+ \sum_{i=1}^n y_{ik}^{*(1)} (1 + \Delta_i^{(n)}) \leq 1,\\
&\sum_{k \notin O_n} y_{ik}^{*(1)} w_k^+ + \sum_{k \in O_n} y_{ik}^{*(1)} (1 + \Delta_i^{(n)}) w_k^+ \leq \sum_{j=1}^m x_{ij}^{*(1)} (1 + \Delta_i^{(n)}) w_j^-,\\
&\qquad\qquad\qquad\qquad \text{for } i = 1, 2, \ldots, n,\\
&\sum_{i=1}^n y_{ik}^{*(1)} \Delta_i^{(n)} \leq \tilde{D}_k, \quad \text{for } k \in O_n,\\
&w_j^+ \geq \varepsilon, \quad \text{for } j = 1, 2, \ldots, m,\\
&w_k^- \geq \varepsilon, \quad \text{for } k = 1, 2, \ldots, p,\\
&\Delta_i^{(n)} \geq -1, \quad \text{for } i = 1, 2, \ldots, n,\\
&\Delta_i^{(n)} \leq -\varepsilon', \quad \text{for } i = 1, 2, \ldots, n.
\end{aligned}
\tag{9.16}
$$

The targets from Eq. 9.16 are given by:

$$
\begin{aligned}
x_{lj}^{*(2)} &= x_{lj}^{*(1)} (1 + \Delta_l^{(n)*}), \text{for } j = 1, 2, \ldots, m,\\
y_{lk}^{*(2)} &= y_{lk}^{*(1)} (1 + \Delta_l^{(n)*}), \text{for } k \in O_n,\\
y_{lk}^{*(2)} &= y_{lk}^{*(1)}, \text{for } k \in \{1, 2, \ldots, p\} - O_n.
\end{aligned}
\tag{9.17}
$$

Or

$$
\begin{aligned}
x_{lj}^{*(2)} &= x_{lj} (1 + \Delta_l^{(n)*})(1 + \Delta_l^{(p)*}), \quad \text{for } j = 1, 2, \ldots, m,\\
y_{lk}^{*(2)} &= y_{lk} (1 + \Delta_l^{(p)*}), \quad \text{for } k \in O_p\\
y_{lk}^{*(2)} &= y_{lk} (1 + \Delta_l^{(n)*}), \quad \text{for } k \in O_n\\
y_{lk}^{*(2)} &= y_{lk}, \quad \text{for } \ k \in O_z.
\end{aligned}
\tag{9.18}
$$

Now, assume that the demand changes for Output 1 and Output 2 in Table 9.3 are predicated as $\tilde{D}_1 = 4$, $\tilde{D}_2 = -3$, in the next production season, that is, the situation of demand increases in the first output and decreases in the second output. The results of applying Eq. 9.12, where $\varepsilon = 0$ and $\varepsilon' = 0.00001$ are illustrated in Tables 9.7 and 9.8, respectively.

Note that, from 'Option' in 'Solver parameters', we changed 'Population Size' to 200 in 'GRG Nonlinear' window and clicked on 'Use Multistart'. This option let solver run repeatedly and start from different starting points to find a better possible solution, and of course, it takes longer than a single run.

**Table 9.7** The results of solving Eq. 9.12 where $\varepsilon = 0$

| DMUs | $\Delta_i^{(p)*}$ | $x_{i1}^{*(1)}$ | $x_{i2}^{*(1)}$ | $y_{i1}^{*(1)}$ | $y_{i2}^{*(1)}$ | CCR-Score |
|---|---|---|---|---|---|---|
| 1 | 0.000000 | 4.000000 | 3.000000 | 2.000000 | 1.000000 | 0.736842 |
| 2 | 0.022879 | 6.137271 | 2.045757 | 1.022879 | 2.000000 | 1.000000 |
| 3 | 2.887869 | 3.887869 | 11.663606 | 3.887869 | 2.000000 | 1.000000 |
| 4 | 0.000000 | 2.000000 | 6.000000 | 1.000000 | 1.000000 | 0.971967 |
| 5 | 1.089253 | 6.267759 | 2.089253 | 2.089253 | 2.000000 | 1.000000 |
| 6 | 0.000000 | 3.000000 | 2.000000 | 2.000000 | 1.000000 | 1.000000 |

**Table 9.8** The results of solving Eq. 9.12 where $\varepsilon = 0.00001$

| DMUs | $\Delta_i^{(n)*}$ | $x_{i1}^{*(2)}$ | $x_{i2}^{*(2)}$ | $y_{i1}^{*(2)}$ | $y_{i2}^{*(2)}$ | CCR-Score |
|---|---|---|---|---|---|---|
| 1 | −0.756565 | 0.973741 | 0.730306 | 2.000000 | 0.243435 | 1.000000 |
| 2 | −0.282374 | 4.404266 | 1.468089 | 1.022879 | 1.435252 | 1.000000 |
| 3 | −0.458243 | 2.106279 | 6.318838 | 3.887869 | 1.083514 | 1.000000 |
| 4 | −0.757366 | 0.485267 | 1.455802 | 1.000000 | 0.242634 | 1.000000 |
| 5 | −0.000010 | 6.267696 | 2.089232 | 2.089253 | 1.999980 | 1.000000 |
| 6 | −0.010916 | 2.967251 | 1.978167 | 2.000000 | 0.989084 | 1.000000 |

It is also suggested that the restaurants are benchmarked before applying Idea 1 for demand changes. In this case, the restaurants will be CCR-efficient and, after panning, will have the same CCR-efficiency score as well.

## 9.4 Idea 2: Max-Min Output-Input

In this section, the three demand change situations, that is, positive, negative and zero demand changes, are incorporated into one single model. In other words, the same problem in production planning is considered, but with a different perspective from Idea 1.

The new plan is to maximize the total values of all output factors which are produced by all DMUs, and simultaneously to minimize the total values of all input factors which are consumed by all individuals. A DEA-based model is illustrated to benchmark all individuals (DMUs) within the original PPS with new input-output plans. The original PPS (See Eq. 4.4) is estimated by the input and output factors of all DMUs, which describes all technically feasible production plans.

Suppose that the forecasted demand change $\tilde{D}_k$, corresponded to the $k$th output factor, is given, for $k = 1, 2, \ldots, p$. In addition, assume that $\tilde{x}_{ij}$, for $j = 1, 2, \ldots, m$, and $\tilde{y}_{ik}$, for $k = 1, 2, \ldots, p$, are the planned targets for input and output factors of $\text{DMU}_i$, for $i = 1, 2, \ldots, n$.

The total values of the $k$th output targets of all DMUs should not be exceed than the total values of the original $k$th outputs of all DMUs plus the forecasted demand change $\tilde{D}_k$ , that is,

$$\sum_{i=1}^{n} \tilde{y}_{ik} \le \sum_{i=1}^{n} y_{ik} + \tilde{D}_k, \text{for } k = 1, 2, \ldots, p. \tag{9.19}$$

Each $\tilde{y}_{ik}$ can at least be 0, and since all demands for outputs can be changed positively or negatively, $\tilde{y}_{ik}$ can be either no more than $y_{ik}$ or no less than $y_{ik}$, for $i = 1$, 2, …, $n$, and $k = 1$, 2, …, $p$. In other words,

$$\begin{aligned} \tilde{y}_{ik} &\le y_{ik}, \quad \text{if } \tilde{D}_k \le 0, \\ \tilde{y}_{ik} &\ge y_{ik}, \quad \text{if } \tilde{D}_k \ge 0. \end{aligned} \tag{9.20}$$

In order to maximize total output production and at the same time minimize total input consumption, the following multi-objective linear programming (MOLP) should be solved.

$$\begin{aligned}
&\max \sum_{i=1}^{n} \sum_{k=1}^{p} \tilde{y}_{ik}, \\
&\min \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{x}_{ij}, \\
&\text{Subject to} \\
&\sum_{i=1}^{n} x_{ij}\lambda_i^{(l)} \le \tilde{x}_{lj}, \quad \text{for } l = 1, 2, \ldots, n, \text{and for } j = 1, 2, \ldots, m, \\
&\sum_{i=1}^{n} y_{ik}\lambda_i^{(l)} \le \tilde{y}_{lk}, \quad \text{for } l = 1, 2, \ldots, n, \text{and for } k = 1, 2, \ldots, p, \\
&\sum_{i=1}^{n} \tilde{y}_{ik} \le \sum_{i=1}^{n} y_{ik} + \tilde{D}_k, \text{for } k = 1, 2, \ldots, p, \\
&\lambda_i^{(l)} \ge 0, \quad \text{for } i = 1, 2, \ldots, n, \text{and for } l = 1, 2, \ldots, n, \\
&\left\{ \begin{array}{ll} \left\{ \begin{array}{l} \tilde{y}_{ik} \le y_{ik} \\ \tilde{y}_{ik} \ge 0 \end{array} \right. & \text{if } \tilde{D}_k \le 0 \\ \tilde{y}_{ik} \ge y_{ik} & \text{if } \tilde{D}_k \ge 0. \end{array} \right. , \quad \text{for } i = 1, 2, \ldots, n \text{ and for } k = 1, 2, \ldots, p,
\end{aligned} \tag{9.21}$$

The unknown variables in Eq. 9.21 are $n \times n$ multipliers, $\lambda_i^{(l)}$ ($i = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, n$), $n \times m$ input targets, $\tilde{x}_{ij}$ ($i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$), and $n \times p$ output targets, $\tilde{y}_{ik}$ ($i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, p$). The above MOLP can be rewritten as follows:

$$\begin{aligned}
&\max \sum_{i=1}^{n} \sum_{k=1}^{p} \tilde{y}_{ik} - \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{x}_{ij}, \\
&\text{Subject to} \\
&\sum_{i=1}^{n} x_{ij}\lambda_i^{(l)} \le \tilde{x}_{lj}, \text{for } l = 1, 2, \ldots, n, \text{and for } j = 1, 2, \ldots, m, \\
&\sum_{i=1}^{n} y_{ik}\lambda_i^{(l)} \ge \tilde{y}_{lk}, \text{and } l = 1, 2, \ldots, n, \text{and for } k = 1, 2, \ldots, p, \\
&\sum_{i=1}^{n} \tilde{y}_{ik} \le \sum_{i=1}^{n} y_{ik} + \tilde{D}_k, \text{for } k = 1, 2, \ldots, p, \\
&\lambda_i^{(l)} \ge 0, \text{for } i = 1, 2, \ldots, n, \text{and for } l = 1, 2, \ldots, n, \\
&\left\{ \begin{array}{ll} \left\{ \begin{array}{l} \tilde{y}_{ik} \le y_{ik} \\ \tilde{y}_{ik} \ge 0 \end{array} \right. & \text{if } \tilde{D}_k \le 0 \\ \tilde{y}_{ik} \ge y_{ik} & \text{if } \tilde{D}_k \ge 0. \end{array} \right. , \quad \text{for } i = 1, 2, \ldots, n, \text{and for } k = 1, 2, \ldots, p,
\end{aligned} \tag{9.22}$$

Equation 9.22 is a linear programming model which is equivalent with Eq. 9.21. The epsilon, $\varepsilon$, in the objective of Eq. 9.22 is also a very small positive real number.

From the first two sets of constraints, that is, $\sum_{i=1}^{n} x_{ij}\lambda_i^{(l)} \leq \tilde{x}_{lj}$, for $l = 1, 2, \ldots, n$, and for $j = 1, 2, \ldots, m$, $\sum_{i=1}^{n} y_{ik}\lambda_i^{(l)} \geq \tilde{y}_{lk}$, for $l = 1, 2, \ldots, n$, and for $k = 1, 2, \ldots, p$, and the objective, that is, max $\sum_{i=1}^{n}\sum_{k=1}^{p} \tilde{y}_{ik} - \varepsilon \sum_{i=1}^{n}\sum_{j=1}^{m} \tilde{x}_{ij}$, we expect that $\tilde{x}_{ij}$ and $\tilde{y}_{ik}$ lie of the DEA frontier generated by DMUs. In other words, the planning results by Eq. 9.22 should completely improve the CCR-efficiency scores of all DMUs to one. We now prove this statement.

Let $\tilde{x}_{ij}^*$ and $\tilde{y}_{ik}^*$, for $i = 1, 2, \ldots, n$, for $j = 1, 2, \ldots, m$, and for $k = 1, 2, \ldots, p$, be the optimal solutions in Eq. 9.22 for $\tilde{x}_{ij}$ and $\tilde{y}_{ik}$, for $i = 1, 2, \ldots, n$, for $j = 1, 2, \ldots, m$, and for $k = 1, 2, \ldots, p$, respectively. Thus, the new input and output plan for DMU$_l$ ($l = 1, 2, \ldots, n$) is $\left(\tilde{x}_{l1}^*, \tilde{x}_{l2}^*, \ldots, \tilde{x}_{lm}^*, \tilde{y}_{l1}^*, \tilde{y}_{l2}^*, \ldots, \tilde{x}_{lp}^*\right)$. The CCR-efficiency of DMU$_l$ with this new plan, which can be simply called the new CCR-efficiency score for DMU$_l$, can be calculated by the following envelopment form of CCR, using the original PPS.

$$\begin{aligned}
&\min \theta_l, \\
&\text{Subject to} \\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i \leq \tilde{x}_{lj}^* \theta_l, \text{for } j = 1, 2, \ldots, m, \\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i \geq \tilde{y}_{lk}^*, \text{for } k = 1, 2, \ldots, p, \\
&\lambda_i \geq 0, i = 1, 2, \ldots, n.
\end{aligned} \tag{9.23}$$

Now, the following theorem is proposed to prove this important property about the optimal targets from Eq. 9.22 which lie on the DEA-frontier generated by the DMUs. In other words, the new CCR-efficiency scores of all DMUs with planned targets measured by Eq. 9.22, can reach one when the original PPS is considered.

**Theorem 9.1** In Eq. 9.23, $\theta_l^* = 1$.

*Proof* Let's suppose that an $\varepsilon > 0$ is given and $\tilde{x}_{ij}^*$, $\tilde{y}_{ik}^*$ and $\lambda_i^{(l)*}$ are optimal solutions for $\tilde{x}_{ij}$, $\tilde{y}_{ik}$ and $\lambda_i^{(l)}$ in Eq. 9.22, respectively, for $i = 1, 2, \ldots, n$, for $j = 1, 2, \ldots, m$, and for $k = 1, 2, \ldots, p$. Assume that $\pi^*$ is the maximum value of the objective in Eq. 9.22, that is,

$$\pi^* = \sum_{i=1}^{n}\sum_{k=1}^{p} \tilde{y}_{ik}^* - \varepsilon \sum_{i=1}^{n}\sum_{j=1}^{m} \tilde{x}_{ij}^*, \tag{9.24}$$

In order to prove that $\theta_l^* = 1$, we prove that the CCR score in Eq. 9.23 for none of DMUs are less than 1, that is, $\nexists l$ ($l = 1, 2, \ldots, n$): $\theta_l^* < 1$.

Suppose that there exist at least one DMU with the CCR score less than 1, that is, assume that $\exists l_0$ ($1 \leq l_0 \leq n$): $\theta_{l_0}^* < 1$.

Let the optimal solutions for the multipliers in Eq. 9.23 related to $\theta_{l_0}^*$ be $\lambda_i^{l_0*}$ ($i = 1, 2, \ldots, n$). Thus, from the constraints of Eq. 9.23 we have

$$\sum_{i=1}^{n} \lambda_i^{l_0 *} x_{ij} \leq \theta_{l_0}^* \tilde{x}_{l_0 j}^*, \quad \text{for } j = 1, 2, \ldots, m, \tag{9.25}$$

And

$$\sum_{i=1}^{n} \lambda_i^{l_0 *} y_{ik} \geq \tilde{y}_{l_0 k}^*, \text{for } k = 1, 2, \ldots, p, \tag{9.26}$$

On the other hand, $\tilde{x}_{l_0 j}^* \theta_{l_0}^*, \tilde{x}_{ij}^*, \tilde{y}_{ik}^*, \lambda_i^{l_0 *}$ and $\lambda_i^{(l)*}$, for $i \neq l_0$, $i = 1, 2, \ldots, n$, $l = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $k = 1, 2, \ldots, p$, is a feasible solution for Eq. 9.22. Thus, the objective of Eq. 9.22 for this feasible solution is given by:

$$\sum_{i=1}^{n} \sum_{k=1}^{p} \tilde{y}_{ik}^* - \varepsilon \sum_{j=1}^{m} \Big( \tilde{x}_{l_0 j}^* \theta_{l_0}^* + \sum_{\substack{i=1 \\ i \neq l_0}}^{n} \tilde{x}_{ij}^* \Big). \tag{9.27}$$

or

$$\sum_{i=1}^{n} \sum_{k=1}^{p} \tilde{y}_{ik}^* - \varepsilon \sum_{j=1}^{m} \tilde{x}_{l_0 j}^* \theta_{l_0}^* - \varepsilon \sum_{j=1}^{m} \sum_{\substack{i=1 \\ i \neq l_0}}^{n} \tilde{x}_{ij}^*. \tag{9.28}$$

From Eq. 9.29, we have

$$\tilde{x}_{l_0 j}^* \theta_{l_0}^* = \tilde{x}_{l_0 j}^* + \tilde{x}_{l_0 j}^* (\theta_{l_0}^* - 1) \tag{9.29}$$

$$\sum_{i=1}^{n} \sum_{k=1}^{p} \tilde{y}_{ik}^* - \varepsilon \sum_{j=1}^{m} \Big( \tilde{x}_{l_0 j}^* + \tilde{x}_{l_0 j}^* (\theta_{l_0}^* - 1) \Big) - \varepsilon \sum_{j=1}^{m} \sum_{i=1,\, i \neq l_0}^{n} \tilde{x}_{ij}^*. \tag{9.30}$$

or

$$\sum_{i=1}^{n} \sum_{k=1}^{p} \tilde{y}_{ik}^* - \varepsilon \Big( \theta_{l_0}^* - 1 \Big) \sum_{j=1}^{m} \tilde{x}_{l_0 j}^* - \varepsilon \sum_{j=1}^{m} \sum_{i=1}^{n} \tilde{x}_{ij}^*. \tag{9.31}$$

or

$$\sum_{i=1}^{n} \sum_{k=1}^{p} \tilde{y}_{ik}^* - \varepsilon \sum_{j=1}^{m} \sum_{i=1}^{n} \tilde{x}_{ij}^* + \varepsilon \Big( 1 - \theta_{l_0}^* \Big) \sum_{j=1}^{m} \tilde{x}_{l_0 j}^*. \tag{9.32}$$

or

$$\pi^* + \varepsilon \Big( 1 - \theta_{l_0}^* \Big) \sum_{j=1}^{m} \tilde{x}_{l_0 j}^*. \tag{9.33}$$

Since $\theta_{l_0}^* < 1$, therefore $\varepsilon \Big( 1 - \theta_{l_0}^* \Big) \sum_{j=1}^{m} \tilde{x}_{l_0 j}^* > 0$ and we have the following contradiction,

$$\pi^* < \pi^* + \varepsilon \Big( 1 - \theta_{l_0}^* \Big) \sum_{j=1}^{m} \tilde{x}_{l_0 j}^*. \tag{9.34}$$

Indeed, we assume that $\pi^*$ is the maximum objective value in Eq. 9.22, but as can be seen in Eq. 9.34, we found a solution for Eq. 9.22, that is, $\theta_{l_0}^* \tilde{x}_{l_0 j}^*, \tilde{x}_{ij}^*, \tilde{y}_{ik}^*, \lambda_i^{l_0 *}$ and

$\lambda_i^{(l)*}$, for $i \neq l_0$, $i = 1, 2, \ldots, n$, $l = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and $k = 1, 2, \ldots, p$, which yields greater value for the objective of Eq. 9.22. Therefore, the assumption that there exists one DMU which has a CCR-score less than one is not valid.

Consequently the CCR- scores for all DMUs with new planned input and output factors by Eq. 9.22 are equal to 1, where the original PPS of DMUs are considered, and the proof is completed. □

The following instructions are proposed to see the results of Eq. 9.22 for data in Table 9.3, where the demand changes for Output 1 and Output 2 are predicated as $\tilde{D}_1 = 4$ and $\tilde{D}_2 = 3$ in the next production season.

1. Copy the 5 columns of Table 9.3 on an Excel sheet into cells A1:E7, as Fig. 9.7 depicts.
2. Label C9 as '$\tilde{D}$', G9 as 'Objective', J9 as 'Epsilon', B11 as 'Target Input 1', C11 as 'Target Input 2', D11 as 'Target Output 1', E11 as 'Target Output 2', F1-K1 as 'Lambdas 1-6', G11-J11 as 'Constraints', C19 as 'Total Target for Output', and C20 as 'Total Output + $\tilde{D}$', as Fig. 9.7 illustrates.
3. Assign 4 and 3 to D9 and E9, respectively.
4. Assign 0.000001 to K9.
5. Assign the following command (without quotations mark) into H9,

   '=Sum(D12:E17)-K9*Sum(B12:C17)'.

6. Assign '=Sum(D12:D17)' into D19.
7. Assign '=Sum(E12:E17)' into E19.
8. Assign '=Sum(D2:D7,D9)' into D20.

H9 | =SUM(D12:E17)-K9*SUM(B12:C17)

| | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | DMUS | Input 1 | Input 2 | Output 1 | Output 2 | $\lambda_i^1$ | $\lambda_i^2$ | $\lambda_i^3$ | $\lambda_i^4$ | $\lambda_i^5$ | $\lambda_i^6$ |
| 2 | 1 | 4 | 3 | 2 | 1 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 3 | 2 | 6 | 2 | 1 | 2 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 4 | 3 | 1 | 3 | 1 | 2 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 5 | 4 | 2 | 6 | 1 | 1 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 6 | 5 | 3 | 1 | 1 | 2 | 0.000000 | 1.000000 | 1.666667 | 0.333333 | 1.000000 | 0.000000 |
| 7 | 6 | 3 | 2 | 2 | 1 | 1.000000 | 0.000000 | 1.666667 | 0.333333 | 0.000000 | 1.000000 |
| 8 | | | | | | | | | | | |
| 9 | | | D~ | 4 | 3 | | Objective | 24.0000 | | Epsilon | 0.000001 |
| 10 | | | | | | | | | | | |
| 11 | | Tin 1 | Tin 2 | Tout 1 | Tout 2 | | Constraints | | | | |
| 12 | | 3 | 2 | 2 | 1 | | 3 | 2 | 2 | 1 | |
| 13 | | 3 | 1 | 1 | 2 | | 3 | 1 | 1 | 2 | |
| 14 | | 10 | 5 | 5 | 5 | | 10 | 5 | 5 | 5 | |
| 15 | | 2 | 1 | 1 | 1 | | 2 | 1 | 1 | 1 | |
| 16 | | 3 | 1 | 1 | 2 | | 3 | 1 | 1 | 2 | |
| 17 | | 3 | 2 | 2 | 1 | | 3 | 2 | 2 | 1 | |
| 18 | | | | | | | | | | | |
| 19 | | | Total target for Output | 12 | 12 | | | | | | |
| 20 | | | Total Output + D~ | 12 | 12 | | | | | | |
| 21 | | | | | | | | | | | |

**Fig. 9.7** Setting Excel sheet to solve Eq. 9.22

9. Assign '=Sum(E2:E7,E9)' into E20.
10. Assign the following command into G12,

    '= Sumproduct (B$2:B$7,$F2:$F7)'.

11. Copy G12, and paste it to H12, I12 and J12.
12. Assign the following command into G13,

    '= Sumproduct (B$2:B$7,$G2:$G7)'.

13. Copy G13, and paste it to H13, I13 and J13.
14. Assign the following command into G14,

    '= Sumproduct (B$2:B$7,$H2:$H7)'.

15. Copy G14, and paste it to H14, I14 and J14.
16. Assign the following command into G15,

    '= Sumproduct (B$2:B$7,$I2:$I7)'.

17. Copy G15, and paste it to H15, I15 and J15.
18. Assign the following command into G16,

    '= Sumproduct (B$2:B$7,$J2:$J7)'.

19. Copy G16, and paste it to H16, I16 and J16.
20. Assign the following command into G17,

    '= Sumproduct (B$2:B$7,$K2:$K7)'.

21. Copy G17, and paste it to H17, I17 and J17.
22. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 9.8 illustrates.
23. Assign 'H9' into 'Set Objective' and choose 'Max'.
24. Assign 'F2:K7, B12:E17' into 'By Changing Variable Cells'.
25. Click on 'Add' and assign 'G12:H17' into 'Cell Reference', then select '<=', and assign 'B12:C17' into 'Constraint'.
26. Click on 'Add' and assign 'I12:J17' into 'Cell Reference', then select '>=' and assign 'D12:E17' into 'Constraint'. Then click on 'OK'.
27. Click on 'Add' and assign 'D12:D17' into 'Cell Reference', then select '>=', and assign 'D2:D7' into 'Constraint'.
28. Click on 'Add' and assign 'E12:E17' into 'Cell Reference', then select '>=', and assign 'E2:E7' into 'Constraint'.
29. Click on 'Add' and assign 'D19:E19' into 'Cell Reference', then select '<=', and assign 'D20:E20' into 'Constraint'.
30. Tick 'Make Unconstrained Variables Non-Negative'.
31. Choose 'Simplex LP' from 'Select a Solving Method'.
32. Click on 'Solve'.

Table 9.9 illustrates the planned input and output factors. DMUs with these planned data are CCR-efficient and lie on the original DEA-frontier generated by

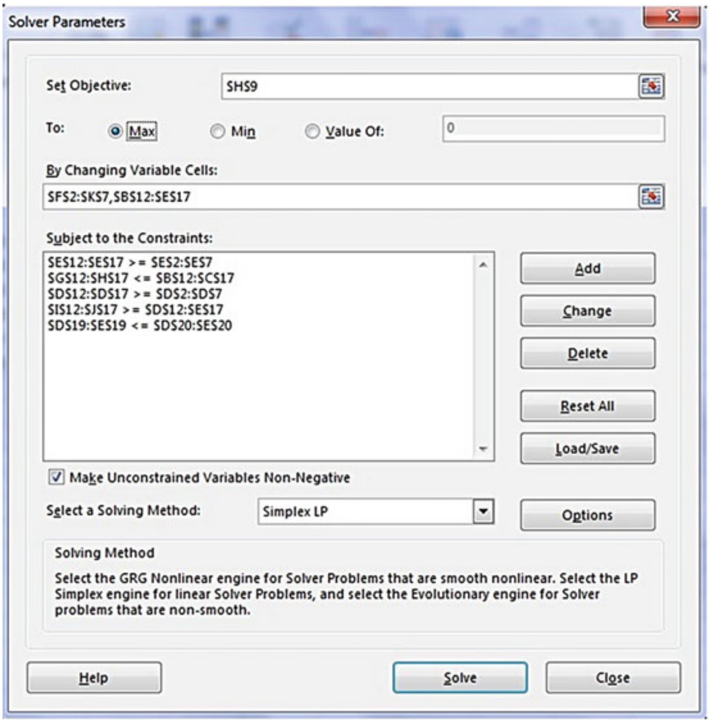



**Fig. 9.8** Setting Solver to solve Eq. 9.22

**Table 9.9** The results of solving Eq. 9.22

| DMUs | $\tilde{x}^*_{i1}$ | $\tilde{x}^*_{i2}$ | $\tilde{y}^*_{i1}$ | $\tilde{y}^*_{i2}$ | CCR-Score |
|---|---|---|---|---|---|
| 1 | 3 | 2 | 2 | 1 | 1.000000 |
| 2 | 3 | 1 | 1 | 2 | 1.000000 |
| 3 | 10 | 5 | 5 | 5 | 1.000000 |
| 4 | 2 | 1 | 1 | 1 | 1.000000 |
| 5 | 3 | 1 | 1 | 2 | 1.000000 |
| 6 | 3 | 2 | 2 | 1 | 1.000000 |

DMUs. It is also obvious that the DMUs with planned data in Table 9.9 lie on the PPS frontier which are generated by themselves.

If the demand changes for Output 1 and Output 2 are predicated as $\tilde{D}_1 = 4$ and $\tilde{D}_2 = -3$, the direction of inequality in Step 28 above should be changed to '<=', that is,

28. Click on 'Add' and assign 'E12:E17' into 'Cell Reference', then select '<=', and assign 'E2:E7' into 'Constraint'.

The results of the model while $\tilde{D}_1 = 4$ and $\tilde{D}_2 = -3$ are represented in Table 9.10.

**Table 9.10** The results of solving Eq. 9.22 by known demands

| DMUs | $\tilde{x}_{i1}^{*}$ | $\tilde{x}_{i2}^{*}$ | $\tilde{y}_{i1}^{*}$ | $\tilde{y}_{i2}^{*}$ | CCR-Score |
|---|---|---|---|---|---|
| 1 | 3 | 2 | 2 | 1 | 1.000000 |
| 2 | 3 | 2 | 2 | 1 | 1.000000 |
| 3 | 6 | 4 | 4 | 2 | 1.000000 |
| 4 | 1.5 | 1 | 1 | 0.5 | 1.000000 |
| 5 | 1.5 | 1 | 1 | 0.5 | 1.000000 |
| 6 | 3 | 2 | 2 | 1 | 1.000000 |

## 9.5 Applying Ideas 1 and 2 for the Chain Restaurants

We now get back to Sect. 9.2 and apply both Ideas 1 and 2 for the chain restaurants. Suppose that the demand changes for meat dish, vegetable dish, soup, noodles and beverage are forecasted as $\tilde{D}_1 = 3,\ \tilde{D}_2 = 2.4,\ \tilde{D}_3 = -3,\ \tilde{D}_4 = 6,\ \tilde{D}_5 = 3\ (10^3$ serving) in the next business month.

We first adapt the related models for these data, and after that we respectively formulize Step 1 of Idea 1, Step 2 of Idea 1, and CCR Envelopment form in three different excel sheets. We then write a macro (Visual Basic Procedure) to run all steps with just one click.

For the first step we have

$$
\begin{aligned}
&\max\left(\sum_{k=1,\ k\neq 3}^{5} w_k^{+}\sum_{i=1}^{20} y_{ik}\left(1+\Delta_i^{(p)}\right)\right)+\left(w_3^{+}\sum_{i=1}^{20} y_{i3}\right),\\
&\text{Subject to}\\
&w_1^{-}\sum_{i=1}^{20} x_{i1}\left(1+\Delta_i^{(p)}\right)+w_2^{-}\sum_{i=1}^{20} x_{i2}\left(1+\Delta_i^{(p)}\right)=1,\\
&\sum_{k=1,\ k\neq 3}^{5} w_k^{+}\sum_{i=1}^{20} y_{ik}\left(1+\Delta_i^{(p)}\right)+w_3^{+}\sum_{i=1}^{20} y_{i3}\le 1,\\
&\left(\sum_{k=1,\ k\neq 3}^{5} w_k^{+} y_{ik}\left(1+\Delta_i^{(p)}\right)\right)+y_{i3}w_3^{+}\le\sum_{j=1}^{2} w_j^{-}x_{ij}\left(1+\Delta_i^{(p)}\right),\\
&\qquad\qquad\qquad\qquad\text{for } i=1,2,\ldots,20,\\
&\sum_{i=1}^{20} y_{i1}\Delta_i^{(p)}\le 3,\\
&\sum_{i=1}^{20} y_{i2}\Delta_i^{(p)}\le 2.4,\\
&\sum_{i=1}^{20} y_{i4}\Delta_i^{(p)}\le 6,\\
&\sum_{i=1}^{20} y_{i5}\Delta_i^{(p)}\le 3,\\
&w_j^{+}\ge\varepsilon,\ \text{for } j=1,2,\\
&w_k^{-}\ge\varepsilon,\ \text{for } k=1,2,\ldots,5,\\
&\Delta_i^{(p)}\ge 0,\ \text{for } i=1,2,\ldots,20.
\end{aligned}
\tag{9.35}
$$

The targets for DMU$_l$ ($l = 1, 2, \ldots, 20$) are:

$$\begin{aligned} x_{l1}^{*(1)} &= x_{l1}(1+\Delta_l^{(p)*}),\\ x_{l2}^{*(1)} &= x_{l2}(1+\Delta_l^{(p)*}),\\ y_{l1}^{*(1)} &= y_{l1}(1+\Delta_l^{(p)*}),\\ y_{l2}^{*(1)} &= y_{l2}(1+\Delta_l^{(p)*}),\\ y_{l3}^{*(1)} &= y_{l3},\\ y_{l4}^{*(1)} &= y_{l4}(1+\Delta_l^{(p)*}),\\ y_{l5}^{*(1)} &= y_{l5}(1+\Delta_l^{(p)*}). \end{aligned} \tag{9.36}$$

The objective in Eq. 9.35 can be written as follows, too. The same action can be applied for the constraints. In other words, when $\tilde{D}_k < 0$, we just need to subtract $w_k^+ \sum_{i=1}^{20} y_{ik}\Delta_i^{(p)}$ in the related equations.

$$\left(\sum_{k=1}^{5} w_k^+ \sum_{i=1}^{20} y_{ik}(1+\Delta_i^{(p)})\right) - (w_3^+ \sum_{i=1}^{20} y_{i3}\Delta_i^{(p)}). \tag{9.37}$$

For the second step, we consider the output changes in group $O_n = \{3\}$ only. Similarly, we suppose that the same negative proportional change, $-1 \le \Delta_i^{(n)} \le 0$, for $i = 1, 2, \ldots, n$, are defined in all input factors and the third output factors which are measured in Step 1.

Equation 9.12 represents the model in Step 2.

$$\begin{aligned}
&\max(\textstyle\sum_{k=1,\,k\neq 3}^{5} w_k^+ \sum_{i=1}^{n} y_{ik}^{*(1)}) + w_3^+ \sum_{i=1}^{5} y_{i3}^{*(1)}(1+\Delta_i^{(n)}),\\
&\text{Subject to}\\
&w_1^- \textstyle\sum_{i=1}^{5} x_{i1}^{*(1)}(1+\Delta_i^{(n)}) + w_2^- \sum_{i=1}^{5} x_{i2}^{*(1)}(1+\Delta_i^{(n)}) = 1,\\
&(\textstyle\sum_{k=1,\,k\neq 3}^{5} w_k^+ \sum_{i=1}^{n} y_{ik}^{*(1)}) + w_3^+ \sum_{i=1}^{n} y_{i3}^{*(1)}(1+\Delta_i^{(n)}) \le 1,\\
&(\textstyle\sum_{k=1,\,k\neq 3}^{5} y_{ik}^{*(1)} w_k^+) + w_3^+(1+\Delta_i^{(n)}) y_{i3}^{*(1)} \le \sum_{j=1}^{2} x_{ij}^{*(1)}(1+\Delta_i^{(n)}) w_j^-, \text{ for } i = 1,2,\ldots,n\\
&\textstyle\sum_{i=1}^{5} \Delta_i^{(n)} y_{i3}^{*(1)} \le \tilde{D}_3,\\
&w_j^+ \ge \varepsilon, \text{ for } j = 1,2,\\
&w_k^- \ge \varepsilon, \text{ for } k = 1,2,\ldots,5,\\
&\Delta_i^{(n)} \ge -1, \text{ for } i = 1,2,\ldots,20,\\
&\Delta_i^{(n)} \le -\varepsilon', \text{ for } i = 1,2,\ldots,20.
\end{aligned} \tag{9.38}$$

The targets from Eq. 9.16 are given by:

$$
\begin{aligned}
x_{l1}^{*(2)} &= x_{l1}^{*(1)}(1+\Delta_l^{(n)*}),\\
x_{l2}^{*(2)} &= x_{l2}^{*(1)}(1+\Delta_l^{(n)*}),\\
y_{l1}^{*(2)} &= y_{l1}^{*(1)},\\
y_{l2}^{*(2)} &= y_{l2}^{*(1)},\\
y_{l3}^{*(2)} &= y_{l3}^{*(1)}(1+\Delta_l^{(n)*}),\\
y_{lk}^{*(2)} &= y_{lk}^{*(1)},\\
y_{lk}^{*(2)} &= y_{lk}^{*(1)}.
\end{aligned} \tag{9.39}
$$

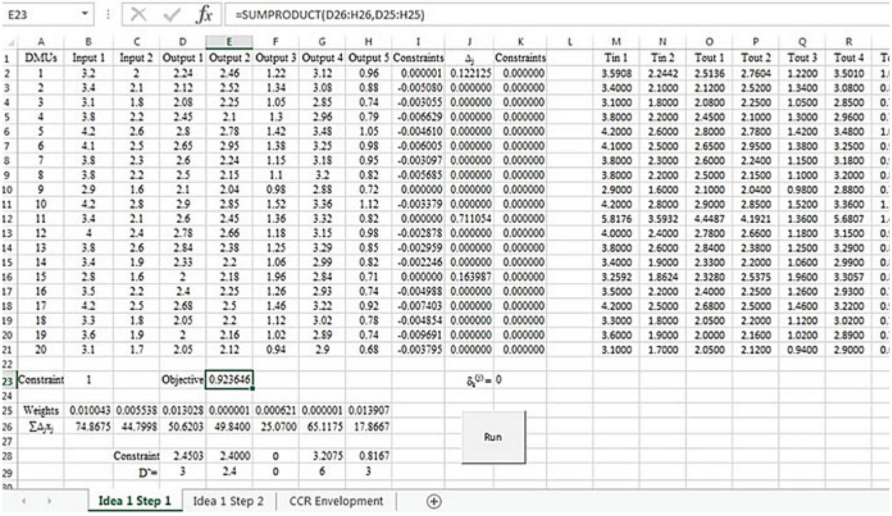
Or

$$
\begin{aligned}
x_{l1}^{*(2)} &= x_{l1}(1+\Delta_l^{(n)*})(1+\Delta_l^{(p)*}),\\
x_{l2}^{*(2)} &= x_{l2}(1+\Delta_l^{(n)*})(1+\Delta_l^{(p)*})\\
y_{l1}^{*(2)} &= y_{l1}(1+\Delta_l^{(p)*}),\\
y_{l2}^{*(2)} &= y_{l2}(1+\Delta_l^{(p)*}),\\
y_{l3}^{*(2)} &= y_{l3}(1+\Delta_l^{(n)*}),\\
y_{l4}^{*(2)} &= y_{l4}(1+\Delta_l^{(p)*}),\\
y_{l5}^{*(2)} &= y_{l5}(1+\Delta_l^{(p)*}).
\end{aligned} \tag{9.40}
$$

The objective in Eq. 9.38 can also be written as $\left(\sum_{k=1}^{5} w_k^+ \sum_{i=1}^{n} y_{ik}^{*(1)}\right) + y_{i3}^{*(1)}\Delta_i^{(n)} w_3^+$. The same action can be applied for the constraints in Eq. 9.38. In other words, when $\tilde{D}_k < 0$, we just need to add $w_k^+ \sum_{i=1}^{20} y_{ik}$ $\Delta_i^{(p)}$ in the related equations.

The following instructions illustrate the steps to run Eqs. 9.35 and 9.38 and CCR Envelopment to optimize the average production performance of all restaurants after planning using Idea 1.

1. Copy the 8 columns of Table 9.1 on an Excel sheet into cells A1:H21, as Fig. 9.9 illustrates.
2. Label A23 as 'Constraint, D23 as 'Objective', A25 as 'Weights', A26 as 'SumDeltasData', C28 as 'Constraints' C29 as '$\tilde{D}$', I1 as 'Constraints', J1 as 'Deltas', K1 as 'Constraint', M1 'Target Input 1', N1 as 'Target Input 2', O1 as 'Target Output 1', P1 as 'Target Output 2', P1 as 'Target Output 2', P1 as 'Target Output 2', and P1 as 'Target Output 2', as Fig. 9.9 represents.



| | DMUs | Input 1 | Input 2 | Output 1 | Output 2 | Output 3 | Output 4 | Output 5 | Constraints | $\Delta_j$ | Constraints |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 3.2 | 2 | 2.24 | 2.46 | 1.22 | 3.12 | 0.96 | 0.000001 | 0.122125 | 0.000000 |
| 3 | 2 | 3.4 | 2.1 | 2.12 | 2.52 | 1.34 | 3.08 | 0.88 | -0.005080 | 0.000000 | 0.000000 |
| 4 | 3 | 3.1 | 1.8 | 2.08 | 2.25 | 1.05 | 2.85 | 0.74 | -0.003055 | 0.000000 | 0.000000 |
| 5 | 4 | 3.8 | 2.2 | 2.45 | 2.1 | 1.3 | 2.96 | 0.79 | -0.006629 | 0.000000 | 0.000000 |
| 6 | 5 | 4.2 | 2.6 | 2.8 | 2.78 | 1.42 | 3.48 | 1.05 | -0.004610 | 0.000000 | 0.000000 |
| 7 | 6 | 4.1 | 2.5 | 2.65 | 2.95 | 1.38 | 3.25 | 0.98 | -0.006005 | 0.000000 | 0.000000 |
| 8 | 7 | 3.8 | 2.3 | 2.6 | 2.24 | 1.15 | 3.18 | 0.95 | -0.003097 | 0.000000 | 0.000000 |
| 9 | 8 | 3.8 | 2.2 | 2.5 | 2.15 | 1.1 | 3.2 | 0.82 | -0.005685 | 0.000000 | 0.000000 |
| 10 | 9 | 2.9 | 1.6 | 2.1 | 2.04 | 0.98 | 2.88 | 0.72 | 0.000000 | 0.000000 | 0.000000 |
| 11 | 10 | 4.2 | 2.8 | 2.9 | 2.85 | 1.52 | 3.36 | 1.12 | -0.003379 | 0.000000 | 0.000000 |
| 12 | 11 | 3.4 | 2.1 | 2.6 | 2.45 | 1.36 | 3.32 | 0.82 | 0.000000 | 0.711054 | 0.000000 |
| 13 | 12 | 4 | 2.4 | 2.78 | 2.66 | 1.18 | 3.15 | 0.98 | -0.002878 | 0.000000 | 0.000000 |
| 14 | 13 | 3.8 | 2.6 | 2.84 | 2.38 | 1.25 | 3.29 | 0.85 | -0.002959 | 0.000000 | 0.000000 |
| 15 | 14 | 3.4 | 1.9 | 2.33 | 2.2 | 1.06 | 2.99 | 0.82 | -0.002246 | 0.000000 | 0.000000 |
| 16 | 15 | 2.8 | 1.6 | 2 | 2.18 | 1.96 | 2.84 | 0.71 | 0.000000 | 0.163987 | 0.000000 |
| 17 | 16 | 3.5 | 2.2 | 2.4 | 2.25 | 1.26 | 2.93 | 0.74 | -0.004988 | 0.000000 | 0.000000 |
| 18 | 17 | 4.2 | 2.5 | 2.68 | 2.5 | 1.46 | 3.22 | 0.92 | -0.007403 | 0.000000 | 0.000000 |
| 19 | 18 | 3.3 | 1.8 | 2.05 | 2.2 | 1.12 | 3.02 | 0.78 | -0.004854 | 0.000000 | 0.000000 |
| 20 | 19 | 3.6 | 1.9 | 2 | 2.16 | 1.02 | 2.89 | 0.74 | -0.009691 | 0.000000 | 0.000000 |
| 21 | 20 | 3.1 | 1.7 | 2.05 | 2.12 | 0.94 | 2.9 | 0.68 | -0.003795 | 0.000000 | 0.000000 |
| 22 | | | | | | | | | | | |
| 23 | Constraint | 1 | | Objective | 0.923646 | | | | | $\delta_k^{(t)} = 0$ | |
| 24 | | | | | | | | | | | |
| 25 | Weights | 0.010043 | 0.005538 | 0.013028 | 0.000001 | 0.000621 | 0.000001 | 0.013907 | | | |
| 26 | $\Sigma\Delta_j x_j$ | 74.8675 | 44.7998 | 50.6203 | 49.8400 | 25.0700 | 65.1175 | 17.8667 | | | |
| 27 | | | | | | | | | | | |
| 28 | | | Constraint | 2.4503 | 2.4000 | 0 | 3.2075 | 0.8167 | | | |
| 29 | | | $D^{*}=$ | 3 | 2.4 | 0 | 6 | 3 | | | |

| | Tin 1 | Tin 2 | Tout 1 | Tout 2 | Tout 3 | Tout 4 |
|---|---|---|---|---|---|---|
| 2 | 3.5908 | 2.2442 | 2.5136 | 2.7604 | 1.2200 | 3.5010 |
| 3 | 3.4000 | 2.1000 | 2.1200 | 2.5200 | 1.3400 | 3.0800 |
| 4 | 3.1000 | 1.8000 | 2.0800 | 2.2500 | 1.0500 | 2.8500 |
| 5 | 3.8000 | 2.2000 | 2.4500 | 2.1000 | 1.3000 | 2.9600 |
| 6 | 4.2000 | 2.6000 | 2.8000 | 2.7800 | 1.4200 | 3.4800 |
| 7 | 4.1000 | 2.5000 | 2.6500 | 2.9500 | 1.3800 | 3.2500 |
| 8 | 3.8000 | 2.3000 | 2.6000 | 2.2400 | 1.1500 | 3.1800 |
| 9 | 3.8000 | 2.2000 | 2.5000 | 2.1500 | 1.1000 | 3.2000 |
| 10 | 2.9000 | 1.6000 | 2.1000 | 2.0400 | 0.9800 | 2.8800 |
| 11 | 4.2000 | 2.8000 | 2.9000 | 2.8500 | 1.5200 | 3.3600 |
| 12 | 5.8176 | 3.5932 | 4.4487 | 4.1921 | 1.3600 | 5.6807 |
| 13 | 4.0000 | 2.4000 | 2.7800 | 2.6600 | 1.1800 | 3.1500 |
| 14 | 3.8000 | 2.6000 | 2.8400 | 2.3800 | 1.2500 | 3.2900 |
| 15 | 3.4000 | 1.9000 | 2.3300 | 2.2000 | 1.0600 | 2.9900 |
| 16 | 3.2592 | 1.8624 | 2.3280 | 2.5375 | 1.9600 | 3.3057 |
| 17 | 3.5000 | 2.2000 | 2.4000 | 2.2500 | 1.2600 | 2.9300 |
| 18 | 4.2000 | 2.5000 | 2.6800 | 2.5000 | 1.4600 | 3.2200 |
| 19 | 3.3000 | 1.8000 | 2.0500 | 2.2000 | 1.1200 | 3.0200 |
| 20 | 3.6000 | 1.9000 | 2.0000 | 2.1600 | 1.0200 | 2.8900 |
| 21 | 3.1000 | 1.7000 | 2.0500 | 2.1200 | 0.9400 | 2.9000 |

**Fig. 9.9** Setting Excel sheet to solve Eqs. 9.35 and 9.38

3. Assign 3, 2.4, 0, 6 and 3 into D29-H29, respectively.
4. Assign 0 to K23.
5. Assign the following command (without quotations mark) into B23,

   '=Sumproduct(B26:C26,B25:C25)'.

6. Assign the following command into E23,

   '=Sumproduct(D26:H26,D25:H25)'.

7. Assign the following command into B26,

   '=Sumproduct(B2:B21,(1+$J$2:$J$21))'.

8. Copy B26, and paste it to C26.
9. Assign the following command into D26,

   '=If(D29>0,Sumproduct(D2:D21,(1+$J$2:$J$21)),Sum(D2:D21))'.

10. Copy D26, and paste it to E26-H26.
11. Assign the following command into D28,

    '=If(D29>0,Sumproduct(D2:D21,$J$2:$J$21),D29)'.

12. Copy D28, and paste it to E28-H28.
13. Assign the following command into I2,

    '=(Sumproduct(D2:H2,D$25:H$25)*(1+J2)-F2*J2*F$25)-Sumproduct(B2:C2,B$25:C$25)*(1+J2)'.

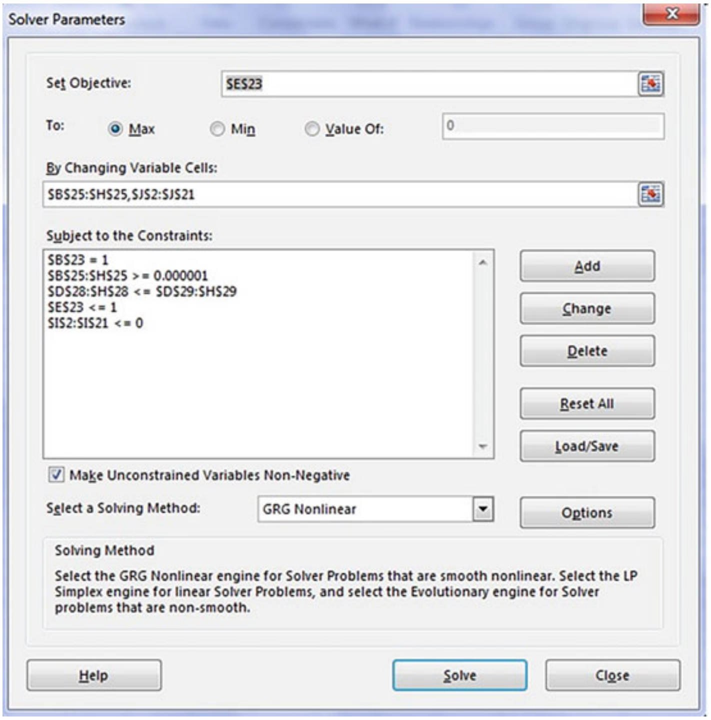



**Fig. 9.10** Setting Solver to solve Eq. 9.35

14. Copy I2, and paste it to I3-I21.
15. Assign the following command into M2,

    '=(1+$J2)*B2'.

16. Copy M2, and paste it to M2 -N21.
17. Assign the following command into O2,

    '=If(D$29>0,(1+$J2)*D2,D2)'.

18. Copy O2, and paste it to O2-S21.
19. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 9.10 illustrates.
20. Assign 'E23' into 'Set Objective' and choose 'Max'.
21. Assign 'B25:H25, J2:J21' into 'By Changing Variable Cells'.
22. Click on 'Add' and assign 'B23' into 'Cell Reference', then select '=', and assign '1' into 'Constraint'.
23. Click on 'Add' and assign 'B25:H25' into 'Cell Reference', then select '>=' and assign '0.000001' into 'Constraint'.
24. Click on 'Add' and assign 'D28:H28' into 'Cell Reference', then select '>=', and assign 'D29:H29' into 'Constraint'.

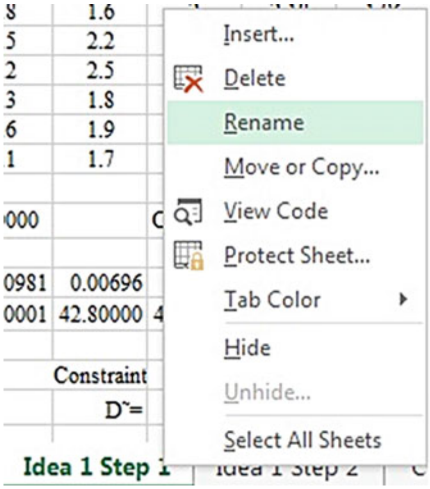



**Fig. 9.11** Rename a worksheet

25. Click on 'Add' and assign 'E23' into 'Cell Reference', then select '<=', and assign '1' into 'Constraint'.
26. Click on 'Add' and assign 'I2:I21' into 'Cell Reference', then select '<=', and assign '0' into 'Constraint'.
27. Tick 'Make Unconstrained Variables Non-Negative'.
28. Choose 'GRG Nonlinear' from 'Select a Solving Method'.
29. Click on 'Solve'.
30. Right click on the worksheet tab to rename the worksheet to 'Idea 1 Step 1', as Fig. 9.11 shows.
31. Right click on the worksheet tab again and select 'Move or Copy . . .', then tick 'Create a copy', as Fig. 9.12 depicts, and click on 'OK'.
32. Rename the new sheet as 'Idea 1 Step 2'.
33. Assign the following command into D26,

    '=If(D29<0,Sumproduct(D2:D21,(1+$J$2:$J$21)),Sum(D2:D21))'.

34. Copy D26, and paste it to E26-H26.
35. Assign the following command into D28,

    '=If(D29<0,Sumproduct(D2:D21,$J$2:$J$21),D29)'.

36. Copy D28, and paste it to E28-H28.
37. Assign the following command into I2,

    '=(Sumproduct(D2:H2,D$25:H$25)+F2*J2*F$25)-Sumproduct(B2:C2,B$25:C$25)*(1+J2)'.

38. Copy I2, and paste it to I3-I21.

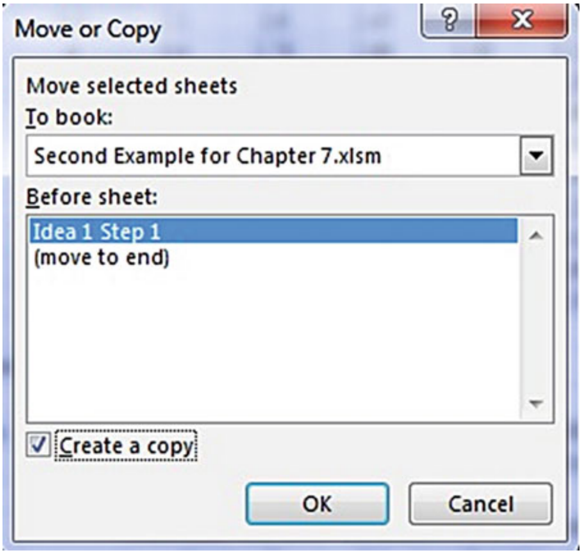



**Fig. 9.12** Copy a worksheet

39. Assign the following command into O2,

    '=If(D$29<0,(1+$J2)*D2,D2)'.

40. Copy O2, and paste it to O2-S21.

The rest of constraints are not changed.

41. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 9.13 illustrates.
42. Click on 'Add' and assign 'J2:J21' into 'Cell Reference', then select '<=', and assign '0' into 'Constraint'.
43. Click on 'Add' and assign 'J2:J21' into 'Cell Reference', then select '>=', and assign '-1' into 'Constraint'.
44. Click on 'Solve'.

Now, we can either use our earlier CCR Multiplier model in Sect. 9.2 or make a new sheet to run CCR Envelopment model. Here we give the instructions to run the CCR Envelopment model.

45. Right click on the worksheet tab and select 'Move or Copy …', then tick 'Create a copy', and click on 'OK'.
46. Rename the new sheet as 'CCR Envelopment'.
47. Delete the formula in cells E23, I2-I21.
48. Assign 1 into B23 and
49. Label A23 as 'Index' and I1 as 'Lambda', as Fig. 9.14 shows.
50. Delete Rows 25-29.
51. Delete Column J-S.

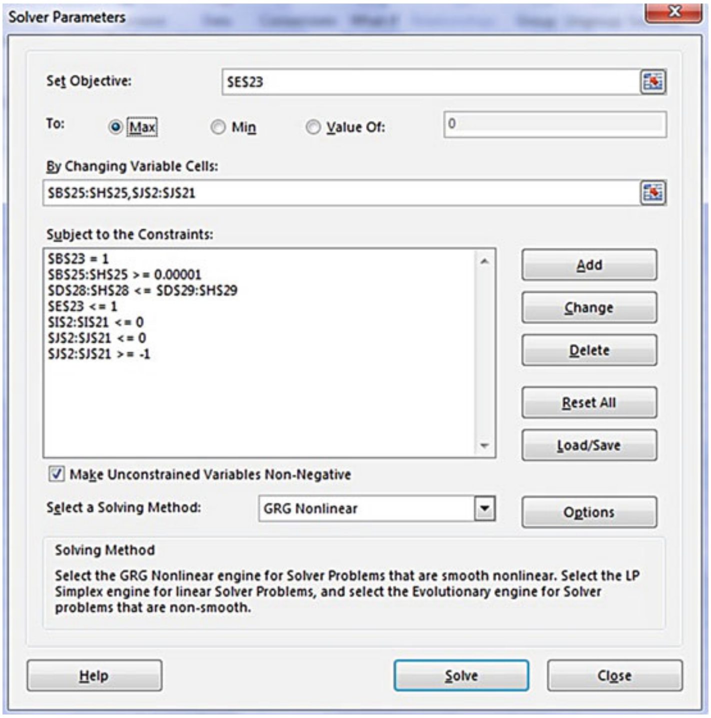



**Fig. 9.13** Setting Solver to solve Eq. 9.38

52. Assign the following command into B25.

    '=Sumproduct(B2:B21,$I2:$I21)'.

53. Copy B25 and paste it into C25-H25.
54. Assign the following command into B26

    '=Index(B2:B21,$B23)*$E23'.

55. Copy B26 and paste it into C26.
56. Assign the following command into D26

    '=Index(B2:B21,$B23)'.

57. Copy D26 and paste it into E26-H26.
58. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 9.15 illustrates.
59. Click on 'Reset All'.
60. Assign 'E23' into 'Set Objective' and choose 'Min'.
61. Assign 'I2:I21, E23' into 'By Changing Variable Cells'.

B26 =INDEX(B2:B21,$B$23)*$E$23

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | DMUS | Input 1 | Input 2 | Output 1 | Output 2 | Output 3 | Output 4 | Output 5 | Lambda | CCR |
| 2 | 1 | 2.981372 | 1.863358 | 2.240001 | 2.460001 | 1.136648 | 3.120001 | 0.96 | 0.000000 | 1 |
| 3 | 2 | 2.845355 | 1.757425 | 2.12 | 2.52 | 1.121405 | 3.08 | 0.88 | 0.000000 | 1 |
| 4 | 3 | 2.781432 | 1.615025 | 2.08 | 2.25 | 0.942098 | 2.85 | 0.74 | 0.000000 | 1 |
| 5 | 4 | 3.163667 | 1.831597 | 2.45 | 2.1 | 1.082307 | 2.96 | 0.79 | 0.000000 | 1 |
| 6 | 5 | 3.584915 | 2.219233 | 2.8 | 2.78 | 1.212043 | 3.48 | 1.05 | 0.000000 | 1 |
| 7 | 6 | 3.407093 | 2.077496 | 2.65 | 2.95 | 1.146778 | 3.25 | 0.98 | 0.000000 | 1 |
| 8 | 7 | 3.345344 | 2.024813 | 2.6 | 2.24 | 1.012407 | 3.18 | 0.95 | 0.000000 | 1 |
| 9 | 8 | 3.2628 | 1.888989 | 2.5 | 2.15 | 0.944495 | 3.2 | 0.82 | 0.000000 | 1 |
| 10 | 9 | 2.87169 | 1.584381 | 2.100001 | 2.040001 | 0.970433 | 2.880001 | 0.72 | 0.000000 | 1 |
| 11 | 10 | 3.538109 | 2.358739 | 2.9 | 2.85 | 1.280458 | 3.36 | 1.12 | 0.000000 | 1 |
| 12 | 11 | 3.258673 | 2.01271 | 2.600001 | 2.450001 | 1.303469 | 3.320002 | 0.82 | 0.000000 | 1 |
| 13 | 12 | 3.531204 | 2.118722 | 2.78 | 2.66 | 1.041705 | 3.150001 | 0.98 | 0.000000 | 1 |
| 14 | 13 | 3.268234 | 2.23616 | 2.84 | 2.38 | 1.075077 | 3.29 | 0.85 | 0.000000 | 1 |
| 15 | 14 | 3.140074 | 1.754747 | 2.33 | 2.2 | 0.978964 | 2.99 | 0.82 | 0.000000 | 1 |
| 16 | 15 | 2.739333 | 1.565333 | 2.000001 | 2.180001 | 1.917532 | 2.840001 | 0.71 | 0.000000 | 1 |
| 17 | 16 | 2.943051 | 1.849918 | 2.4 | 2.25 | 1.059499 | 2.93 | 0.74 | 0.000000 | 1 |
| 18 | 17 | 3.439043 | 2.047049 | 2.68 | 2.5 | 1.195477 | 3.22 | 0.92 | 0.000000 | 1 |
| 19 | 18 | 2.907318 | 1.58581 | 2.05 | 2.2 | 0.986726 | 3.02 | 0.78 | 0.000000 | 1 |
| 20 | 19 | 2.859463 | 1.509161 | 2 | 2.16 | 0.810181 | 2.89 | 0.74 | 0.000000 | 1 |
| 21 | 20 | 2.810777 | 1.541394 | 2.05 | 2.12 | 0.8523 | 2.9 | 0.68 | 1.000000 | 1 |
| 22 | | | | | | | | | | |
| 23 | Index | 20 | | Theta | 1.00000 | | | | | |
| 24 | | | | | | | | | | |
| 25 | | 2.8108 | 1.5414 | 2.0500 | 2.1200 | 0.8523 | 2.9000 | 0.6800 | | |
| 26 | | 2.8108 | 1.5414 | 2.0500 | 2.1200 | 0.8523 | 2.9000 | 0.6800 | | |

**Fig. 9.14** Setting Excel sheet to solve CCR envelopment

62. Click on 'Add' and assign 'B25:C25' into 'Cell Reference', then select '<=', and assign 'B26:C26' into 'Constraint'.
63. Click on 'Add' and assign 'D25:H25' into 'Cell Reference', then select '>=', and assign 'D26:H26' into 'Constraint'.
64. Tick 'Make Unconstrained Variables Non-Negative'.
65. Choose 'Simplex LP' from 'Select a Solving Method'.
66. Click on 'Solve'.
67. Click on the worksheet labeled 'Idea 1 Step 1' to get back to the first step.
68. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.
69. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet.
70. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.
71. Inside of the 'Microsoft Visual Basic for Applications' window, write the commands in Fig. 9.16 between 'Sub Button1_Click ()' and 'End Sub'.

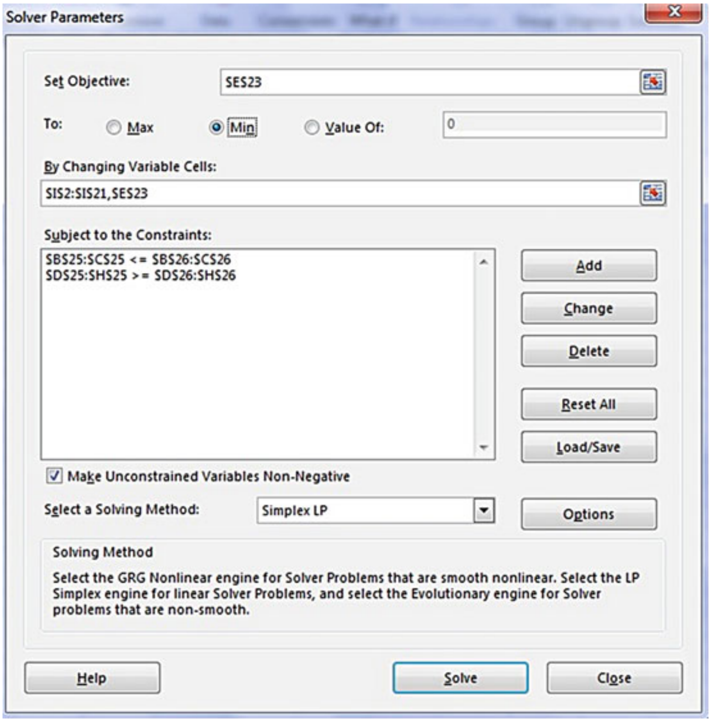



**Fig. 9.15** Setting Solver to solve CCR envelopment

72. Close the 'Microsoft Visual Basic for Applications' window.
73. Click on the small rectangle which was automatically made on the Excel sheet and created by step 69.

The model in worksheet 'Idea 1 Step 1' is run and its results are copied into B2: H21 in worksheet 'Idea 1 Step 2'. The results of Step 2 are copied into B2:H21 in worksheet 'CCR Envelopment' and the model is run for every DMU. As can be seen with the planned input and output factors, the CCR scores of all DMUs are 1, as Fig. 9.14 illustrates.

Tables 9.11 demonstrates the results from Idea 1 Steps 1 and 2 as well as the results by CCR Envelopment model.

For Idea 2, we have the following model.

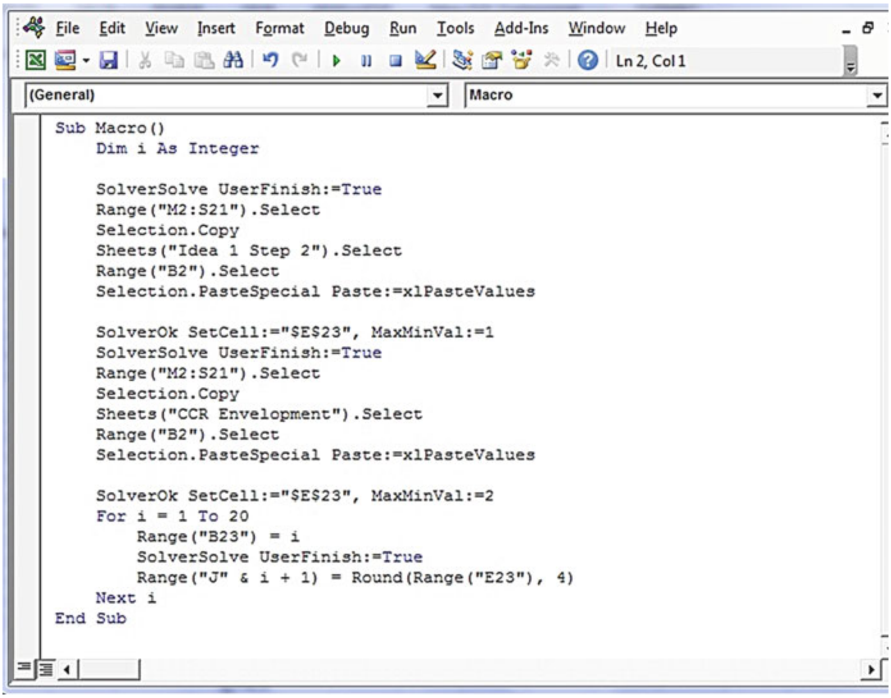



**Fig. 9.16** Setting VBA to solve Eqs. 9.35 and 9.38

**Table 9.11** The results of solving Eqs. 9.35 and 9.38

| DMUs | $\Delta_i^{(p)*}$ | $\Delta_i^{(n)*}$ | $x_{i1}^{(2)*}$ | $x_{i2}^{(2)*}$ | $y_{i1}^{(2)*}$ | $y_{i2}^{(2)*}$ | $y_{i3}^{(2)*}$ | $y_{i4}^{(2)*}$ | $y_{i5}^{(2)*}$ | CCR |
|---|---|---|---|---|---|---|---|---|---|---|
| R01 | 0.1221 | −0.00015 | 3.59 | 2.24 | 2.51 | 2.76 | 1.22 | 3.50 | 1.08 | 1 |
| R02 | 0.0000 | −0.11761 | 3.00 | 1.85 | 2.12 | 2.52 | 1.18 | 3.08 | 0.88 | 1 |
| R03 | 0.0000 | −0.09680 | 2.80 | 1.63 | 2.08 | 2.25 | 0.95 | 2.85 | 0.74 | 1 |
| R04 | 0.0000 | −0.19112 | 3.07 | 1.78 | 2.45 | 2.10 | 1.05 | 2.96 | 0.79 | 1 |
| R05 | 0.0000 | −0.13069 | 3.65 | 2.26 | 2.80 | 2.78 | 1.23 | 3.48 | 1.05 | 1 |
| R06 | 0.0000 | −0.15299 | 3.47 | 2.12 | 2.65 | 2.95 | 1.17 | 3.25 | 0.98 | 1 |
| R07 | 0.0000 | −0.10483 | 3.40 | 2.06 | 2.60 | 2.24 | 1.03 | 3.18 | 0.95 | 1 |
| R08 | 0.0000 | −0.16284 | 3.18 | 1.84 | 2.50 | 2.15 | 0.92 | 3.20 | 0.82 | 1 |
| R09 | 0.0000 | 0.00000 | 2.90 | 1.60 | 2.10 | 2.04 | 0.98 | 2.88 | 0.72 | 1 |
| R10 | 0.0000 | −0.14870 | 3.58 | 2.38 | 2.90 | 2.85 | 1.29 | 3.36 | 1.12 | 1 |
| R11 | 0.7111 | −0.11155 | 5.17 | 3.19 | 4.45 | 4.19 | 1.21 | 5.68 | 1.40 | 1 |
| R12 | 0.0000 | −0.10804 | 3.57 | 2.14 | 2.78 | 2.66 | 1.05 | 3.15 | 0.98 | 1 |
| R13 | 0.0000 | −0.24644 | 2.86 | 1.96 | 2.84 | 2.38 | 0.94 | 3.29 | 0.85 | 1 |
| R14 | 0.0000 | −0.05181 | 3.22 | 1.80 | 2.33 | 2.20 | 1.01 | 2.99 | 0.82 | 1 |
| R15 | 0.1640 | −0.02256 | 3.19 | 1.82 | 2.33 | 2.54 | 1.92 | 3.31 | 0.83 | 1 |
| R16 | 0.0000 | −0.23045 | 2.69 | 1.69 | 2.40 | 2.25 | 0.97 | 2.93 | 0.74 | 1 |
| R17 | 0.0000 | −0.18755 | 3.41 | 2.03 | 2.68 | 2.50 | 1.19 | 3.22 | 0.92 | 1 |
| R18 | 0.0000 | −0.06396 | 3.09 | 1.68 | 2.05 | 2.20 | 1.05 | 3.02 | 0.78 | 1 |
| R19 | 0.0000 | −0.15245 | 3.05 | 1.61 | 2.00 | 2.16 | 0.86 | 2.89 | 0.74 | 1 |
| R20 | 0.0000 | −0.10014 | 2.79 | 1.53 | 2.05 | 2.12 | 0.85 | 2.90 | 0.68 | 1 |

$$
\begin{aligned}
&\max\ \left(\sum_{i=1}^{20} \sum_{k=1}^{5} \tilde{y}_{ik}\right) - \varepsilon \sum_{i=1}^{20} (\tilde{x}_{i1} + \tilde{x}_{i2}),\\
&\text{Subject to}\\
&\sum_{i=1}^{20} x_{i1}\lambda_i^{(l)} \le \tilde{x}_{l1}, \quad \text{for } l = 1, 2, \ldots, 20,\\
&\sum_{i=1}^{20} x_{i2}\lambda_i^{(l)} \le \tilde{x}_{l2}, \quad \text{for } l = 1, 2, \ldots, 20,\\
&\sum_{i=1}^{n} y_{i1}\lambda_i^{(l)} \ge \tilde{y}_{l1}, \quad \text{for } l = 1, 2, \ldots, 20,\\
&\sum_{i=1}^{n} y_{i2}\lambda_i^{(l)} \ge \tilde{y}_{l2}, \quad \text{for } l = 1, 2, \ldots, 20,\\
&\sum_{i=1}^{n} y_{i3}\lambda_i^{(l)} \ge \tilde{y}_{l3}, \quad \text{for } l = 1, 2, \ldots, 20,\\
&\sum_{i=1}^{n} y_{i4}\lambda_i^{(l)} \ge \tilde{y}_{l4}, \quad \text{for } l = 1, 2, \ldots, 20,\\
&\sum_{i=1}^{n} y_{i5}\lambda_i^{(l)} \ge \tilde{y}_{l5}, \quad \text{for } l = 1, 2, \ldots, 20,\\
&\sum_{i=1}^{n} \tilde{y}_{i1} \le \sum_{i=1}^{n} y_{ik} + \tilde{D}_1,\\
&\sum_{i=1}^{n} \tilde{y}_{i2} \le \sum_{i=1}^{n} y_{ik} + \tilde{D}_2,\\
&\sum_{i=1}^{n} \tilde{y}_{i3} \le \sum_{i=1}^{n} y_{ik} + \tilde{D}_3,\\
&\sum_{i=1}^{n} \tilde{y}_{i4} \le \sum_{i=1}^{n} y_{ik} + \tilde{D}_4,\\
&\sum_{i=1}^{n} \tilde{y}_{i5} \le \sum_{i=1}^{n} y_{ik} + \tilde{D}_5,\\
&\lambda_i^{(l)} \ge 0, \quad \text{for } i = 1, 2, \ldots, 20, \quad \text{and for } l = 1, 2, \ldots, 20,\\
&\tilde{y}_{i1} \ge y_{i1}, \quad \text{for } i = 1, 2, \ldots, 20,\\
&\tilde{y}_{i2} \ge y_{i2}, \quad \text{for } i = 1, 2, \ldots, 20,\\
&\tilde{y}_{i3} \le y_{i3}, \quad \text{for } i = 1, 2, \ldots, 20,\\
&\tilde{y}_{i3} \ge 0, \quad \text{for } i = 1, 2, \ldots, 20,\\
&\tilde{y}_{i4} \ge y_{i4}, \quad \text{for } i = 1, 2, \ldots, 20,\\
&\tilde{y}_{i5} \ge y_{i5}, \quad \text{for } i = 1, 2, \ldots, 20.
\end{aligned}
\tag{9.41}
$$

Equation 9.41 has 540 decision variables or changing cells and 286 constraints, whereas the limit for the number of changing variables and constraints of the Standard Microsoft Excel Solver are at most 200 and 100, respectively. To deal with this limitation, we use Frontline Solver Products which has a limit of 2000 decision variables.

The second to eighth columns in Table 9.12 illustrate the planned input and output factors by Idea 2 and the ninth column represents the CCR scores for the restaurants. As can be seen, the CCR score is equal to 1 for each restaurant. The results in table 9.12 are suggested by the plan to maximize the total values of all output factors which are produced by all restaurants, and simultaneously to minimize the total values of all input factors which are consumed by all restaurants.

**Table 9.12** The results of applying Idea 2

| DMUs | $x_{i1}^{(2)*}$ | $x_{i2}^{(2)*}$ | $y_{i1}^{(2)*}$ | $y_{i2}^{(2)*}$ | $y_{i3}^{(2)*}$ | $y_{i4}^{(2)*}$ | $y_{i5}^{(2)*}$ | CCR |
|---|---|---|---|---|---|---|---|---|
| R01 | 3.520 | 2.178 | 2.471 | 2.460 | 1.049 | 3.310 | 1.041 | 1 |
| R02 | 3.713 | 2.296 | 2.607 | 2.520 | 1.248 | 3.488 | 1.096 | 1 |
| R03 | 3.157 | 1.946 | 2.219 | 2.251 | 0.994 | 2.955 | 0.928 | 1 |
| R04 | 3.484 | 2.053 | 2.480 | 2.297 | 1.244 | 3.316 | 0.957 | 1 |
| R05 | 3.974 | 2.417 | 2.804 | 2.784 | 1.420 | 3.744 | 1.145 | 1 |
| R06 | 4.473 | 2.679 | 3.170 | 2.950 | 1.303 | 4.261 | 1.260 | 1 |
| R07 | 3.723 | 2.117 | 2.675 | 2.453 | 1.037 | 3.523 | 0.969 | 1 |
| R08 | 3.566 | 2.031 | 2.562 | 2.342 | 0.890 | 3.290 | 0.930 | 1 |
| R09 | 3.119 | 1.891 | 2.203 | 2.194 | 0.957 | 2.889 | 0.895 | 1 |
| R10 | 4.122 | 2.512 | 2.907 | 2.961 | 1.246 | 3.936 | 1.191 | 1 |
| R11 | 3.771 | 2.132 | 2.714 | 2.608 | 1.248 | 3.516 | 0.973 | 1 |
| R12 | 4.016 | 2.259 | 2.894 | 2.760 | 1.156 | 3.857 | 1.027 | 1 |
| R13 | 4.207 | 2.563 | 2.967 | 2.794 | 0.965 | 4.116 | 1.215 | 1 |
| R14 | 3.312 | 1.979 | 2.349 | 2.340 | 0.989 | 3.113 | 0.929 | 1 |
| R15 | 3.040 | 1.822 | 2.153 | 2.191 | 0.970 | 2.841 | 0.857 | 1 |
| R16 | 3.393 | 1.930 | 2.438 | 2.368 | 1.181 | 3.227 | 0.883 | 1 |
| R17 | 3.845 | 2.184 | 2.764 | 2.692 | 1.271 | 3.669 | 0.999 | 1 |
| R18 | 3.300 | 2.009 | 2.328 | 2.355 | 1.107 | 3.021 | 0.952 | 1 |
| R19 | 3.131 | 1.893 | 2.213 | 2.240 | 0.936 | 2.899 | 0.894 | 1 |
| R20 | 3.187 | 1.926 | 2.252 | 2.281 | 0.860 | 2.937 | 0.910 | 1 |

In this application, both planning designs lead to a technical efficiency score equal to one for all newly expanded restaurants input–output plans, which determines the reasonability of both methods. Note that, the second input, shop size, is a non-controllable input in real-life (see Sect. 12.3). This means that the rental floor space of the restaurants cannot be changed in a common sense. To fit the real condition more suitably, one can basically apply the introduced two Ideas 1 and 2 into such application where the shop size input unchanged.

## 9.6 Conclusion

In this chapter, two approaches are introduced which allow making future production plans when demand changes can be predicted in a centralized decision-making situation. The first plan is to optimize the overall or average production performance in the whole organization after planning using CCR efficiency scores. The second plan is to maximize the total output productions and simultaneously minimize the total input consumptions in the entire organization. All the individual DMUs are supposed to be able to modify their output productions and input usages. The approaches are exemplified with a simple numerical example and a real world data set. The production possibility set is characterized to indicate the production plans that are technically feasible. After that, the set is estimated by the observed performances under consideration and supposed to be unchanged during a planning period. The planning results from this method for all observations, lie on the original empirical production frontier with a CCR score of 1.

## 9.7 Exercises

9.1. Are 'the original PPS generated by observed DMUs' and 'the PPS generated by the DMUs with planned data', by Ideas 1 and 2, the same? Why?

9.2. Improve the VBA procedures to create the new worksheets, set Solver, and run Eqs. 9.35 and 9.38 and CCR Envelopment to optimize the average production performance of all restaurants in Table 9.1 after planning using Idea 1.

9.3. Using variable returns to scale and given data in Table 9.1

  9.3.1. Apply Idea 1.
  9.3.2. Apply Idea 2.

# Chapter 10
# Context-Dependent DEA

## 10.1 Introduction

In this chapter, the context-dependent DEA is discussed. Since a product can appear attractive in comparison with a contextual of less attractive or unattractive alternatives, the performance of firms can be influenced by the context. For an example, twenty-three Tokyo public libraries are considered and a context-dependent DEA proposed by Chen et al. (2005) is discussed. The attractiveness of each library on a particular performance level in comparison with other libraries are measured. Libraries are classified on several empirical efficient frontiers, where each frontier is used to evaluate the attractiveness. The performance of the technically efficient libraries changes as the technically inefficient libraries change their performance. The context-dependent DEA also represents another view to differentiate the performance of efficient DMUs. When DMUs in a particular level are observed as having the same performance, the attractiveness measure lets us discriminate the "equal performance" based upon the third option or the same particular evaluation context. We also develop the VBA procedure to measure the attractiveness with just one click.

## 10.2 Context-Dependent

As discussed in the previous chapters, adding or excluding a technically inefficient DMU or a set of technically inefficient DMUs does not change the technical efficiencies of the existing DMUs or the estimated production frontier. The technical inefficiency scores change only if the estimated production frontier is changed. In other words, the performance of DMUs by DEA approach completely depends on the estimated production frontier, where the performance of technically efficient



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DMUs is not influenced by the performance of technically inefficient DMUs. Nevertheless, in some real-life applications, consumer choices can be influenced by the context as well as presence of technically inefficient DMUs. Thus, to obtain the attractiveness within the context-dependent, CCR is modified to a situation where the performance is calculated with respect to a specific evaluation context.

Suppose that there are $n$ DMUs $A_i$, for $i = 1, 2, \ldots, n$, in which each DMU has $m$ positive input factors $x_{ij}$, for $j = 1, 2, \ldots, m$, and $p$ positive output factors $y_{ik}$, for $k = 1, 2, \ldots, p$. Assume that $A_l$ is evaluated, for $l = 1, 2, \ldots, n$. The input oriented envelopment form of CCR is given by:

$$\begin{aligned} &\theta_l^* = \min\ \theta_l, \\ &\text{Subject to} \\ &\textstyle\sum_{i=1}^n x_{ij}\lambda_i \le x_{lj}\theta_l, \quad \text{for } j = 1, 2, \ldots, m, \\ &\textstyle\sum_{i=1}^n y_{ik}\lambda_i \ge y_{lk}, \quad \text{for } k = 1, 2, \ldots, p, \\ &\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n, \end{aligned} \tag{10.1}$$

After solving the CCR model, a non-empty set of technically efficient DMUs is obtained. Suppose that the observed set of DMUs is shown by $I^1 = \{A_i : i = 1, 2, \ldots, n\}$ and the set of DMUs with CCR scores of 1 from $I^1$ is shown by $L^1 = \{A_l \in I^1 : \theta_l^* = 1\}$. This first empirical production frontier generated by DMUs in $I^1$ provides an evaluation context to measure the attractiveness. The set $L^1$ is called 'Level 1' empirical production frontier.

Now, we exclude the DMUs with the CCR scores equal to 1 from the observed DMUs to find 'Level 2', the second empirical production frontier. In other words, the second set of DMUs is $I^2 = I^1 - L^1$, and the set $L^2$ is defined as the set of DMUs in $I^2$ with CCR scores equal to 1, that is, $L^2 = \{A_l \in I^2 : \theta_l^* = 1\}$.

Consequently, the set $I^{t+1}$ represents the $t + 1$th set of DMUs after excluding the set of $L^t$ from $I^t$, that is, $I^{t+1} = I^t - L^t$, where $L^t$ indicates 'Level $t$', the $t^{\text{th}}$ empirical production frontier. The generating of these levels is stopped when $I^t = \varnothing$. In other words, we have the following algorithm to identify the levels:

Step 1: Set $t = 1$, and $I^1$ as the set of observed DMUs.
Step 2: Evaluate the set $I^t$ by Eq. 10.1 to obtain Level $t$ technically efficient DMUs, or Level $t$ production frontier, $L^t$.
Step 2. Exclude DMUs in the set $L^t$ from $I^t$ to obtain $I^{t+1}$.
Step 3: If $I^{t+1} = \varnothing$, go to Step 5.
Step 4: Set $t = t + 1$ and go to Step 2.
Step 5: Stop the algorithm.

From the above illustrations, the evaluation contexts are found by subdividing a set of DMUs into several levels of empirical production frontiers. Each production frontier offers an evaluation context for measuring the attractiveness. For example, Level 2 production frontier, generated by $L^2$, is considered as the evaluation context for measuring the attractiveness of the DMUs in $L^1$. In addition, the presence, absence, or the shape of the Level 2 production frontier affects the attractiveness of DMUs on the level 1 production frontier.

The introduced nest-sets of DMUs have the following properties:

(i) $I^1 = \cup_{t=1}^{s} L^t$, where $L^t \cap L^{t'} = \emptyset$ for $t \neq t'$, where $L^{s+1} = \emptyset$.
(ii) The DMUs in $L^{t'}$ are dominated by the DMUs in $L^t$ or a linear combination of DMUs in $L^t$, where $t' \geq t$.
(iii) Each DMU in set $L^t$ is technically efficient with respect to the DMUs in set $L^{t'}$ for all $t' \geq t$.
(iv) Each DMU in set $L^t$ is technically inefficient with respect to the DMUs in set $L^{t'}$ for all $t' \leq t$.

As a result, each DMU belongs to one of the calculated levels. DMUs in $L^t$, that is, the DMUs which lie on level $t$ production frontier, are attractive to themselves. They are least attractive in comparison with DMUs in $L^{t-1}$, and most attractive in comparison with DMUs in $L^{t+1}$.

After classifying the levels and partitioning DMUs, we can measure the performance of each DMU with respect to each level of production frontier. Eq. 10.2 illustrates the input oriented CCR model to measure the attractiveness of DMU$_l$ in set $L^t$ based upon the empirical production frontier in Level $t'$, that is, the frontier which is made by DMUs in $L^{t'}$. In other words, $x_{lj}^t$ and $y_{lk}^t$ are the input and output factors of DMU$_l$ in set $L^t$, and $x_{ij}^{t'}$ and $y_{ik}^{t'}$ are the input and output factors of DMUs in set $L^{t'}$, for $j = 1, 2, \ldots, m$, for $k = 1, 2, \ldots, p$ and $i = 1, 2\ldots, n_{t'}$ where $n_{t'}$ is the number of DMUs in set $L^{t'}$.

$$\begin{aligned}
&\theta_l^{*t} = \min\ \theta_l^t,\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n_{t'}} x_{ij}^{t'}\lambda_i \leq x_{lj}^t\theta_l^t, \quad \text{for } j = 1,2,\ldots,m,\\
&\textstyle\sum_{i=1}^{n_{t'}} y_{ik}^{t'}\lambda_i \geq y_{lk}^t, \quad \text{for } k = 1,2,\ldots,p,\\
&\lambda_i \geq 0, \quad \text{for } i = 1,2,\ldots,n.,
\end{aligned} \tag{10.2}$$

If $t < t'$, then $\theta_l^{*t} > 1$, because in this case, DMUs in $L^{t'}$ are dominated by a linear combination of DMUs in $L^t$. It is obvious that when $t = t'$, then $\theta_l^{*t} = 1$, because the DMUs in $L^t$ are technically efficient and lie on the Level $t$ empirical production frontier.

**Definition 10.1** The score, $\theta_l^{*t}$, measured for DMU$_l$ from set $L^t$ by Eq. 10.2 with respect to the generated PPS by DMUs in set $L^{t'}$, is called the input oriented $t$-degree attractiveness of DMU$_l$ from level $t'$.

The bigger $\theta_l^{*t}$ represents the more attractive DMU$_l$, for $l = 1, 2, \ldots, n_t$ in comparison with DMUs in $L^t$. Eq. 10.2 determines the attractiveness score for DMU$_l$ when outputs are fixed at their current levels. Similarly, we can have the output-oriented version of the context-dependent DEA to determine the attractiveness score for DMU$_l$ when inputs are fixed at their current levels.

As explained in Chaps. 4 and 8, in constant retunes to scale technology, $\theta^* = 1/\varphi^*$, and the measured attractiveness in the input oriented model can easily be calculated in the output-oriented model as well. Nonetheless, if the variable returns to scale technology is considered, the related models should be solved separately. Eq. 10.3 represent the output-oriented CCR model to measure the $t$-degree attractiveness of $\mathrm{DMU}_l$ from set $L^t$ according to the generated PPS by DMUs in set $L^{t'}$.

$$
\begin{aligned}
&\varphi_l^{*t} = \max \varphi_l^t, \\
&\text{Subject to} \\
&\textstyle\sum_{i=1}^{n_{t'}} x_{ij}^{t'} \lambda_i \le x_{lj}^t, \quad \text{for } j = 1, 2, \ldots, m, \\
&\textstyle\sum_{i=1}^{n_{t'}} y_{ik}^{t'} \lambda_i \ge y_{lk}^t \varphi_l^t, \quad \text{for } k = 1, 2, \ldots, p, \\
&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n.,
\end{aligned}
\tag{10.3}
$$

**Definition 10.2** The $1/\varphi_l^{*t}$, where $\varphi_l^{*t}$ is the measured score for $\mathrm{DMU}_l$ from set $L^t$ by Eq. 10.2 with respect to the generated PPS by DMUs in set $L^{t'}$, is called *the output-oriented t-degree attractiveness* of $\mathrm{DMU}_l$ from level $t'$.

The larger the value of $1/\varphi_l^{*t}$, the more attractive $\mathrm{DMU}_l$ is, because this makes itself more distinctive from the evaluation context. We are also able to rank the DMUs in $L^t$ based upon their attractiveness scores and identify the best one. In the next section, an application of the context-dependent is discussed.

## 10.3 An Example of Twenty-Three Public Libraries in Tokyo

In this section, we apply the illustrated context-dependent model in the previous section to measure the attractiveness of 23 public libraries in Tokyo (Cooper et al. 2007).

Table 10.1 shows the data for the 23 public libraries in the Tokyo Metropolitan Area. The input factors are floor area in $1000m^2$, the number of books in 1000 unit measurement, the number of staff in 1000 unit measurement, and the population in 1000 unit measurement. The output factor are the number of registered residents in 1000 unit measurement and the number of borrowed books in 1000 unit measurement.

In order to measure the attractiveness of these libraries, we need to solve Eq. 10.2 several times to find the levels and after that measure the attractiveness score of each DMU based upon each level. In this example, more than 20 times Eq. 10.2 should be solved. The previous way of solving models requires programming in 20 different sheets, which is not user-friendly. Thus, we develop the VBA programming to solve all process with just one click. The following instructions illustrate how to measure the attractiveness of the 23 libraries in Table 10.1 by Microsoft Excel Solver with one click only.

**Table 10.1** Data of twenty three libraries in Tokyo

| DMUs | Area | Books | Staff | Population | Regist. | Borrow. |
|---|---|---|---|---|---|---|
| 1 | 2249 | 163,523 | 26 | 49,196 | 5561 | 105,321 |
| 2 | 4617 | 338,671 | 30 | 78,599 | 18,106 | 314,682 |
| 3 | 3873 | 281,655 | 51 | 176,381 | 16,498 | 542,349 |
| 4 | 5541 | 400,993 | 78 | 189,397 | 30,810 | 847,872 |
| 5 | 11,381 | 363,116 | 69 | 192,235 | 57,279 | 758,704 |
| 6 | 10,086 | 541,658 | 114 | 194,091 | 66,137 | 1,438,746 |
| 7 | 5434 | 508,141 | 61 | 228,535 | 35,295 | 839,597 |
| 8 | 7524 | 338,804 | 74 | 238,691 | 33,188 | 540,821 |
| 9 | 5077 | 511,467 | 84 | 267,385 | 65,391 | 1,562,274 |
| 10 | 7029 | 393,815 | 68 | 277,402 | 41,197 | 978,117 |
| 11 | 11,121 | 509,682 | 96 | 330,609 | 47,032 | 930,437 |
| 12 | 7072 | 527,457 | 92 | 332,609 | 56,064 | 1,345,185 |
| 13 | 9348 | 601,594 | 127 | 356,504 | 69,536 | 1,164,801 |
| 14 | 7781 | 528,799 | 96 | 365,844 | 37,467 | 1,348,588 |
| 15 | 6235 | 394,158 | 77 | 389,894 | 57,727 | 1,100,779 |
| 16 | 10,593 | 515,624 | 101 | 417,513 | 46,160 | 1,070,488 |
| 17 | 10,866 | 566,708 | 118 | 503,914 | 102,967 | 1,707,645 |
| 18 | 6500 | 467,617 | 74 | 517,318 | 47,236 | 1,223,026 |
| 19 | 11,469 | 768,484 | 103 | 537,746 | 84,510 | 2,299,694 |
| 20 | 10,868 | 669,996 | 107 | 590,601 | 69,576 | 1,901,465 |
| 21 | 10,717 | 844,949 | 120 | 622,550 | 89,401 | 1,909,698 |
| 22 | 19,716 | 1,258,981 | 242 | 660,164 | 97,941 | 3,055,193 |
| 23 | 10,888 | 1,148,863 | 202 | 808,369 | 191,166 | 4,096,300 |

1. Copy the 7 columns of Table 10.1 on an Excel sheet into cells A1:G24, as Fig. 10.1 depicts.
2. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.
3. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet.
4. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.
5. From the toolbar menu, click on 'Tools> References...>' and make sure 'Solver' is ticked, and then 'OK' (Fig. 10.2).
6. Inside of the 'Microsoft Visual Basic for Applications' window, write the following commands between 'Sub Button1_Click ()' and 'End Sub'.

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 1 | No. | Area | Books | Staff | Population | Regist. | Borrow. | | Lambdas |
| 2 | 1 | 2249 | 163523 | 26 | 49196 | 5561 | 105321 | | 0.000 |
| 3 | 2 | 4617 | 338671 | 30 | 78599 | 18106 | 314682 | | 0.000 |
| 4 | 3 | 3873 | 281655 | 51 | 176381 | 16498 | 542349 | | 0.000 |
| 5 | 4 | 5541 | 400993 | 78 | 189397 | 30810 | 847872 | | 0.000 |
| 6 | 5 | 11381 | 363116 | 69 | 192235 | 57279 | 758704 | | 0.031 |
| 7 | 6 | 10086 | 541658 | 114 | 194091 | 66137 | 1438746 | | 0.042 |
| 8 | 7 | 5434 | 508141 | 61 | 228535 | 35295 | 839597 | | 0.000 |
| 9 | 8 | 7524 | 338804 | 74 | 238691 | 33188 | 540821 | | 0.000 |
| 10 | 9 | 5077 | 511467 | 84 | 267385 | 65391 | 1562274 | | 0.000 |
| 11 | 10 | 7029 | 393815 | 68 | 277402 | 41197 | 978117 | | 0.000 |
| 12 | 11 | 11121 | 509682 | 96 | 330609 | 47032 | 930437 | | 0.000 |
| 13 | 12 | 7072 | 527457 | 92 | 332609 | 56064 | 1345185 | | 0.000 |
| 14 | 13 | 9348 | 601594 | 127 | 356504 | 69536 | 1164801 | | 0.000 |
| 15 | 14 | 7781 | 528799 | 96 | 365844 | 37467 | 1348588 | | 0.000 |
| 16 | 15 | 6235 | 394158 | 77 | 389894 | 57727 | 1100779 | | 0.000 |
| 17 | 16 | 10593 | 515624 | 101 | 417513 | 46160 | 1070488 | | 0.000 |
| 18 | 17 | 10866 | 566708 | 118 | 503914 | 102967 | 1707645 | | 0.000 |
| 19 | 18 | 6500 | 467617 | 74 | 517318 | 47236 | 1223026 | | 0.000 |
| 20 | 19 | 11469 | 768484 | 103 | 537746 | 84510 | 2299694 | | 0.000 |
| 21 | 20 | 10868 | 669996 | 107 | 590601 | 69576 | 1901465 | | 0.000 |
| 22 | 21 | 10717 | 844949 | 120 | 622550 | 89401 | 1909698 | | 0.000 |
| 23 | 22 | 19716 | 1258981 | 242 | 660164 | 97941 | 3055193 | | 0.000 |
| 24 | 23 | 10888 | 1148863 | 202 | 808369 | 191166 | 4096300 | | 0.005 |
| 25 | | | | | | | | | |
| 26 | Index | 1 | | | Theta | 0.372764 | | | |
| 27 | | | | | | | | | |
| 28 | | 838.347 | 40087.253 | 8.001 | 18338.51126 | 5561 | 105321 | | |
| 29 | | 838.347 | 60955.532 | 9.692 | 18338.51126 | 5561 | 105321 | | |
| 30 | | | | | | | | | |
| 31 | | | | | | | | | |

**Fig. 10.1** Copying data into an Excel sheet

```
Dim i, j, k, t, r As Integer
Dim Store(1 To 10), Level1() As Variant

Range("B26") = 1

Range("B28:G28").Formula = "=SUMPRODUCT(B2:B24,$I2:$I24)"
Range("B29:E29").Formula = "=INDEX(B2:B24,$B26)*$F26"
Range("F29:G29").Formula = "=INDEX(F2:F24,$B26)"

SolverReset
SolverAdd CellRef:="$B$28:$E$28", relation:=1, _
    Formulatext:="$B$29:$E$29"
```

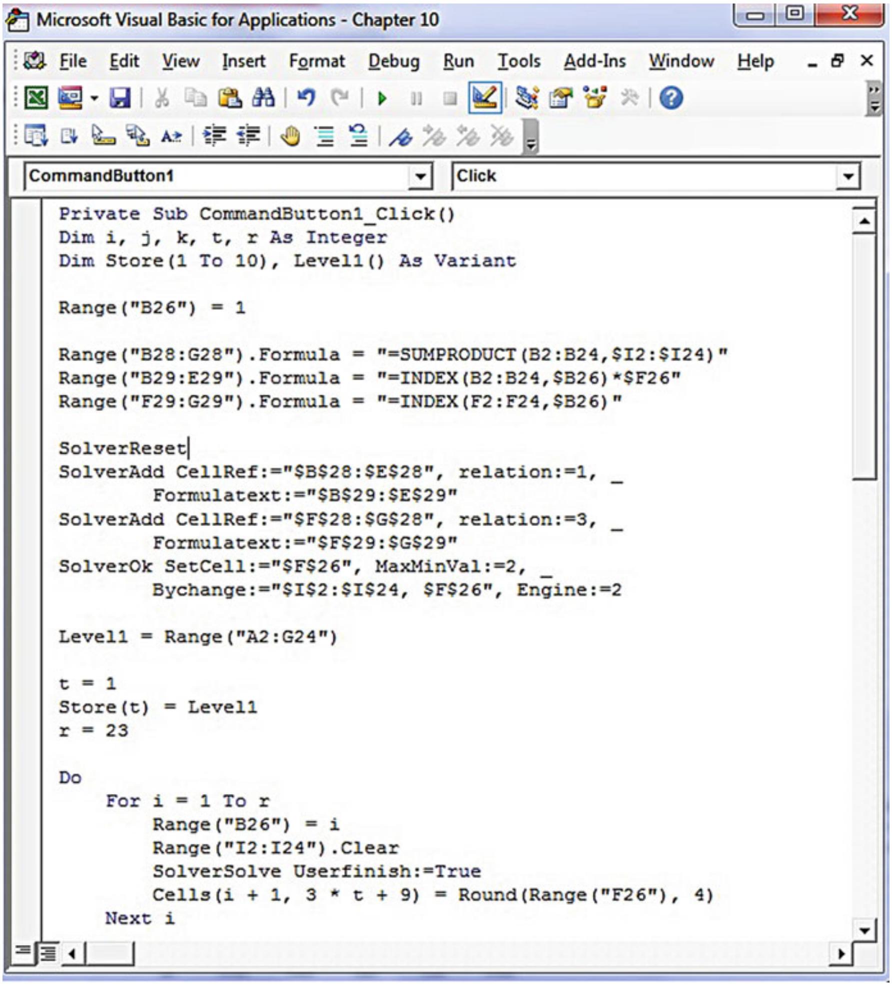



**Fig. 10.2** Setting VBA for context-dependent with one click

```
SolverAdd CellRef:="$F$28:$G$28", relation:=3, _
        Formulatext:="$F$29:$G$29"
SolverOk SetCell:="$F$26", MaxMinVal:=2, _
        Bychange:="$I$2:$I$24, $F$26", Engine:=2
Level1 = Range("A2:G24")

t = 1
Store(t) = Level1
r = 23

Do
   For i = 1 To r
```

```
        Range("B26") = i
        Range("I2:I24").Clear
        SolverSolve Userfinish:=True
        Cells(i + 1, 3 * t + 9) = Round(Range("F26"), 4)
    Next i

    j = 0
    Level1 = Store(t)

    For i = 1 To r
        If Cells(i + 1, 3 * t + 9) <> 1 Then
            j = j + 1
            For k = 1 To 7
                Level1(j, k) = Level1(i, k)
            Next k
        Else
            Cells(i + 1, 3 * t + 8) = Level1(i, 1)
        End If
    Next i

    For i = j + 1 To r
        For k = 1 To 7
            Level1(i, k) = ""
        Next k
    Next i

    r = j
    Range("A2:G24") = Level1
    Store(t + 1) = Level1
    t = t + 1

Loop Until r = 0

Range("B29:E29").Formula = "=B31*$F26"
Range("F29:G29").Formula = "=F31"

For s = 1 To t - 1
    Level1 = Store(1)
    For r = 1 To t - 1
        Range("A2:G24") = Store(r)
        j = 1
        For i = 1 To 23
            If Cells(i + 1, 3 * s + 8) <> "" Then
                For k = 1 To 7
                    Cells(31, k) = Level1(Cells(i + 1, 3 * s + 8), k)
                Next k
                SolverSolve Userfinish:=True
                Cells(j + 26, 7 * (s - 1) + r + 11) = Round(Range("F26"), 4)
                Cells(j + 26, 7 * (s - 1) + 11) = Cells(i + 1, 3 * s + 8)
                j = j + 1
            End If
        Next i
```

```
    Next r
Next s

Range("I2:I24").Clear
Range("A2:G24") = Store(1)
```

7. Close the ‘Microsoft Visual Basic for Applications’ window.
8. Click on the small rectangle which was automatically made on the Excel sheet and created by step 19. The results are represented to cells K2:AR34.

From the above procedure, DMUs are partitioned in cells K2:X24. The DMUs in Level 1 are shown in cells K2:K24, where the corresponding scores of DMUs in Level 1 are presented in cells L2:L24, as demonstrated in columns 2 and 3 of Table 10.2. Thus, $L^1$ is $\{5, 6, 9, 17, 19, 23\}$.

In order to find the DMUs in Level 2, the DMUs in Level 1 are eliminated from the observed DMUs. The DMUs in Level 2 are shown in cells N2:

**Table 10.2** Classifying DMUs and indicating the levels

| | Level 1 | | Level 2 | | Level 3 | | Level 4 | | Level 5 | |
|---|---|---|---|---|---|---|---|---|---|---|
| DMUs | No. | Score | No. | Score | No. | Score | No. | Score | No. | Score |
| 1 | | 0.373 | | 0.539 | | 0.732 | | 0.799 | 1 | 1.000 |
| 2 | | 0.792 | 2 | 1.000 | | 0.835 | 3 | 1.000 | | |
| 3 | | 0.573 | | 0.757 | 7 | 1.000 | 8 | 1.000 | | |
| 4 | | 0.719 | 4 | 1.000 | | 0.936 | 11 | 1.000 | | |
| 5 | 5 | 1.000 | | 0.934 | 10 | 1.000 | 16 | 1.000 | | |
| 6 | 6 | 1.000 | | 0.788 | | 0.952 | | | | |
| 7 | | 0.697 | | 0.955 | 14 | 1.000 | | | | |
| 8 | | 0.580 | | 0.813 | | 0.856 | | | | |
| 9 | 9 | 1.000 | 12 | 1.000 | | | | | | |
| 10 | | 0.705 | 13 | 1.000 | | | | | | |
| 11 | | 0.569 | | 0.973 | | | | | | |
| 12 | | 0.758 | 15 | 1.000 | | | | | | |
| 13 | | 0.867 | | 0.795 | | | | | | |
| 14 | | 0.722 | 18 | 1.000 | | | | | | |
| 15 | | 0.844 | 20 | 1.000 | | | | | | |
| 16 | | 0.582 | 21 | 1.000 | | | | | | |
| 17 | 17 | 1.000 | 22 | 1.000 | | | | | | |
| 18 | | 0.787 | | | | | | | | |
| 19 | 19 | 1.000 | | | | | | | | |
| 20 | | 0.849 | | | | | | | | |
| 21 | | 0.787 | | | | | | | | |
| 22 | | 0.785 | | | | | | | | |
| 23 | 23 | 1.000 | | | | | | | | |

N18, where the corresponding scores of DMUs in Level 1 are presented in cells O2:O18, as demonstrated in columns 4 and 5 of Table 10.2. The DMUs in Level 2 are $L^2 = \{2, 4, 12, 13, 15, 18, 20, 21, 22\}$.

The same step is followed for DMUs in Level 3 and 4. The last partition is a set of one DMU, that is, $L^5 = \{1\}$. The first library, $DMU_1$ is the least attractive library among the other libraries in Tokyo.

Tables 10.3, 10.4, 10.5, 10.6, 10.7, 10.8, 10.9, 10.10 and 10.11 demonstrate the attractiveness scores of each DMU in Level $t$ in comparison with the generated PPS by DMUs in Level $t'$. For example, columns 2–6 in Table 10.3 illustrate the attractiveness scores for each DMU in set $L^1$ according to generated PPS by DMUs in sets $L^1$, $L^2$, $L^3$, $L^4$ and $L^5$.

The ranks of DMUs in $L^1 = \{5, 6, 9, 17, 19, 23\}$ in comparison with the contexts in Levels 1–5 are also represented in Table 10.4. From the results in Tables 10.3 and 10.4, the library numbered 23, is the best library among the observed libraries, followed by libraries 9 and 6. As can be seen in Table 10.4, the ranking of DMUs is

**Table 10.3** Attractiveness for the libraries in level 1 vs. levels 1–5

| DMUs | Level 1 | Level 2 | Level 3 | Level 4 | Level 5 |
|---|---|---|---|---|---|
| 5 | 1.000 | 1.507 | 1.961 | 2.095 | 4.639 |
| 6 | 1.000 | 1.786 | 2.206 | 2.583 | 4.124 |
| 9 | 1.000 | 1.627 | 1.989 | 2.961 | 6.571 |
| 17 | 1.000 | 1.310 | 1.737 | 2.149 | 5.343 |
| 19 | 1.000 | 1.295 | 1.566 | 2.103 | 5.512 |
| 23 | 1.000 | 2.049 | 2.703 | 4.017 | 8.034 |

**Table 10.4** Attractiveness' ranks for the libraries in Table 10.3

| DMUs | Level 1 | Level 2 | Level 3 | Level 4 | Level 5 |
|---|---|---|---|---|---|
| 5 | 1 | 4 | 4 | 6 | 5 |
| 6 | 1 | 2 | 2 | 3 | 6 |
| 9 | 1 | 3 | 3 | 2 | 2 |
| 17 | 1 | 5 | 5 | 4 | 4 |
| 19 | 1 | 6 | 6 | 5 | 3 |
| 23 | 1 | 1 | 1 | 1 | 1 |

**Table 10.5** Attractiveness for the libraries in level 2 vs. levels 1–5

| DMUs | Level 1 | Level 2 | Level 3 | Level 4 | Level 5 |
|---|---|---|---|---|---|
| 2 | 0.792 | 1.000 | 1.492 | 1.645 | 2.822 |
| 4 | 0.719 | 1.000 | 1.217 | 1.496 | 3.283 |
| 12 | 0.758 | 1.000 | 1.290 | 1.823 | 4.062 |
| 13 | 0.867 | 1.000 | 1.296 | 1.688 | 3.399 |
| 15 | 0.844 | 1.000 | 1.548 | 2.109 | 4.336 |
| 18 | 0.787 | 1.000 | 1.239 | 1.678 | 4.080 |
| 20 | 0.849 | 1.000 | 1.247 | 1.676 | 4.406 |
| 21 | 0.787 | 1.000 | 1.335 | 1.908 | 3.929 |
| 22 | 0.785 | 1.000 | 1.257 | 1.516 | 3.768 |

**Table 10.6** Attractiveness' ranks for the libraries in Table 10.5

| DMUs | Level 1 | Level 2 | Level 3 | Level 4 | Level 5 |
|---|---|---|---|---|---|
| 2 | 4 | 1 | 2 | 7 | 9 |
| 4 | 9 | 1 | 9 | 9 | 8 |
| 12 | 8 | 1 | 5 | 3 | 4 |
| 13 | 1 | 1 | 4 | 4 | 7 |
| 15 | 3 | 1 | 1 | 1 | 2 |
| 18 | 6 | 1 | 8 | 5 | 3 |
| 20 | 2 | 1 | 7 | 6 | 1 |
| 21 | 5 | 1 | 3 | 2 | 5 |
| 22 | 7 | 1 | 6 | 8 | 6 |

**Table 10.7** Attractiveness for the libraries in level 3 vs. levels 1–5

| DMUs | Level 1 | Level 2 | Level 3 | Level 4 | Level 5 |
|---|---|---|---|---|---|
| 7 | 0.697 | 0.934 | 1.000 | 1.493 | 3.398 |
| 10 | 0.705 | 0.955 | 1.000 | 1.357 | 3.856 |
| 14 | 0.722 | 0.973 | 1.000 | 1.322 | 3.960 |

**Table 10.8** Attractiveness' ranks for the libraries in Table 10.7

| DMUs | Level 1 | Level 2 | Level 3 | Level 4 | Level 5 |
|---|---|---|---|---|---|
| 7 | 3 | 3 | 1 | 1 | 3 |
| 10 | 2 | 2 | 1 | 2 | 2 |
| 14 | 1 | 1 | 1 | 3 | 1 |

**Table 10.9** Attractiveness for the libraries in level 4 vs. levels 1–5

| DMUs | Level 1 | Level 2 | Level 3 | Level 4 | Level 5 |
|---|---|---|---|---|---|
| 3 | 0.573 | 0.757 | 0.835 | 1.000 | 2.990 |
| 8 | 0.580 | 0.788 | 0.936 | 1.000 | 2.880 |
| 11 | 0.569 | 0.813 | 0.952 | 1.000 | 2.834 |
| 16 | 0.582 | 0.795 | 0.856 | 1.000 | 3.223 |

**Table 10.10** Attractiveness' ranks for the libraries in Table 10.9

| DMUs | Level 1 | Level 2 | Level 3 | Level 4 | Level 5 |
|---|---|---|---|---|---|
| 3 | 3 | 4 | 4 | 1 | 2 |
| 8 | 2 | 3 | 2 | 1 | 3 |
| 11 | 4 | 1 | 1 | 1 | 4 |
| 16 | 1 | 2 | 3 | 1 | 1 |

**Table 10.11** Attractiveness for the libraries in level 1 vs. levels 1–5

| DMUs | Level 1 | Level 2 | Level 3 | Level 4 | Level 5 |
|---|---|---|---|---|---|
| 1 | 0.373 | 0.539 | 0.732 | 0.799 | 1.000 |

changed as the evaluation context is changed. In other words, the performance of a library is dependent on the evaluation context and background. For instance, when the evaluation context is $L^4$, the rank of $DMU_5$ is 6 among DMUs in $L^1$, whereas when the evaluation context is $L^2$, the rank of $DMU_5$ is 4 among DMUs in $L^1$.

The same illustration can be discussed for Tables 10.5–10.11. The scores in Table 10.11 represent how $DMU_1$ is not attractive with respect to a different evaluation context.

We can also calculate the average attractiveness scores of each DMUs among the Levels 1–5, as shown in Table 10.12. The rank of each DMU is also shown in the last column. As discussed, $DMU_{23}$ is the best library followed by $DMU_9$ and $DMU_6$. The least average attractiveness scores belong to $DMU_1$, $DMU_3$ and $DMU_{11}$. The interesting result to measure the average is that $DMU_{10}$ in Level 3 has a larger average attractiveness in comparison with $DMU_2$ in Level 2. As a results, the performance evaluation of a library can be different where the evaluation context is changed.

Note that in the above procedure, the command "relation:=1" means "<=", "relation:=2" means "=", and "relation:=3" means ">=". In addition, "MaxMinVal:=1" means maximization, and "MaxMinVal:=2" means minimization. These information are needed in Exercises 10.1 and 10.2.

**Table 10.12** Average attractiveness score

| No. | Average attractiveness score | Rank |
|---|---|---|
| 1 | 0.6886 | 23 |
| 2 | 1.5501 | 16 |
| 3 | 1.2310 | 22 |
| 4 | 1.5428 | 17 |
| 5 | 2.2401 | 6 |
| 6 | 2.3399 | 3 |
| 7 | 1.5042 | 18 |
| 8 | 1.2370 | 20 |
| 9 | 2.8295 | 2 |
| 10 | 1.5746 | 15 |
| 11 | 1.2336 | 21 |
| 12 | 1.7866 | 10 |
| 13 | 1.6500 | 13 |
| 14 | 1.5952 | 14 |
| 15 | 1.9674 | 7 |
| 16 | 1.2912 | 19 |
| 17 | 2.3078 | 4 |
| 18 | 1.7568 | 11 |
| 19 | 2.2951 | 5 |
| 20 | 1.8356 | 8 |
| 21 | 1.7919 | 9 |
| 22 | 1.6652 | 12 |
| 23 | 3.5605 | 1 |

## 10.4 Conclusion

In this chapter, a DEA-based context-dependent model is presented. The context-dependent captures situations where the performance of DMUs depends on the absence or presence of a third option, and is called attractiveness. The attractiveness of a set of public libraries in Tokyo is measured with respect to different measured evaluation contexts. Unlike the traditional first-level of empirical production frontier, different strata of empirical production frontiers are measured step by step, and considered as evaluation contexts. The context-dependent performance depends on both technically inefficient and efficient DMUs. Such change makes DEA more flexible and allows DEA to globally and locally detect better options. In particular, the attractiveness measure can be used to identify DMUs that have outstanding performance, and to differentiate the performance of DEA technically efficient DMUs.

## 10.5 Exercises

10.1 Update the procedure in Sect. 10.3 to measure the attractiveness scores of libraries by CCR output-oriented.

10.2 Use the BCC input oriented to partition the libraries, and after that use BCC output-oriented to measure the attractiveness of each library with respect to different evaluation context.
Note that "changing cells" in Solver should be changed for each level, when the variable returns to scale technology is considered.

10.3 Use the BCC output-oriented to partition the libraries, and after that use BCC input oriented to measure the attractiveness of each library with respect to different evaluation context.

10.4 Compare the results in Exercises 10.2 and 10.3. Are the outcomes the same? In other words, are the output degree attractiveness is the same as the input degree attractiveness for each libraries, when variable returns to scale technology is used?

10.5 Use SBM to partition the libraries, and to measure the attractiveness of each library with respect to different evaluation context.

10.6 Use Eq. 3.31 to partition the libraries, and to measure the attractiveness of each library with respect to different evaluation context.

# Chapter 11
# Efficiency Change Over Different Times

## 11.1 Introduction

In the literature of macroeconomics and the business economic press, there is a strong interest in the variation of efficiency over different times (Ray 2004). There are two methods to address this issue, such as, the Tornqvist and the Fisher productivity indexes. These two indexes use the price information of input and output factors without requiring the production technology of firms. Caves et al. (1982) introduced an index called the Malmquist efficiency index to construct a production frontier to represent the production technology of firms. Färe et al. (1992) combined ideas on the Farrell measurement and Caves et al. (1982) efficiency measurement to construct a Malmquist efficiency index directly from the input and output factors using DEA. The Malmquist efficiency index can be decomposed into two components. One component is to measure the technical change and the other component is to measure the frontier shift. In this chapter, we first illustrate the basic Malmquist index. After that, we illustrate the works of Chen and Ali (2004) to provide an extension to the Malmquist index by analyzing the above two components with an example of computer industry. Finally, a non-linear Malmquist index, proposed by Chen (2003), is discussed.

## 11.2 The Basic Theory on the Malmquist Index

In this section we illustrate the basic knowledge to measure the Malmquist efficiency index by a simple example. Suppose that there are five homogenous firms in which each firm has one single input and one single output factor over two different time periods, as data in Table 11.1 represents. For instance, $A^1$ represents the firm A in the first period and $A^2$ represents the same firm in the second period.



D. Khezrimotlagh, Y. Chen, *Decision Making and Performance Evaluation Using Data Envelopment Analysis*, International Series in Operations Research & Management Science 269, https://doi.org/10.1007/978-3-319-76345-3_11

**Table 11.1** Example of four DMUs

| Firm | Input | Output | Firm | Input | Output |
|---|---|---|---|---|---|
| $A^1$ | 5.00 | 4.00 | $A^2$ | 2.00 | 4.00 |
| $B^1$ | 7.00 | 2.00 | $B^2$ | 7.00 | 4.00 |
| $C^1$ | 3.00 | 2.00 | $C^2$ | 3.00 | 4.00 |
| $D^1$ | 5.00 | 2.00 | $D^2$ | 5.00 | 2.00 |

**Table 11.2** Efficiency of the four DMUs

| Firm | Efficiency | Firm | Output |
|---|---|---|---|
| $A^1$ | 4/5 | $A^2$ | 4/2 |
| $B^1$ | 2/7 | $B^2$ | 4/7 |
| $C^1$ | 2/3 | $C^2$ | 4/3 |
| $D^1$ | 2/5 | $D^2$ | 2/5 |

The efficiency of each firm is measured by Eq. 1.2, that is, output/input. Table 11.2 shows the efficiency of each firm corresponding to the first and the second periods.

Four different relative efficiency scores can be assigned for a firm. One of the relative efficiency score for firm A is obtained by comparing the efficiency score of firm A in the first period with the efficiency scores of all firms in the first period. We denote this relative efficiency score with $D^1_{A^1}$.

As discussed in Sect. 1.5 this relative efficiency score is measured by the ratio of the efficiency score of A in the first period to the maximum efficiency scores of firms A–D in the first period. In other words,

$$D^1_{A^1} = \frac{4/5}{\max\left\{\frac{4}{5},\frac{2}{7},\frac{2}{3},\frac{2}{5}\right\}} = \frac{4/5}{4/5} = 1. \tag{11.1}$$

This relative efficiency score can also be measured by sketching the straight line which passes the origin and $A^1$, as Fig. 11.1 illustrates the relative efficiency score of $A^1$ is 1, as $A^1$ lies on the frontier or since the ratio of $TA^1$ to $TA^1 = 1$. In addition, the relative efficiency score of $B^1$ is the ratio of HP to $HB^1$.

Another relative efficiency score for firm A is obtained by comparing the efficiency score of firm A in the first period with the efficiency scores of all firms in the second period. We denote this relative efficiency score with $D^2_{A^1}$. Similarly, this relative efficiency score is measured by the ratio of the efficiency score of A in the first period to the maximum efficiency scores of firms A-D in the second period. In other words,

$$D^2_{A^1} = \frac{4/5}{\max\left\{\frac{4}{2},\frac{4}{7},\frac{4}{3},\frac{2}{5}\right\}} = \frac{4/5}{4/2} = 0.4. \tag{11.2}$$

As Fig. 11.2 represents, the relative efficiency score of $A^1$ is measured by the ratio of TR to $TA^1$, and so on for the other firms B-D.

The third relative efficiency score for firm A is obtained by comparing the efficiency score of firm A in the second period with the efficiency scores of all

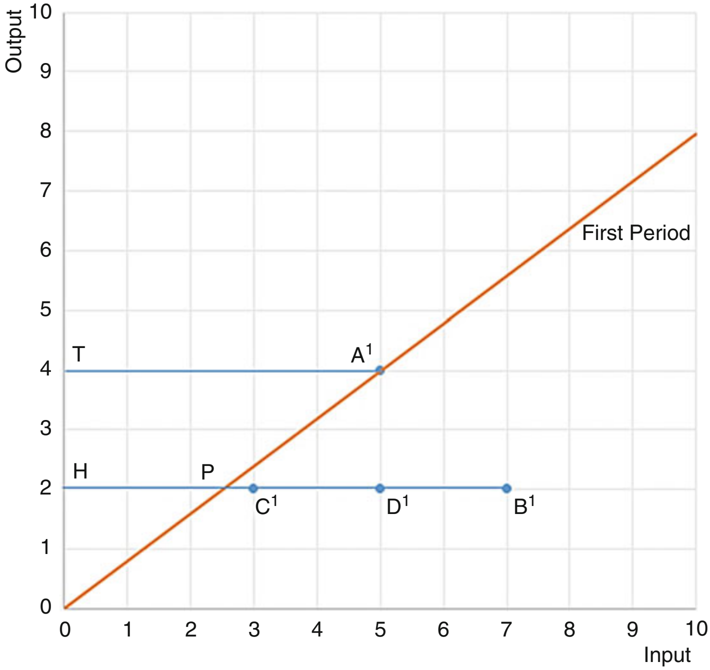



**Fig. 11.1** Measuring the first component

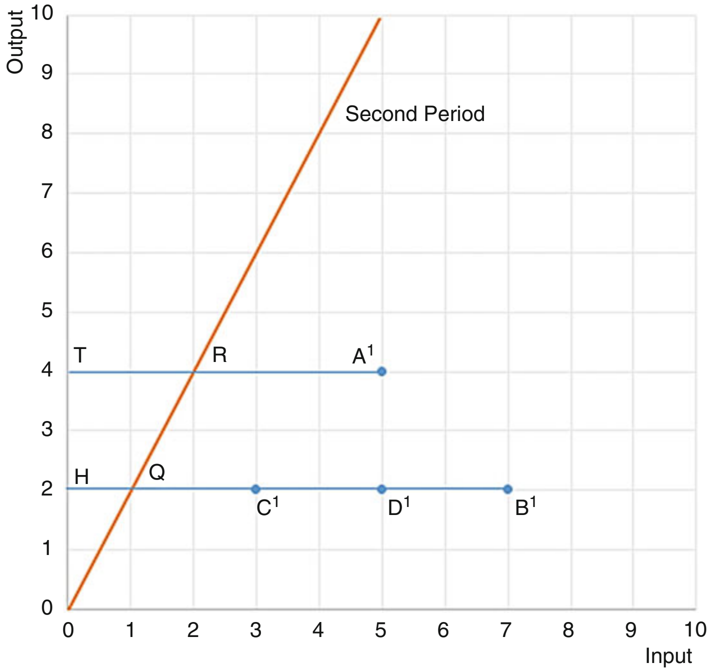



**Fig. 11.2** Measuring the second component

firms in the first period. We denote this relative efficiency score with $D^1_{A^2}$. Correspondingly, this relative efficiency score is measured by the ratio of the efficiency score of A in the second period to the maximum efficiency scores of firms A-D in the first period. In other words,

$$D^1_{A^2} = \frac{4/2}{\max\left\{\frac{4}{5}, \frac{2}{7}, \frac{2}{3}, \frac{2}{5}\right\}} = \frac{4/2}{4/5} = 2.5. \tag{11.3}$$

Figure 11.3 illustrates the frontier generated by firms $A^1$-$D^1$. The firms $B^2$ and $D^2$ are under the line which passes the origin and $A^1$ and the firms $A^2$ and $C^2$ are above the line. The relative efficiency scores of $A^2$ and $C^2$, which are measured by the ratio of $TA^1$ to $TA^2$ and $TA^1$ to $TC^2$, respectively. Both of these scores are greater than 1, as they are above the frontier. Similarly, the relative efficiency scores of $B^2$ and $D^2$ are less than 1 and they are measured by the ratio of $TA^1$ to $TB^2$ and HP to $HD^2$, respectively.

The last relative efficiency score for firm A is obtained by comparing the efficiency score of firm A in the second period with the efficiency scores of all firms in the second period, as Fig. 11.4 depicts. We denote this relative efficiency score with $D^2_{A^2}$.

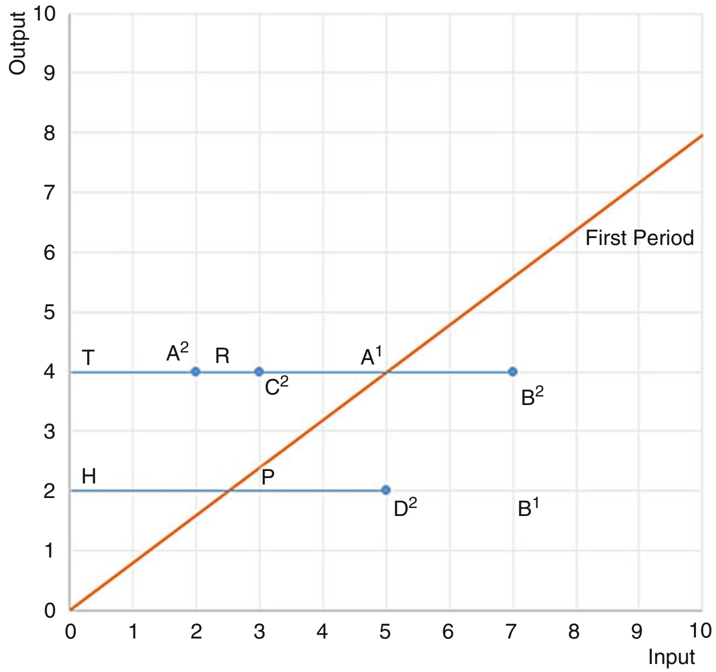



**Fig. 11.3** Measuring the third component

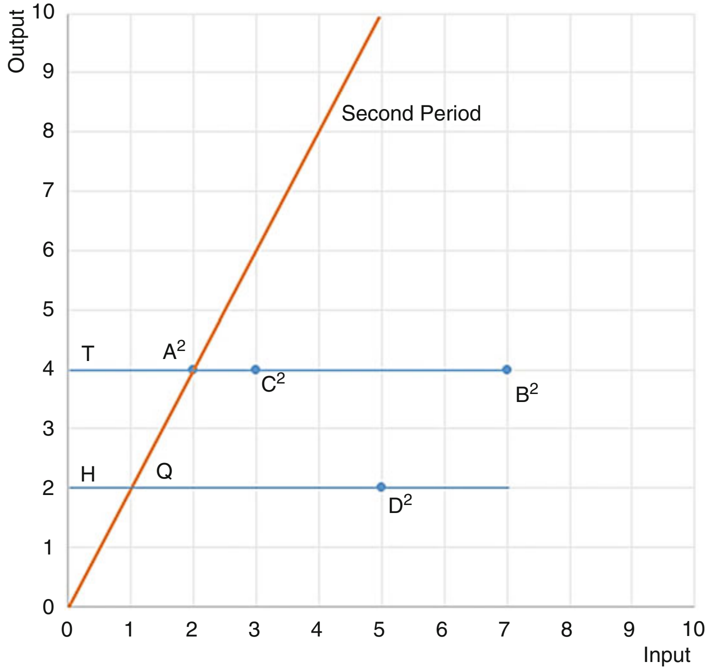



**Fig. 11.4** Measuring the fourth component

**Table 11.3** The measured four components

| Firm | $D_1^1$ | $D_1^2$ | $D_2^1$ | $D_2^2$ |
|---|---|---|---|---|
| A | 1.00 | 0.40 | 2.50 | 1.00 |
| B | 0.36 | 0.14 | 0.71 | 0.29 |
| C | 0.83 | 0.33 | 1.67 | 0.67 |
| D | 0.50 | 0.20 | 0.50 | 0.20 |

This relative efficiency score is measured by the ratio of the efficiency score of A in the second period to the maximum efficiency scores of firms A-D in the second period. In other words,

$$D_{A^2}^2 = \frac{4/2}{\max\left\{\frac{4}{2}, \frac{4}{7}, \frac{4}{3}, \frac{2}{5}\right\}} = \frac{4/2}{4/2} = 1. \tag{11.4}$$

Table 11.3 represents the relative efficiency scores of firms A-D regarding each category and time period.

Here $D_2^1$ denotes the efficiency score of firms in the second category related to the first period. For instance, $D_2^1$ for firm A refers to Eq. 11.3, that is, $D_{A^2}^1$.

The Malmquist efficiency index is defined as follows

**Table 11.4** The Malmquist efficiency index of firms A-D

| Firm | *M* |
|---|---|
| A | 2.50 |
| B | 2.00 |
| C | 2.00 |
| D | 1.00 |

$$M_A = \sqrt{\frac{D^1_{A^2} \times D^2_{A^2}}{D^1_{A^1} \times D^2_{A^1}}}. \tag{11.5}$$

Equation 11.5 demonstrates the geometric mean of $D^1_{A^2}/D^1_{A^1}$ and $D^2_{A^2}/D^2_{A^1}$. Table 11.4 shows the DEA Malmquist efficiency index for firms A-D.

The Malmquist efficiency index in Eq. 11.5 can also be addressed using the CCR Envelopment model. Note that CCR-efficiency radially calculates the technical efficiency scores of firms which are the same as the efficiency scores of firms where the firms have only one single input and one single output factors. These four different CCR Envelopment models given by Models 11.6–11.9.

The first model, Model 11.6 measures the CCR efficiency score of $A^1$ in the generated PPS by $A^1$- $D^1$. Consequently, Models 11.7–11.9 measure the CCR efficiency scores of $A^1$ in the generated PPS by $A^2$- $D^2$, the CCR efficiency scores of $A^2$ in the generated PPS by $A^1$- $D^1$, and the CCR efficiency scores of $A^2$ in the generated PPS by $A^2$- $D^2$.

$$\begin{aligned} & D^1_{A^1} = \min\ \theta, \\ & \text{Subject to} \\ & x_{A^1}\lambda_1 + x_{B^1}\lambda_2 + x_{C^1}\lambda_3 + x_{D^1}\lambda_4 \le x_{A^1}\theta, \\ & y_{A^1}\lambda_1 + y_{B^1}\lambda_2 + y_{C^1}\lambda_3 + y_{D^1}\lambda_4 \ge y_{A^1}, \\ & \lambda_i \ge 0, \quad \text{for } i = 1, 2, 3, 4 \end{aligned} \tag{11.6}$$

$$\begin{aligned} & D^2_{A^1} = \min\ \theta, \\ & \text{Subject to} \\ & x_{A^2}\lambda_1 + x_{B^2}\lambda_2 + x_{C^2}\lambda_3 + x_{D^2}\lambda_4 \le x_{A^1}\theta, \\ & y_{A^2}\lambda_1 + y_{B^2}\lambda_2 + y_{C^2}\lambda_3 + y_{D^2}\lambda_4 \ge y_{A^1}, \\ & \lambda_i \ge 0, \quad \text{for } i = 1, 2, 3, 4. \end{aligned} \tag{11.7}$$

$$\begin{aligned} & D^1_{A^2} = \min\ \theta, \\ & \text{Subject to} \\ & x_{A^1}\lambda_1 + x_{B^1}\lambda_2 + x_{C^1}\lambda_3 + x_{D^1}\lambda_4 \le x_{A^2}\theta, \\ & y_{A^1}\lambda_1 + y_{B^1}\lambda_2 + y_{C^1}\lambda_3 + y_{D^1}\lambda_4 \ge y_{A^2}, \\ & \lambda_i \ge 0, \quad \text{for } i = 1, 2, 3, 4 \end{aligned} \tag{11.8}$$

**Fig. 11.5** Coping data in excel

| | A | B | C |
|---|---|---|---|
| 1 | Firm | Input | Output |
| 2 | A[1] | 5.00 | 4.00 |
| 3 | B[1] | 7.00 | 2.00 |
| 4 | C[1] | 3.00 | 2.00 |
| 5 | D[1] | 5.00 | 2.00 |
| 6 | A[2] | 2.00 | 4.00 |
| 7 | B[2] | 7.00 | 4.00 |
| 8 | C[2] | 3.00 | 4.00 |
| 9 | D[2] | 5.00 | 2.00 |

$$
\begin{aligned}
&D_{A^2}^2 = \min\, \theta, \\
&\text{Subject to} \\
&x_{A^2}\lambda_1 + x_{B^2}\lambda_2 + x_{C^2}\lambda_3 + x_{D^2}\lambda_4 \le x_{A^2}\theta, \\
&y_{A^2}\lambda_1 + y_{B^2}\lambda_2 + y_{C^2}\lambda_3 + y_{D^2}\lambda_4 \ge y_{A^2}, \\
&\lambda_i \ge 0, \quad \text{for } i = 1, 2, 3, 4
\end{aligned}
\tag{11.9}
$$

We now formulize an Excel sheet to solve the above four models with one single click, as the following instructions illustrate.

1. Copy data of Table 7.1 on an Excel sheet into cells A1:C9, as Fig. 11.5 illustrates.
2. Label D1 as ‘Lambdas’, E1 as ‘Firm’, F1 as ‘$D_1^1$’, G1 as ‘$D_1^2$’, H1 as ‘$D_2^1$’, I1 as ‘$D_2^2$’, J1 as ‘Malmquist Index’, A11 as ‘Index of Firm’, D11 as ‘Theta’, A13 as Models, B13-C13 as ‘Input Constraints’, E13-F13 as ‘Output Constraints’, A19 as ‘Index of Model’, and A21 as ‘Selected Model’ (Fig. 11.6).
3. Assign number 1 to B11 and B19.
4. Assign the following command into B14 and B16,

   ‘=Sumproduct(B2:B5,D2:D5)’.

5. Assign the following command into B15 and B17,

   ‘=Sumproduct(B6:B9,D6:D9)’.

6. Assign the following command into C14 and C15,

   ‘=Index(B2:B5,B11)*E11’.

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Firm | Input | Output | Lambdas | Firm | $D_1^1$ | $D_1^2$ | $D_2^1$ | $D_2^2$ | M |
| 2 | $A^1$ | 5.00 | 4.00 | 0.00 | A | 1.00 | 0.40 | 2.50 | 1.00 | 2.50 |
| 3 | $B^1$ | 7.00 | 2.00 | 0.00 | B | 0.36 | 0.14 | 0.71 | 0.29 | 2.00 |
| 4 | $C^1$ | 3.00 | 2.00 | 0.00 | C | 0.83 | 0.33 | 1.67 | 0.67 | 2.00 |
| 5 | $D^1$ | 5.00 | 2.00 | 0.00 | D | 0.50 | 0.20 | 0.50 | 0.20 | 1.00 |
| 6 | $A^2$ | 2.00 | 4.00 | 0.50 | | | | | | |
| 7 | $B^2$ | 7.00 | 4.00 | 0.00 | | | | | | |
| 8 | $C^2$ | 3.00 | 4.00 | 0.00 | | | Run | | | |
| 9 | $D^2$ | 5.00 | 2.00 | 0.00 | | | | | | |
| 10 | | | | | | | | | | |
| 11 | Index Firm | 4 | | Theta | 0.2 | | | | | |
| 12 | | | | | | | | | | |
| 13 | Models | Input Constraints | | | Output Constraints | | | | | |
| 14 | 1 | 0 | 1 | | 0 | 2 | | | | |
| 15 | 2 | 1 | 1 | | 2 | 2 | | | | |
| 16 | 3 | 0 | 1 | | 0 | 2 | | | | |
| 17 | 4 | 1 | 1 | | 2 | 2 | | | | |
| 18 | | | | | | | | | | |
| 19 | Index Model | 4 | | | | | | | | |
| 20 | | | | | | | | | | |
| 21 | Selected Model | 1 | 1 | | 2 | 2 | | | | |

**Fig. 11.6** Setting Excel to solve Eqs. 11.6, 11.7, 11.8 and 11.9

7. Assign the following command into C16 and C17,

   '=Index(B6:B9,B11)*E11'.

8. Assign the following command into E14 and E16,

   '=Sumproduct(C2:C5,D2:D5)'.

9. Assign the following command into E15 and E17,

   '=Sumproduct(C6:C9,D6:D9)'.

10. Assign the following command into F14 and F15,

    '=Index(C2:C5,B11)'.

11. Assign the following command into F16 and F17,

    '=Index(C6:C9,B11)'.

12. Assign the following command into B21,

    '=Index(B14:B17,$B19)'.

13. Copy B21 and paste it to C21, E21 and F21.
14. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 11.7 illustrates.

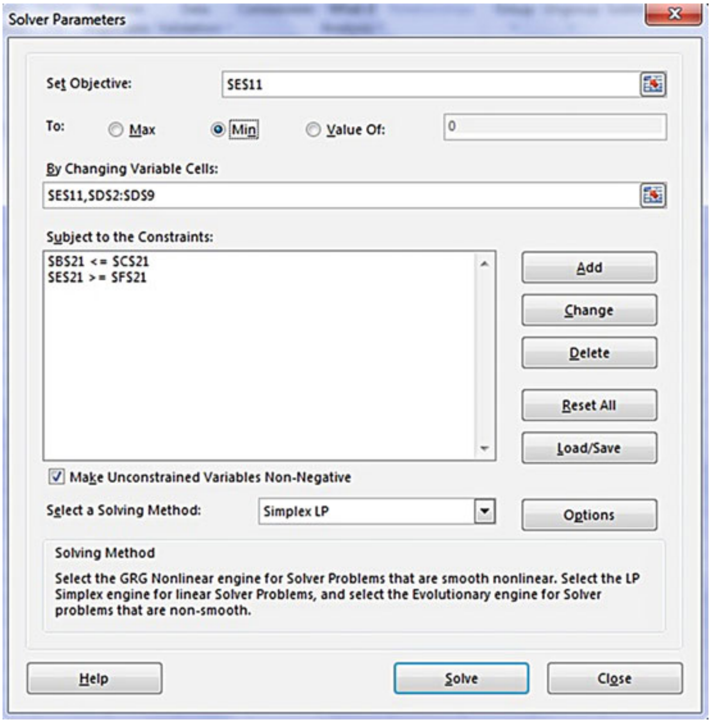



**Fig. 11.7** Setting solver to solve Eqs. 11.6, 11.7, 11.8 and 11.9

15. Assign ‘E11’ into ‘Set Objective’ and choose ‘Min’.
16. Assign ‘E11, D2:D9’ into ‘By Changing Variable Cells’.
17. Click on ‘Add’ and assign ‘B21’ into ‘Cell Reference’, then select ‘<=’, and assign ‘C21’ into ‘Constraint’.
18. Click on ‘Add’ and assign ‘E21’ into ‘Cell Reference’, then select ‘>=’ and assign ‘F21’ into ‘Constraint’. Then click on ‘OK’.
19. Tick ‘Make Unconstrained Variables Non-Negative’.
20. Choose ‘Simplex LP’ from ‘Select a Solving Method’.
21. Click on ‘Solve’.
22. From ‘Developer’ in the toolbar menu, click on the ‘Insert’ icon to open the ‘Form Control’ window.
23. Click on the first icon, ‘Button (Form Control)’, and then click on a place on the Excel sheet.
24. In the opened window with the title ‘Assign Macro’, click on ‘New’. So, the ‘Microsoft Visual Basic for Applications’ window is opened.
25. Inside of the ‘Microsoft Visual Basic for Applications’ window, write the following commands between ‘Sub Button1_Click ()’ and ‘End Sub’, as Fig. 11.8 shows.

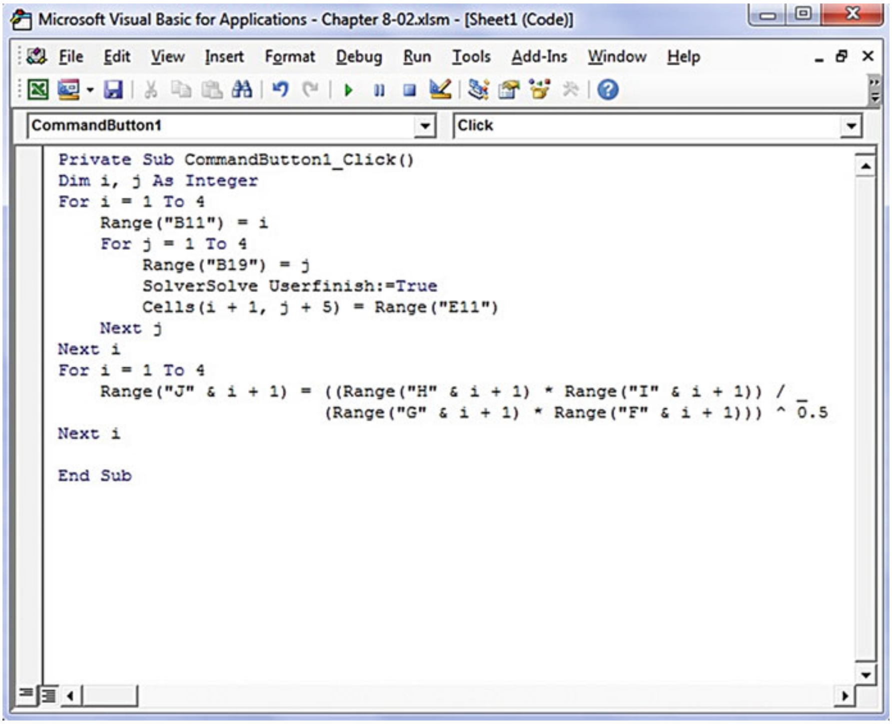



**Fig. 11.8** Setting VBA to solve Eqs. 11.6, 11.7, 11.8 and 11.9

```
Dim i, j As Integer
For i = 1 To 4
    Range("B11") = i
    For j = 1 To 4
        Range("A19") = j
        SolverSolve Userfinish:=True
        Cells(i + 1, j + 5) = Range("E11")
    Next j
Next i
For i = 1 To 4
Range("J" & i + 1) = ((Range("G" & i + 1) * Range("I" & i + 1)) / (Range("F" & i + 1)
* Range("H" & i + 1))) ^ 0.5
Next i
```

26. Close the ‘Microsoft Visual Basic for Applications’ window.
27. Click on the small rectangle which was automatically made on the Excel sheet and created by step 23. The results are represented to cells F2:J5.

In general case, the four CCR Envelopment models, where $\mathrm{DMU}_l\,(l = 1, 2, \ldots, n)$ is evaluated, are given by:

$$\begin{aligned}&D_{l^t}^{t} = \min \theta_l,\\&\text{Subject to}\\&\textstyle\sum_{i=1}^{n} x_{ij}^{t}\lambda_i \le x_{lj}^{t}\theta_l, \quad \text{for } j = 1, 2, \ldots, m,\\&\textstyle\sum_{i=1}^{n} y_{ik}^{t}\lambda_i \ge y_{lk}^{t}, \quad \text{for } k = 1, 2, \ldots, p,\\&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n.\end{aligned} \tag{11.10}$$

$$\begin{aligned}&D_{l^t}^{t+1} = \min \theta_l,\\&\text{Subject to}\\&\textstyle\sum_{i=1}^{n} x_{ij}^{t+1}\lambda_i \le x_{lj}^{t}\theta_l, \quad \text{for } j = 1, 2, \ldots, m,\\&\textstyle\sum_{i=1}^{n} y_{ik}^{t+1}\lambda_i \ge y_{lk}^{t}, \quad \text{for } k = 1, 2, \ldots, p,\\&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n,\end{aligned} \tag{11.11}$$

$$\begin{aligned}&D_{l^{t+1}}^{t} = \min \theta_l,\\&\text{Subject to}\\&\textstyle\sum_{i=1}^{n} x_{ij}^{t}\lambda_i \le x_{lj}^{t+1}\theta_l, \quad \text{for } j = 1, 2, \ldots, m\\&\textstyle\sum_{i=1}^{n} y_{ik}^{t}\lambda_i \ge y_{lk}^{t+1}, \quad \text{for } k = 1, 2, \ldots, p,\\&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n.\end{aligned} \tag{11.12}$$

$$\begin{aligned}&D_{l^{t+1}}^{t+1} = \min \theta_l,\\&\text{Subject to}\\&\textstyle\sum_{i=1}^{n} x_{ij}^{t+1}\lambda_i \le x_{lj}^{t+1}\theta_l, \quad \text{for } j = 1, 2, \ldots, m,\\&\textstyle\sum_{i=1}^{n} y_{ik}^{t+1}\lambda_i \ge y_{lk}^{t+1}, \quad \text{for } k = 1, 2, \ldots, p,\\&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n.\end{aligned} \tag{11.13}$$

In models 11.10–11.13, the index $t$ shows the first time period and $t + 1$ shows the second time period. The Malmquist CCR-Efficiency index is also given by

$$M_l = \sqrt{\frac{D_{l^{t+1}}^{t} \times D_{l^{t+1}}^{t+1}}{D_{l^t}^{t} \times D_{l^t}^{t+1}}}. \tag{11.14}$$

Färe et al. (1992) defined that if $M_l > 1$ the efficiency gain; if $M_l < 1$ indicates efficiency loss; and if $M_l = 1$ means no change in efficiency from the first time $t$ to the second time $t + 1$.

## 11.3 The Component of the Malmquist Index

Färe et al. (1992) decomposed the Malmquist efficiency index into two different components as follows:

$$M_l = \left(\frac{D_{l^{t+1}}^{t+1}}{D_{l^t}^{t}}\right)\left(\frac{D_{l^t}^{t} \times D_{l^{t+1}}^{t}}{D_{l^t}^{t+1} \times D_{l^{t+1}}^{t+1}}\right)^{1/2}. \tag{11.15}$$

The first fraction, $D_{l^{t+1}}^{t+1}/D_{l^t}^{t}$, is the component that measures the change in the technical efficiencies and the second component measures the production technology. If the value of the second component is greater (smaller) than 1, this indicates that we have a positive (negative) shift or a technical progress. If the value of the second component is equal to 1, this means that we have no shift in production technology.

In this section, the two components are examined to reveal sources and designs of efficiency change that are hidden by the aggregated nature of the discussed Malmquist index. Indeed, it is discussed that more information can be obtained from each of the Malmquist components.

Note that, when the prices (weights) of input and output factors are available, the allocative efficiency in the characterization of *strategy shift* as well as scale efficiency should also be considered, because a better *strategy* choice of a firm reflects considerations of all technical, scale and allocative efficiencies.

Let's call the first component in the Malmquist efficiency index in Eq. 11.15 the Technical Efficiency Change (TEC) for $\text{DMU}_l$ from time $t$ to $t+1$, and denoted by $TEC_l$, as shown in Eq. 11.16.

$$TEC_l = \frac{D_{l^{t+1}}^{t+1}}{D_{l^t}^{t}} \tag{11.16}$$

It is obvious that $TEC_l$ can be greater than 1, less than 1 or equal to 1. In other words,

$$\frac{D_{l^{t+1}}^{t+1}}{D_{l^t}^{t}} > 1 \text{ or } \frac{D_{l^{t+1}}^{t+1}}{D_{l^t}^{t}} < 1 \text{ or } \frac{D_{l^{t+1}}^{t+1}}{D_{l^t}^{t}} = 1 \tag{11.17}$$

When the value $TEC_l$ is greater than 1, this means that the radial distance of $\text{DMU}_l$ in time $t+1$ to the production frontier in time $t+1$ is closer than the radial distance of $\text{DMU}_l$ in time $t$ to the production frontier in time $t$. When the value $TEC_l$ is less than 1, this means that the radial distance of $\text{DMU}_l$ in time $t+1$ to the production frontier in time $t+1$ is greater than the radial distance of $\text{DMU}_l$ in time $t$ to the production frontier in time $t$. When the value $TEC_l$ is equal to 1, this means that $\text{DMU}_l$ in time $t+1$ is as close to the production frontier in time $t+1$ as $\text{DMU}_l$ in time $t$ to the production frontier in time $t$. Of course, it is a very rare situation to have $TEC_l = 1$, except when $\text{DMU}_l$ is equally efficient in both time $t$ and $t+1$.

We also call the second component in the Malmquist efficiency index in Eq. 11.15 as the Efficiency Frontier Change (EFC) for $\text{DMU}_l$ from time $t$ to $t+1$, as shown in Eq. 11.18.

$$EFC_l = \left( \frac{D_{l^t}^t}{D_{l^t}^{t+1}} \times \frac{D_{l^{t+1}}^t}{D_{l^{t+1}}^{t+1}} \right)^{1/2}. \tag{11.18}$$

As a result, the Malmquist efficiency index is the $TEC_l$ index times to the $EFC_l$ index, that is,

$$M_l = TEC_l \times EFC_l. \tag{11.19}$$

As can be seen, the Malmquist efficiency index only shows the average of efficiency change, which is oversimplified or over-aggregated.

The $EFC_l$ index can also be greater than 1, less than 1 or equal to 1. When the $EFC_l$ value is greater than 1, this means a positive shift in $\text{DMU}_l$ or a technical progress in $\text{DMU}_l$. If the $EFC_l$ value is less than 1, this means a negative shift in $\text{DMU}_l$ or a technical regress in $\text{DMU}_l$. There is no shift in production technology if the $EFC_l$ value is equal to 1.

The $EFC_l$ index can be described as an average aggregated change in production technology of $\text{DMU}_l$ from time period $t$ to $t + 1$. It can also be decomposed into two different fractions, and each fraction can also be greater than 1, less than 1 or equal to 1, as Eqs. 11.20 and 11.21 represent.

$$\frac{D_{l^t}^t}{D_{l^t}^{t+1}} > 1 \quad \text{or} \quad \frac{D_{l^t}^t}{D_{l^t}^{t+1}} < 1 \quad \text{or} \quad \frac{D_{l^t}^t}{D_{l^t}^{t+1}} = 1 \tag{11.20}$$

$$\frac{D_{l^{t+1}}^t}{D_{l^{t+1}}^{t+1}} > 1 \quad \text{or} \quad \frac{D_{l^{t+1}}^t}{D_{l^{t+1}}^{t+1}} < 1 \quad \text{or} \quad \frac{D_{l^{t+1}}^t}{D_{l^{t+1}}^{t+1}} = 1 \tag{11.21}$$

Since the frontier from time to time can have a downward shift in one region and an upward shift in another, the average frontier shift index, $EFC_l$, oversimplifies or over-aggregates the frontier shift. This can lead to the omission of some very important managerial information. To describe this issue, suppose that we have three DMUs, $A_1^t$, $A_2^t$ and $A_3^t$ in which each $A_i^t$ $(i = 1, 2, 3)$ has two input factors at time period $t$, as Fig. 11.9 shows.

As depicted in Fig. 11.10, let's select $A_1^t$ and assume that the production frontier from time period $t$ is shifted to time period $t + 1$.

Figure 11.11 illustrates three different locations that DMU $A_1$ may have in time period $t + 1$, such as, $A_1^{t+1}$, $A_2^{t+1}$ or $A_3^{t+1}$.

For example, assume that $A_1^t$ in time period $t$ will have a location at $A_3^{t+1}$ in time period $t + 1$. The technical efficiencies of $A_1^t$ and $A_3^{t+1}$ in time $t$ and $t + 1$ are calculated as follows:

$$D_{A_1^t}^t = \frac{OP}{OA_1^t}, \tag{11.22}$$

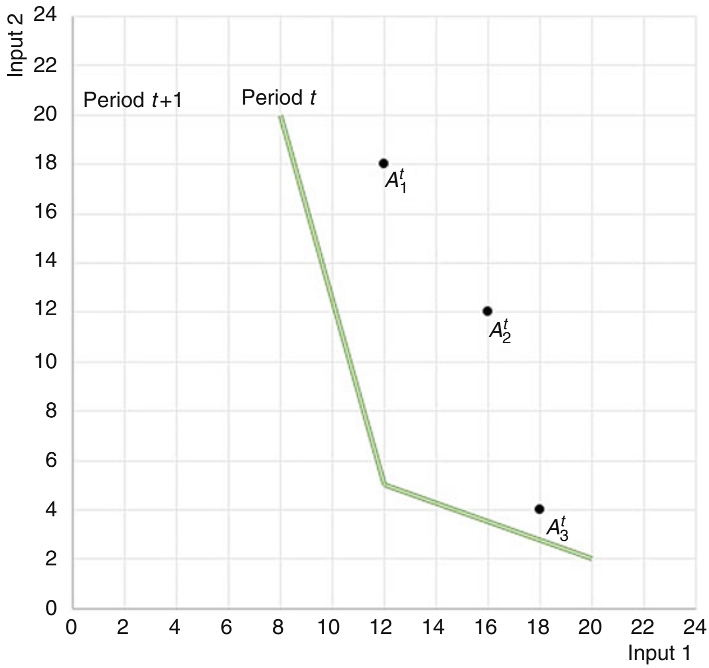



**Fig. 11.9** Example of three DMUs

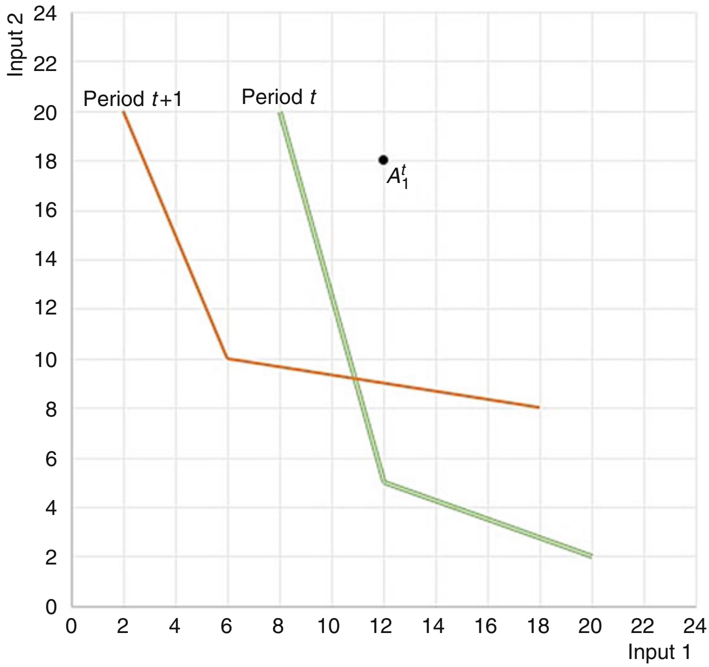



**Fig. 11.10** Frontiers in two period time

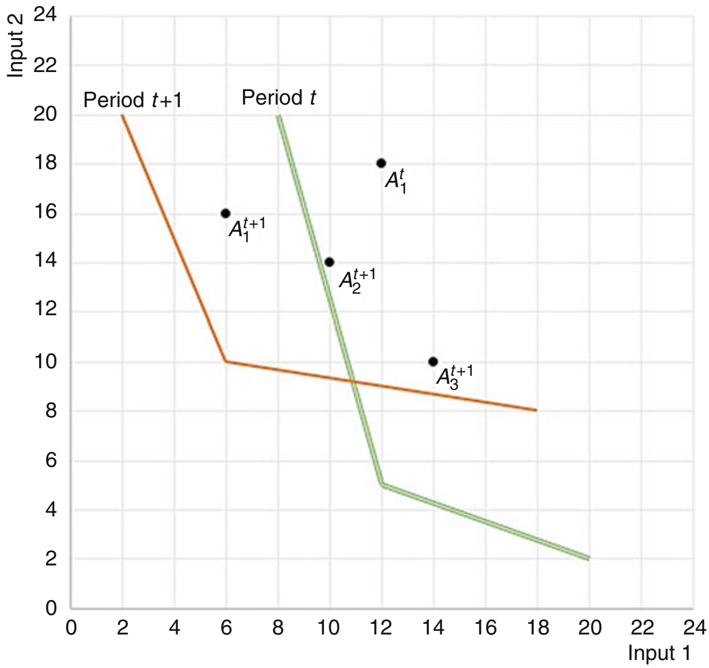



**Fig. 11.11** Possible locations for movement

$$D_{A_1^t}^{t+1} = \frac{OQ}{OA_1^t}, \tag{11.23}$$

$$D_{A_3^{t+1}}^{t} = \frac{OS}{OA_1^t}, \tag{11.24}$$

$$D_{A_3^{t+1}}^{t+1} = \frac{OR}{OA_1^t}. \tag{11.25}$$

From Fig. 11.12, it is clear that $D_{A_1^t}^t > D_{A_1^t}^{t+1}$ and $D_{A_3^{t+1}}^t < D_{A_3^{t+1}}^{t+1}$. Thus,

$$\frac{D_{A_1^t}^t}{D_{A_1^t}^{t+1}} > 1 \quad \& \quad \frac{D_{A_3^{t+1}}^t}{D_{A_3^{t+1}}^{t+1}} < 1. \tag{11.26}$$

From the inequalities in Eq. 11.26, we cannot directly conclude whether $EFC_l > 1$ or $EFC_l < 1$. However, if $A_1^t$ have a location at $A_2^{t+1}$ (or $A_1^{t+1}$) in time period $t + 1$, we will have the following inequalities which result that $EFC_{A_1^t} > 1$.

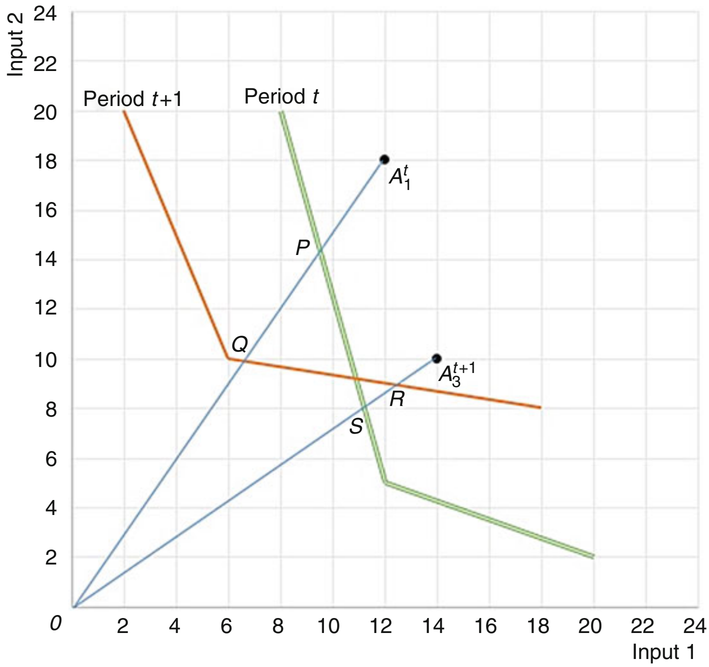



**Fig. 11.12** Measuring the radial scores

Indeed, both $A_1^t$ and $A_2^{t+1}$ are closer to the production frontier in time period $t$ than the production frontier in time period $t+1$, as Fig. 11.13 illustrates. Therefore, we have the following inequalities which yields that $EFC_{A_1^t} > 1$.

$$\frac{D_{A_1^t}^t}{D_{A_1^t}^{t+1}} > 1 \quad \& \quad \frac{D_{A_2^{t+1}}^t}{D_{A_2^{t+1}}^{t+1}} > 1. \tag{11.27}$$

In Fig. 11.14, a downward (upward) shift in the production frontier, that is, shift towards (away) the origin, from period $t$ to period $t+1$ represents a positive (negative) shift or indicates a production technology progress (decline). The above illustration can also be discussed for $A_2^t$ and $A_3^t$. Each one of the three DMUs, $A_1^t$, $A_2^t$ and $A_3^t$ may find one of the locations $A_1^{t+1}$, $A_2^{t+1}$ and $A_3^{t+1}$ in time period $t+1$. Thus, nine different inequalities can be considered, as the tree diagram in Fig. 11.15 represents.

There are 5 branches of the 9 branches in Fig. 11.15 that do not have a certain answer as to whether the value of $EFC_l$ is less or greater than 1. Overall four different cases can be seen in Fig. 11.15.

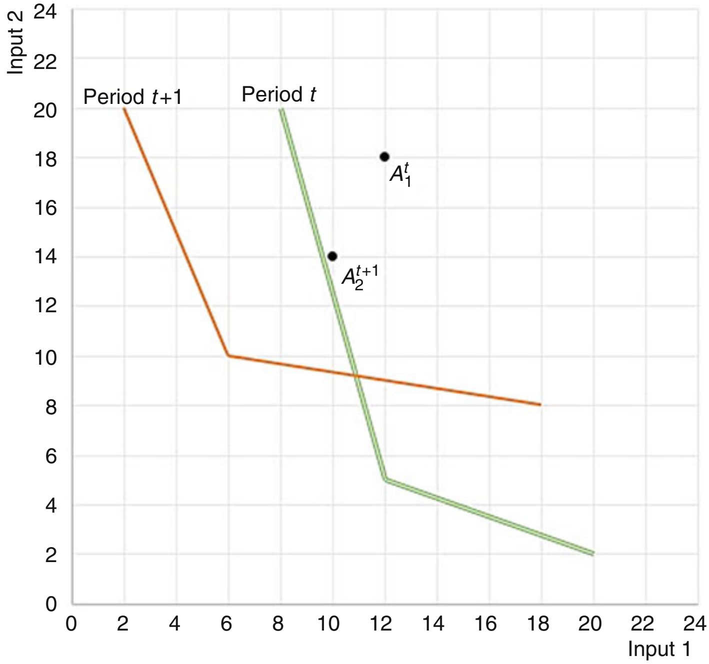



**Fig. 11.13** Locations closer to time period *t*

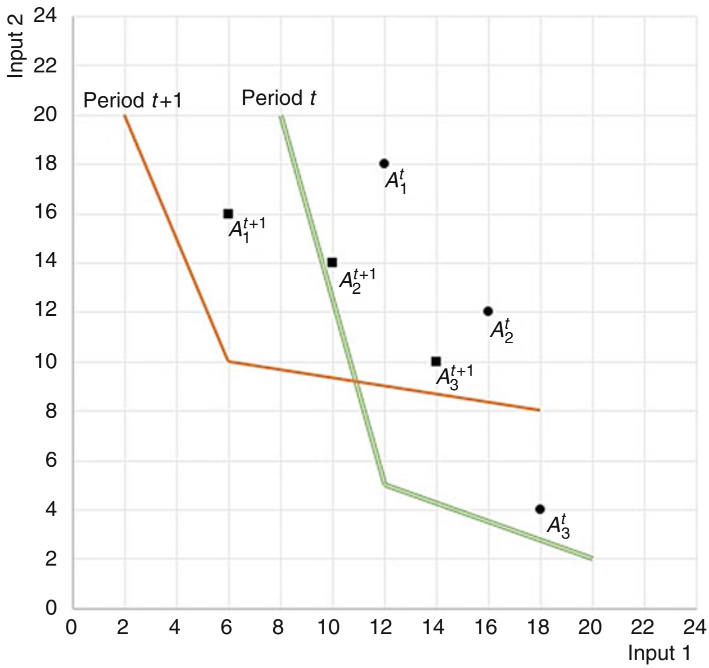



**Fig. 11.14** Downward (upward) shift in the frontiers

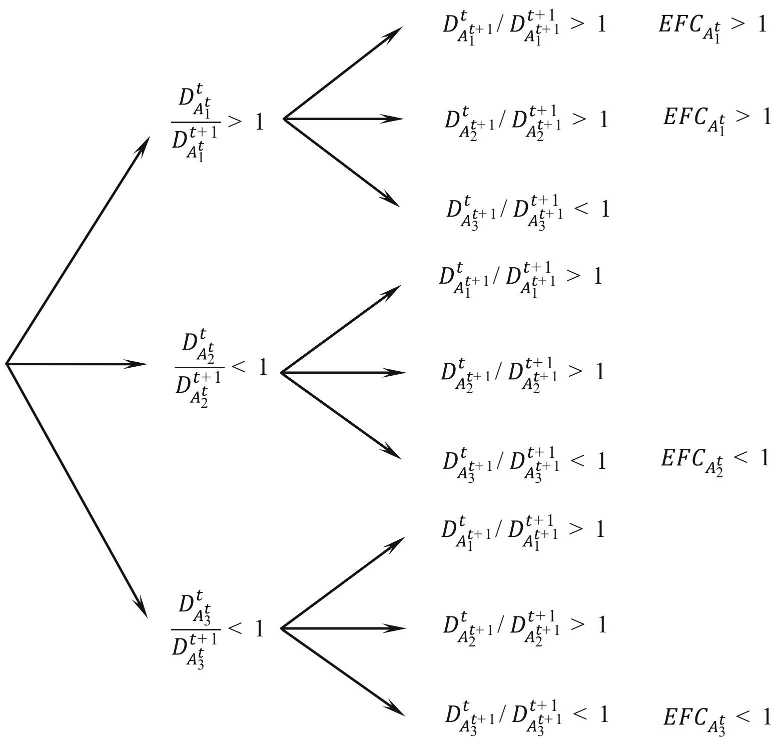



**Fig. 11.15** Nine different inequalities for the Malmquist index

*Case (a)* both inequalities are greater than 1, that is,

$$\frac{D^t_{l^t}}{D^{t+1}_{l^t}} > 1 \quad \& \quad \frac{D^t_{l^{t+1}}}{D^{t+1}_{l^{t+1}}} > 1. \tag{11.28}$$

In this case, the value of $EFC_l$ is larger than 1, and indicates that $\mathrm{DMU}_l$ is moved onto a side which has a positive shift as well as progresses in the production technology of $\mathrm{DMU}_l$.

*Case (b)* both inequalities are less than 1, that is,

$$\frac{D^t_{l^t}}{D^{t+1}_{l^t}} < 1 \quad \& \quad \frac{D^t_{l^{t+1}}}{D^{t+1}_{l^{t+1}}} < 1. \tag{11.29}$$

In this case, the value of $EFC_l$ is less than 1, and indicates that $\mathrm{DMU}_l$ is moved onto a side which has a negative shift as well as declines in the production technology of $\mathrm{DMU}_l$.

*Case (c)* The first inequality is less than 1 and the second inequality is greater than 1, that is,

$$\frac{D_{l^t}^t}{D_{l^t}^{t+1}} < 1 \quad \& \quad \frac{D_{l^{t+1}}^t}{D_{l^{t+1}}^{t+1}} > 1. \tag{11.30}$$

In this case, the value of $EFC_l$ can be larger or less than 1, and indicates that the production technology of $\mathrm{DMU}_l$ projected from a negative shift side towards a positive shift side. In other words, there is a tradeoff change between the two inputs. The change obtained from the positive shift side is greater than that of the negative shift side, if $EFC_l$ is greater than 1, that is, $EFC_l > 1$. This shows that the production technology of $\mathrm{DMU}_l$ improves on average. The change obtained from the negative shift side is greater than that of the positive shift side, if $EFC_l$ is less than 1, that is, $EFC_l < 1$. This shows that the production technology of $\mathrm{DMU}_l$ drops on average. The production technology of $\mathrm{DMU}_l$ is not changed on average, if $EFC_l = 1$.

*Case (d)* The second inequality is less than 1 and the first inequality is greater than 1, that is,

$$\frac{D_{l^t}^t}{D_{l^t}^{t+1}} > 1 \quad \& \quad \frac{D_{l^{t+1}}^t}{D_{l^{t+1}}^{t+1}} < 1. \tag{11.31}$$

Similarly, the value of $EFC_l$ can be less or larger than 1. This indicates that the production technology of $\mathrm{DMU}_l$ is projected from a positive shift side towards a negative shift side. As a result, the change obtained from the positive shift side is less than that of the negative shift side, if $EFC_l$ is less than 1, that is, $EFC_l < 1$. This shows that the production technology of $\mathrm{DMU}_l$ declines on average. The change obtained from the positive shift side is greater than that of the negative shift side if $EFC_l$ is greater than 1, that is, $EFC_l > 1$. This shows that the production technology of $\mathrm{DMU}_l$ progresses on average. The production technology of $\mathrm{DMU}_l$ remains the same on average, if $EFC_l = 1$.

As can be seen, a DMU changes its strategy if cases (c) or (d) occurs, that is, a tradeoff change between the two inputs occur. From a productivity perspective, case (c) is more favorable than case (d) for a DMU.

Now, the geometric mean of all $EFC_l$ values, $l = 1, 2, \ldots, n$, in an industry provides an estimation of the productive frontier change in the industry, and is called the Malmquist Productive Frontier Shift (MPFS) index for that industry. In other words,

$$MPFS = \sqrt[n]{\prod_{l=1}^{n} EFC_l}. \tag{11.32}$$

From the above discussions, there are three main cases for the industry view, that is, considering all $EFC_l$ values where $l = 1, 2, \ldots, n$.

*Case (A)* all inequalities for $l = 1, 2, \ldots, n$ are greater than 1, that is,

$$\begin{aligned} &\frac{D^t_{l^t}}{D^{t+1}_{l^t}} > 1, \text{for}\, l = 1, 2, \ldots, n, \\ &\qquad\& \\ &\frac{D^t_{l^{t+1}}}{D^{t+1}_{l^{t+1}}} > 1, \text{for}\, l = 1, 2, \ldots, n. \end{aligned} \tag{11.33}$$

Since the value of $EFC_l$ is larger than 1, for $l = 1, 2, \ldots, n$, thus *MPFS* is greater than 1. This shows a pure positive shift of the entire production frontier, and indicates that the production technology of the industry progresses.

*Case (B)* all inequalities for $l = 1, 2, \ldots, n$ are less than 1, that is,

$$\begin{aligned} &\frac{D^t_{l^t}}{D^{t+1}_{l^t}} < 1, \text{for}\, l = 1, 2, \ldots, n, \\ &\qquad\& \\ &\frac{D^t_{l^{t+1}}}{D^{t+1}_{l^{t+1}}} < 1, \text{for}\, l = 1, 2, \ldots, n \end{aligned} \tag{11.34}$$

Since the value of $EFC_l$ is less than 1, for $l = 1, 2, \ldots, n$, thus *MPFS* is less than 1. This shows a pure negative shift of the entire production frontier, and indicates that the production technology of the industry declines.

*Case (C)* Some of the inequalities are less than 1 and some are greater than 1, that is,

$$\begin{aligned} &\frac{D^t_{l^t}}{D^{t+1}_{l^t}} < 1, \text{for some}\, l = 1, 2, \ldots, n, \\ &\qquad\& \\ &\frac{D^t_{l^t}}{D^{t+1}_{l^t}} > 1, \text{for some}\, l = 1, 2, \ldots, n, \\ &\qquad\& \\ &\frac{D^t_{l^{t+1}}}{D^{t+1}_{l^{t+1}}} < 1, \text{for some}\, l = 1, 2, \ldots, n, \\ &\qquad\& \\ &\frac{D^t_{l^{t+1}}}{D^{t+1}_{l^{t+1}}} > 1, \text{for some}\, l = 1, 2, \ldots, n. \end{aligned} \tag{11.35}$$

In this case, the frontier shift is neither pure positive nor pure negative. As a result, there is a cross-frontier shift. If $MPFS > 1$, average the industry production technology progresses on average, and if $MPFS < 1$ the industry production technology declines.

In Eq. 11.19, the Malmquist efficiency index is defined as $TEC_l \times EFC_l$, for $l = 1, 2, \ldots, n$. The $TEC_l$ index is $D^{t+1}_{l^{t+1}}/D^t_{l^t}$ and the $EFC_l$ index is also the geometry mean of $D^t_{l^t}/D^{t+1}_{l^t}$ and $D^t_{l^{t+1}}/D^{t+1}_{l^{t+1}}$, for $l = 1, 2, \ldots, n$. Similarly, the components of the Malmquist efficiency index can be analyzed, as Fig. 11.16 illustrates.

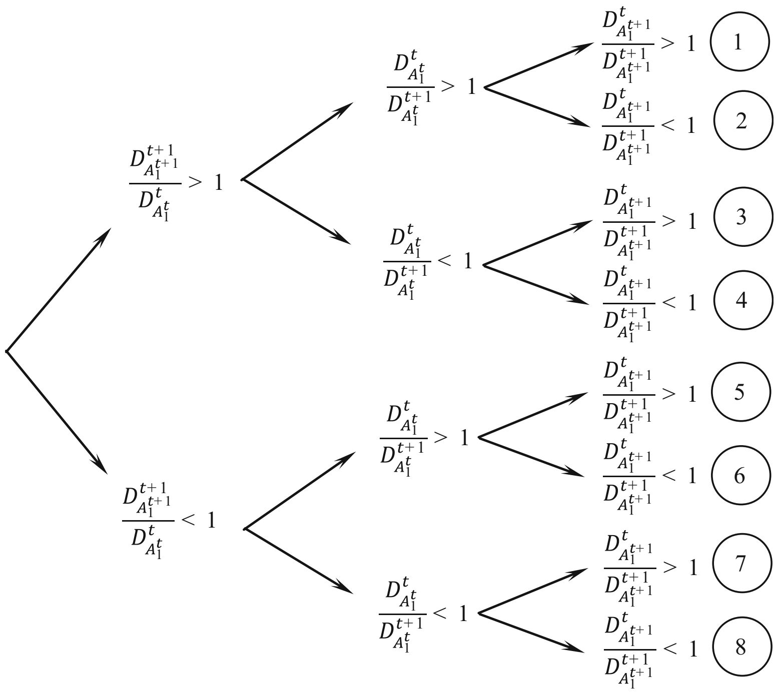


**Fig. 11.16** Positive (negative) production shifts or technical changes

The tree diagram in Fig. 11.16 shows the different cases for the Malmquist efficiency index for $A_1$. The certain answer to find out whether the Malmquist efficiency index for $A_1$ is greater or less than 1, that is, $M_{A_1} > 1$ or $M_{A_1} < 1$, is obtained from Cases 1 and 8, given by

$$\frac{D^{t+1}_{A_1^{t+1}}}{D^{t}_{A_1^{t}}} > 1 \quad \& \quad \frac{D^{t}_{A_1^{t}}}{D^{t+1}_{A_1^{t}}} > 1 \quad \& \quad \frac{D^{t}_{A_1^{t+1}}}{D^{t+1}_{A_1^{t+1}}} > 1. \tag{11.36}$$

$$\frac{D^{t+1}_{A_1^{t+1}}}{D^{t}_{A_1^{t}}} < 1 \quad \& \quad \frac{D^{t}_{A_1^{t}}}{D^{t+1}_{A_1^{t}}} < 1 \quad \& \quad \frac{D^{t}_{A_1^{t+1}}}{D^{t+1}_{A_1^{t+1}}} < 1. \tag{11.37}$$

For the rest of cases, the efficiency change is mixed either with a positive or a negative production technology shift and an increase or a decrease in technical efficiency. Thus, it is essential to investigate the indications of each individual component.

*Case 1* This case is the best efficiency improvement scenario. As can be seen, the efficiency change is related to a positive production frontier shift as well as development in technical efficiency.

*Case 2* In this case, the production technology changes from a positive shift side to negative shift side which shows that $A_1$ has an unfavorable policy change, as explained in case (d). The joint effect of the production technology growth and the technical efficiency improvement occurs, if $M_{A_1} > 1$, that is, the efficiency gains. If $M_{A_1} < 1$, the efficiency loss is from the joint effect of the production technology regress and the technical efficiency improvement.

*Case 3* This case is one of the favorable situations, because if $M_{A_1} > 1$, the Malmquist efficiency gain is from an efficiency progress and a production technology shift from negative shift sector to positive shift sector of the frontier. This indicates that $A_1$ has a favorable policy shift as well as technical efficiency improvement with regard to a positively shift frontier.

In the other hand, if $M_{A_1} < 1$, an extreme condition occurs. Indeed, only the decline in the production technology causes the efficiency loss. The production technology also changes from a negative shift side towards a positive shift side, indicating a favorable strategy adjustment. As a result, the combined Malmquist efficiency score may provide misleading information in this case.

*Case 4* This case says that there is progress in the technical efficiency, that is, $A_1$ is closer to its production frontier in time period $t + 1$ than to its production frontier in time period $t$. However, the performance of $A_1$ is related to a negative production frontier shift. An improvement in the efficiency occurs if the negative frontier shift cannot take in the progress in the technical efficiency.

*Case 5* This case says that the performance of $A_1$ is related to a positive production frontier shift, but there is a decline in the technical efficiency, that is, $A_1$ is farer to its production frontier in time period $t + 1$ than to its production frontier in time period $t$. An improvement in the efficiency occurs if the positive frontier shift can take in the decline in the technical efficiency.

*Case 6* This case is one of the least favorable conditions, because if $M_{A_1} < 1$, the Malmquist efficiency decrease is due to an efficiency decline and a production technology change from a positive shift sector of the frontier to a negative shift sector of the frontier. This indicates that $A_1$ has an unfavorable policy shift and loses the technical efficiency with regard to a negative shift frontier.

On the other hand, if $M_{A_1} > 1$, an unfavorable strategy change occurs. Indeed, only the increase in the production technology causes the efficiency gain. The production technology also changes from a positive shift side towards a negative shift side. As a result, the combined Malmquist efficiency score in this case may provide misleading information, as well.

*Case 7* In this case, $A_1$ has a favorable policy change, because the production technology changes from a negative shift side to a positive shift side. If $M_{A_1} > 1$, the efficiency gain is from the joint effects of the average production technology growth and the technical efficiency drop. If $M_{A_1} < 1$, the efficiency loss is from the joint effects of the average production technology regress and the technical efficiency decrease.

*Case 8* Without a doubt, this case is the worst scenario for $A_1$. The efficiency decline is associated with a negative production frontier shift and a drop in the technical efficiency.

As discussed earlier, if the prices are available or the isocost line is known, the interpretation could be changed. For instance, if the technology shift yields an input value with a relatively low price in Case 6, a change could possibly yield to lower cost. Thus, similar to differences between efficiency and technical efficiency, the above favorable or unfavorable policy change is based upon the isoquant changes only. In order to fully characterize the strategy change, the price information should be considered to study overall efficiency or allocative efficiency.

## 11.4 A Computer Industry Example

The DEA Malmquist efficiency index has been used in many real-life applications and there is an extensive form of applications that uses the DEA Malmquist efficiency index. For example, efficiency growths in Swedish hospitals, deregulation's effects on Spanish saving banks, variations in agricultural efficiency in 18 developing countries, an experiential study of the catch-up hypothesis for a group of high and low income countries, telecommunications efficiency, a Swedish eye-care facility delivery, production machinery catch-up and invention in 74 countries, and so on (Chen and Ali 2004). As can be seen, the DEA Malmquist efficiency index has confirmed itself to be an exceptional tool for measuring the efficiency change of DMUs.

In this section, a set of Fortune Global 500 Computer and Office Equipment companies from 1991 to 1997 which was examined by Chen and Ali (2004) is discussed. They estimated whether or not the strategy shifts of companies were favorable. There were eight companies, such as APPLE, CANON, COMPAQ, DIGITAL, FUJITSU, HP, IBM, and RICOH. Four factors were selected for each company, three input factors and one output factor. The three input factors were assets, shareholder's equity and the number of employees. The single output factor was revenue. In their study, they did not have data for APPLE in 1997, since Apple did not appear on the list in 1997.

We illustrated in Sect. 11.1 how to calculate the Malmquist efficiency index with one single click by using Microsoft Excel Solver. Here we discuss the outcomes. Table 11.5 illustrates the technical efficiency scores of companies in 1991–1997.

APPLE was the only CCR technically efficient company in each period between 1991 and 1996. COMPAQ was also CCR technically efficient between 1991 and 1997. The last row in Table 11.5 also represents the average of CCR efficiency scores of companies.

Table 11.6 displays the Malmquist technical efficiency changes for each company as well as the average of technical efficiency changes for all companies from 1991 to 1997. There was no improvement in technical efficiency of APPLE from 1991 to 1996, but it does not mean that APPLE was not the best practice, as APPLE lies on the

**Table 11.5** The CCR efficiency scores of companies (1991–1997)

| Company | 1991 | 1992 | 1993 | 1994 | 1995 | 1996 | 1997 |
|---|---|---|---|---|---|---|---|
| APPLE | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | – |
| CANON | 0.7462 | 0.8358 | 0.6493 | 0.6200 | 0.7280 | 0.5815 | 0.8218 |
| COMPAQ | 0.7484 | 0.9012 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| DIGITAL | 0.6540 | 0.7039 | 0.8261 | 1.0000 | 1.0000 | 0.8452 | 1.0000 |
| FUJITSU | 0.7209 | 0.8926 | 0.7154 | 0.6537 | 0.9310 | 0.8786 | 1.0000 |
| HP | 0.6725 | 0.7147 | 0.7394 | 0.7322 | 0.7199 | 0.7567 | 0.8967 |
| IBM | 0.4850 | 0.7210 | 0.8072 | 0.6988 | 0.8306 | 0.7349 | 1.0000 |
| RICOH | 0.7873 | 0.8283 | 0.6671 | 0.6006 | 0.7877 | 0.7173 | 0.7815 |
| Average | 0.7268 | 0.8247 | 0.8005 | 0.7882 | 0.8746 | 0.8143 | 0.9286 |

**Table 11.6** The Malmquist technical efficiency changes for companies

| Company | 91 → 92 | 92 → 93 | 93 → 94 | 94 → 95 | 95 → 96 | 96 → 97 |
|---|---|---|---|---|---|---|
| APPLE | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | – |
| CANON | 1.1200 | 0.7768 | 0.9549 | 1.1742 | 0.7986 | 1.4133 |
| COMPAQ | 1.2041 | 1.1096 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| DIGITAL | 1.0762 | 1.1735 | 1.2105 | 1.0000 | 0.8451 | 1.1831 |
| FUJITSU | 1.2382 | 0.8014 | 0.9137 | 1.4242 | 0.9437 | 1.1381 |
| HP | 1.0626 | 1.0345 | 0.9902 | 0.9832 | 1.0511 | 1.1851 |
| IBM | 1.4865 | 1.1195 | 0.8657 | 1.1885 | 0.8848 | 1.3606 |
| RICOH | 1.0520 | 0.8053 | 0.9004 | 1.3114 | 0.9106 | 1.0895 |
| Average | 1.1466 | 0.9656 | 0.9748 | 1.1352 | 0.9292 | 1.1957 |

estimated production frontier in time period $t$ and time period $t + 1$. Conversely, a score greater than 1 in Table 11.6 only indicates that there was progress in the CCR technical efficiency score. But, this does not necessarily mean that there was an improvement in performance. For example, the technical efficiency change for DIGITAL is greater than 1 from 1991 to 1992, 1992 to 1993 and 1993 to 1994. These scores do not indicate that the performance of DIGITAL in technical efficiency progress is better than the performance of APPLE in technical efficiency progress.

Both progress and decline in Malmquist technical efficiency changes can be seen for the companies CANON, DIGITAL FUJITSU, HP, IBM and RICOH. No Malmquist technical efficiency decline existed for APPLE and COMPAQ companies. The average of Malmquist technical efficiency change of all companies progresses from 1991 to 1992 by 14.7%, from 1994 to 1995 by 13.5% and from 1996 to 1997 by 19.6%, respectively. In addition, the average of Malmquist technical efficiency change of all companies declines from 1992 to 1993 by 3.4%, from 1993 to 1994 by 2.5% and from 1995 to 1996 by 7.1%, respectively. It is also interesting to note that none of the companies had a decline in their technical efficiency scores in time period 1996–1997.

Table 11.7 also represents the Malmquist frontier shift for each company from time period $t$ to time period $t + 1$ as well as the geometrical mean of the scores in each shift.

**Table 11.7** The Malmquist frontier shift for companies

| Company | 91 → 92 | 92 → 93 | 93 → 94 | 94 → 95 | 95 → 96 | 96 → 97 |
|---|---|---|---|---|---|---|
| APPLE | 0.9969 | 1.0931 | 1.0428 | 0.9608 | 1.1367 | – |
| CANON | 0.9072 | 1.2151 | 0.9907 | 0.9007 | 1.2482 | 0.7783 |
| COMPAQ | 1.0901 | 1.3155 | 1.0509 | 1.0107 | 1.1373 | 0.9600 |
| DIGITAL | 0.9291 | 1.0724 | 1.0625 | 0.9159 | 1.2366 | 0.9072 |
| FUJITSU | 0.9072 | 1.2151 | 1.0087 | 0.8897 | 1.2394 | 0.9921 |
| HP | 0.9291 | 0.9974 | 1.0628 | 0.9790 | 1.0225 | 0.8331 |
| IBM | 0.9072 | 1.2151 | 0.9991 | 0.8963 | 1.2449 | 0.8542 |
| RICOH | 0.9072 | 1.2151 | 1.0050 | 0.8917 | 1.2386 | 1.0124 |
| Average | 0.9449 | 1.1632 | 1.0274 | 0.9296 | 1.1854 | 0.9017 |

**Table 11.8** Malmquist technology shift from periods 91–93

| Time | 91 → 92 | | 92 → 93 | |
|---|---|---|---|---|
| Company | I | II | I | II |
| APPLE | 0.9704 | 0.9772 | 1.0609 | 1.0831 |
| CANON | 0.9628 | 0.9622 | 1.0889 | 1.0894 |
| COMPAQ | 0.9830 | 0.9880 | 1.0862 | 1.1055 |
| DIGITAL | 0.9475 | 0.9475 | 0.9984 | 1.0445 |
| FUJITSU | 0.9501 | 0.9520 | 1.0941 | 1.0938 |
| HP | 0.9548 | 0.9557 | 1.0254 | 1.0378 |
| IBM | 0.9687 | 0.9654 | 1.0913 | 1.1052 |
| RICOH | 0.9507 | 0.9544 | 1.0956 | 1.0964 |

From Table 11.7, on average, there was 5.5% decline in the industry technology frontier from 1991 to 1992, 7.0% decline from 1994 to 1995, and declined 10% decline from 1996 to 1997, respectively. There was also 16.3% progress from 1992 to 1993, 3.03% progress from 1993 to 1994, and 18.5% progress from 1995 to 1996.

In addition, from 1991 to 1992, all the companies had a negative shift in the technology frontier, except COMPAQ. From 1992 to 1993, except HP, all the companies had a positive technology frontier shift. From 1993 to 1994, six of the companies had a positive technology frontier shift, and the other two companies had a negative technology frontier shift. From 1994 to 1995, all the companies had a negative technology frontier shift, except COMPAQ. From 1995 to 1996, all the companies had a positive technology frontier shift. And from 1996 to 1997 except RICOH, all the companies show a negative technology frontier shift in the technology frontier. Note that, we could also employ a modified method where period $t$ technology is created from input and output factors of all companies in all periods before period $t$ and period $t$ itself. In this case, the technologies in place in earlier periods are also remembered, and will remain available to adopt with the current period.

Tables 11.8, 11.9 and 11.10 illustrates the Malmquist component shifts in technology frontier based upon the ratios discussed in Fig. 11.16. Here, (I) and (II) represents the ratios, $D_{t'}^{t}/D_{t'}^{t+1}$ and $D_{t'^{+1}}^{t}/D_{t'^{+1}}^{t+1}$, respectively.

**Table 11.9** Malmquist technology shift from periods 93–95

| Time | 93 → 94 | | 94 → 95 | |
|---|---|---|---|---|
| Company | I | II | I | II |
| APPLE | 1.0178 | 1.0980 | 0.9057 | 1.0608 |
| CANON | 0.9131 | 0.9600 | 1.1034 | 0.9915 |
| COMPAQ | 1.0629 | 1.0650 | 1.0318 | 1.0346 |
| DIGITAL | 1.0817 | 0.7299 | 1.0175 | 0.9613 |
| FUJITSU | 0.7971 | 0.8414 | 1.2172 | 0.9827 |
| HP | 1.0750 | 1.0768 | 0.9178 | 1.0450 |
| IBM | 0.8333 | 0.8972 | 1.1559 | 0.9878 |
| RICOH | 0.8040 | 0.8686 | 1.1852 | 0.9820 |

**Table 11.10** Malmquist technology shift from periods 95–97

| Time | 95 → 96 | | 96 → 97 | |
|---|---|---|---|---|
| Company | I | II | I | II |
| APPLE | 1.1304 | 1.1832 | – | – |
| CANON | 1.3576 | 1.1398 | 0.6008 | 0.6035 |
| COMPAQ | 1.1586 | 1.0550 | 0.8334 | 0.7610 |
| DIGITAL | 1.6186 | 1.1435 | 0.6029 | 0.5993 |
| FUJITSU | 1.6045 | 1.1594 | 0.5925 | 0.6039 |
| HP | 1.1381 | 1.1382 | 0.6305 | 0.6421 |
| IBM | 1.4777 | 1.1548 | 0.5732 | 0.5715 |
| RICOH | 1.6178 | 1.1583 | 0.5955 | 0.5957 |

As can be seen, the industry technology frontier has a pure negative shift from 1991 to 1992 and 1996 to 1997. The industry technology frontier has a cross-frontier shift from 1992 to 1993, 1993 to 1994, and 1994 to 1995. The industry technology frontier has a pure positive shift from 1995 to 1996 only.

For example, the technology frontier change at the company level indicates that for COMPAQ, the two ratios, $D_{I^t}^t / D_{I^t}^{t+1}$ and $D_{I^{t+1}}^t / D_{I^{t+1}}^{t+1}$, are all larger than 1 from 1994 to 1995. This shows that COMPAQ has a consistent operations strategy from 1994 to 1995. Other companies in this period of time, had a move between two sides, which indicates that the companies had a variation in their operations strategy. For instance, the technology frontier for Apple and HP changes from a negative shift side towards a positive shift side. This change means that there is a favorable strategy change for these two companies.

Table 11.11 represents the Malmquist efficiency indexes for the companies as well as the average Malmquist efficiency index for all companies. From the table, the efficiency of computer industry improves on average from time period $t$ to $t + 1$ by 8.2%, 12.3%, 0.16%, 4.5%, 9.7% and 7.1%, for $t = 1991, 1992, \ldots$, and 1996, respectively.

As illustrated earlier, the average Malmquist efficiency variation is the product of the geometry mean of the technical efficiency change and the technology frontier shift. In order to analyze the results in Table 11.11, we should refer to Tables 11.6, 11.7, 11.8, 11.9 and 11.10. Based upon these tables, for instance, the average

**Table 11.11** The Malmquist efficiency indexes for the companies

| Company | 91 → 92 | 92 → 93 | 93 → 94 | 94 → 95 | 95 → 96 | 96 → 97 |
|---|---|---|---|---|---|---|
| APPLE | 0.9969 | 1.0930 | 1.0428 | 0.9608 | 1.1367 | – |
| CANON | 1.0161 | 0.9439 | 0.9460 | 1.0576 | 0.9969 | 1.1000 |
| COMPAQ | 1.3127 | 1.4597 | 1.0509 | 1.0107 | 1.1373 | 0.9600 |
| DIGITAL | 1.0000 | 1.2585 | 1.2863 | 0.9159 | 1.0451 | 1.0734 |
| FUJITSU | 1.1233 | 0.9738 | 0.9217 | 1.2672 | 1.1696 | 1.1292 |
| HP | 0.9872 | 1.0319 | 1.0525 | 0.9625 | 1.0748 | 0.9874 |
| IBM | 1.3340 | 1.3603 | 0.8649 | 1.0653 | 1.1016 | 1.1624 |
| RICOH | 0.9544 | 0.9785 | 0.9049 | 1.1693 | 1.1280 | 1.1031 |
| Average | 1.0819 | 1.1233 | 1.0015 | 1.0457 | 1.0974 | 1.0714 |

**Table 11.12** The Malmquist components in periods 91–93

| Time | 91 → 92 | | | | 92 → 93 | | | |
|---|---|---|---|---|---|---|---|---|
| Company/ratio | I | II | III | IV | I | II | III | IV |
| APPLE | =1 | <1 | <1 | <1 | =1 | >1 | >1 | >1 |
| CANON | >1 | <1 | <1 | >1 | <1 | >1 | >1 | <1 |
| COMPAQ | >1 | <1 | <1 | >1 | >1 | >1 | >1 | >1 |
| DIGITAL | >1 | <1 | <1 | <1 | >1 | <1 | >1 | >1 |
| FUJITSU | >1 | <1 | <1 | >1 | <1 | >1 | >1 | <1 |
| HP | >1 | <1 | <1 | >1 | >1 | >1 | >1 | >1 |
| IBM | >1 | <1 | <1 | >1 | >1 | >1 | >1 | >1 |
| RICOH | >1 | <1 | <1 | <1 | <1 | >1 | >1 | <1 |

efficiency gain from 1991 to 1992 is a joint effect of an improvement in the technical efficiency change on average and a negative shift in the technology frontier on average. Therefore, the progress in the technical efficiency is the only source of the efficiency gain from 1991 to 1992.

The average efficiency progress from 1992 to 1993 is a mutual effect of an average drop in the technical efficiency and an average positive shift in the technology frontier. As a result, an improvement in the technology frontier shift is the only source of the efficiency progress.

The average efficiency from 1993 to 1994 slightly gains, due to the joint effect of an average drop in the technical efficiency and an average positive shift in the technology frontier. Consequently, a slight efficiency progress is due to a positive shift in the technology frontier. The same analysis can be made for time periods 1994 to 1995, 1995 to 1996, and 1996 to 1997.

Tables 11.12, 11.13 and 11.14 illustrates the components of the Malmquist efficiency indexes associated with the efficiency change for each company from 1991 to 1997. In the tables, ratios I, II, III and IV indicate the ratios $D_{l^{t+1}}^{t+1}/D_{l^t}^{t}$, $D_{l^t}^{t}/D_{l^t}^{t+1}$, $D_{l^{t+1}}^{t}/D_{l^{t+1}}^{t+1}$ and $M_l$, respectively.

There are two cases which need specific consideration in the previous tables. The first case is that DIGITAL had the efficiency gain ($M_l > 1$) from 1992 to 1993 and

**Table 11.13** The Malmquist components in periods 93–95

| Time | 93 → 94 | | | | 94 → 95 | | | |
|---|---|---|---|---|---|---|---|---|
| Company/ratio | I | II | III | IV | I | II | III | IV |
| APPLE | =1 | >1 | >1 | >1 | =1 | <1 | >1 | <1 |
| CANON | >1 | <1 | <1 | <1 | >1 | >1 | <1 | >1 |
| COMPAQ | =1 | >1 | >1 | >1 | =1 | >1 | >1 | >1 |
| DIGITAL | >1 | >1 | <1 | >1 | =1 | >1 | <1 | <1 |
| FUJITSU | >1 | <1 | <1 | <1 | >1 | >1 | <1 | >1 |
| HP | >1 | >1 | >1 | >1 | >1 | <1 | >1 | <1 |
| IBM | >1 | <1 | <1 | <1 | >1 | >1 | <1 | >1 |
| RICOH | >1 | <1 | <1 | <1 | >1 | >1 | <1 | >1 |

**Table 11.14** The Malmquist components in periods 95–97

| Time | 95 → 96 | | | | 96 → 97 | | | |
|---|---|---|---|---|---|---|---|---|
| Company/ratio | I | II | III | IV | I | II | III | IV |
| APPLE | =1 | >1 | >1 | >1 | – | – | – | – |
| CANON | <1 | >1 | >1 | <1 | >1 | <1 | <1 | <1 |
| COMPAQ | =1 | >1 | >1 | >1 | =1 | <1 | <1 | <1 |
| DIGITAL | <1 | >1 | >1 | <1 | >1 | <1 | <1 | <1 |
| FUJITSU | <1 | >1 | >1 | <1 | >1 | <1 | <1 | >1 |
| HP | <1 | >1 | >1 | >1 | >1 | <1 | <1 | <1 |
| IBM | <1 | >1 | >1 | <1 | >1 | <1 | <1 | >1 |
| RICOH | <1 | >1 | >1 | <1 | >1 | <1 | <1 | <1 |

from 1993 to 1994. Nevertheless, the technology frontier of DIGITAL changes from a negative shift side towards a positive shift side from 1992 to 1993; while from 1993 to 1994, the technology frontier of DIGITAL changes from a positive shift side towards a negative shift side. As can be seen, the cause for the efficiency gain is different which should be considered for interpretation.

The second case is that HP had an efficiency loss ($M_l < 1$) from 1994 to 1995. The only reason that causes the efficiency loss is the average technology frontier decline. Nevertheless, the technology of HP changes from a negative shift side towards a positive shift side, indicating a favorable policy change.

The above discussion shows that more insights for the efficiency changes can be obtained by analyzing the Malmquist efficiency components.

## 11.5 A Non-radial Malmquist Efficiency Index

In this section, the Malmquist radial efficiency index is extended into a Malmquist non-radial efficiency index, proposed by Chen (2003). The new index incorporates with the preferences of decision makers over performance progress, and measures all inefficiencies which might be represented by non-zero slacks. The radial approach does not measure possible slacks, as shown in Fig. 11.17.

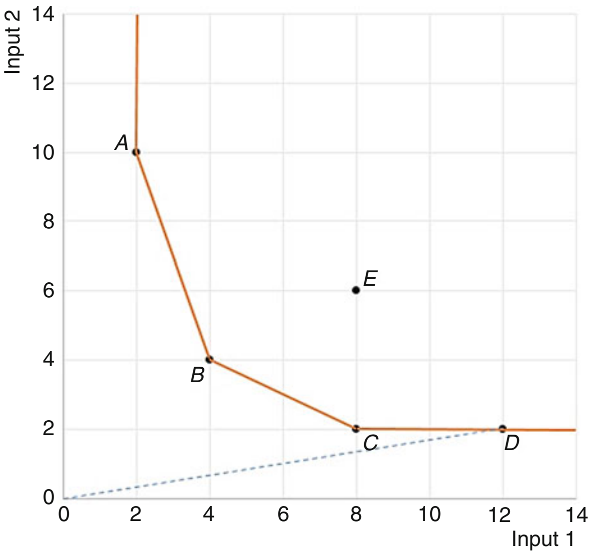



**Fig. 11.17** Weak technical efficiency

There are five DMUs in Fig. 11.17 in which each DMU has two input factors. Note that the output factor in this case is considered as a single constant value such as 1. DMUs A-D are on the estimated production frontier, thus their CCR scores are 1. As can be seen, D is dominated by C, because D has two units more than C in the first input and the same amount in the second input. In other words, there is a slack in the first input of D, that is, $s_1^- = 2$, which cannot be measured by the radial approach. In other words, the CCR-efficiency is measured by a radial approach and possible non-zero input or output slacks. The introduced radial Malmquist efficiency index in the previous section is also based upon the CCR scores only. Disregarding non-zero input (output) slacks in input (output) oriented model clearly cannot completely characterize the efficiency change. The previous radial Malmquist efficiency index also fails to corporate the preference over the performance progress of individual input and output factors by decision-makers. DEA analysis with incorporation of value judgment is vital in applications to avoid incorrect results and false implications. We can, of course, use the ADD DEA model to measure the possible slacks after measuring the CCR scores. This technique is called the Two-Phase CCR model, shown in Eq. 11.38.

$$\begin{aligned}
&\min \theta_l - \varepsilon\left(\textstyle\sum_{j=1}^{m} s_{lj}^- + \sum_{k=1}^{p} s_{lk}^+\right),\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i + s_{lj}^- = x_{lj}\theta_l, \quad \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i - s_{lk}^+ = y_{lk}, \quad \text{for } k = 1, 2, \ldots, p,\\
&\lambda_i \geq 0, \quad \text{for } i = 1, 2, \ldots, n,\\
&s_{lj}^- \geq 0, \quad \text{for } j = 1, 2, \ldots, m,\\
&s_{lk}^+ \geq 0, \quad \text{for } k = 1, 2, \ldots, p.
\end{aligned} \tag{11.38}$$

The above model is two models which are combined into one single model. In order to solve Eq. 11.38, one should first solve Eq. 11.39 to calculate $\theta_l^*$, for $l = 1, 2, \ldots, n$, and after that solve Eq. 11.40 to calculate the optimal slacks.

$$\begin{aligned} & \min \theta_l, \\ & \text{Subject to} \\ & \sum_{i=1}^{n} x_{ij}\lambda_i \leq x_{lj}\theta_l, \quad \text{for } j = 1, 2, \ldots, m, \\ & \sum_{i=1}^{n} y_{ik}\lambda_i \geq y_{lk}, \quad \text{for } k = 1, 2, \ldots, p \\ & \lambda_i \geq 0, \quad \text{for } i = 1, 2, \ldots, n. \end{aligned} \tag{11.39}$$

$$\begin{aligned} & \max \sum_{j=1}^{m} s_{lj}^- + \sum_{k=1}^{p} s_{lk}^+, \\ & \text{Subject to} \\ & \sum_{i=1}^{n} x_{ij}\lambda_i + s_{lj}^- = x_{lj}\theta_l^*, \quad \text{for } j = 1, 2, \ldots, m \\ & \sum_{i=1}^{n} y_{ik}\lambda_i - s_{lk}^+ = y_{lk}, \quad \text{for } k = 1, 2, \ldots, p, \\ & \lambda_i \geq 0, \quad \text{for } i = 1, 2, \ldots, n, \\ & s_{lj}^- \geq 0, \quad \text{for } j = 1, 2, \ldots, m, \\ & s_{lk}^+ \geq 0, \quad \text{for } k = 1, 2, \ldots, p. \end{aligned} \tag{11.40}$$

After that, for an $\varepsilon$ value, which is a very small positive real number, the score for is estimated by the Two-Phase CCR, given by:

$$\theta_l^* - \varepsilon\left(\sum_{j=1}^{m} s_{lj}^{-*} + \sum_{k=1}^{p} s_{lk}^{+*}\right).$$

We can also use SBM or KAM to fully measure the inefficiencies in both input and output orientations, but we leave it as exercises. Here we use an improvement of the Russell Measure (RM) proposed by Zhu (1996) to incorporate the preferences of decision makers over performance progress as well as measuring all inefficiencies which might be represented by non-zero slacks. The use of this non-radial Malmquist efficiency index correctly measures the efficiency changes while the radial approach may deliver distorted information.

Suppose that there are $n$ DMUs, $A_i$, $i = 1, 2, \ldots, n$, in which each DMU has $m$ input factors, $x_{ij}$, $j = 1, 2, \ldots, m$, and $p$ output factors, $y_{ik}$, $k = 1, 2, \ldots, p$. Also assume that $\alpha_j$, $j = 1, 2, \ldots, m$, are the user-specified weights to indicate preferences over the input progresses. The following input oriented DEA model was established by Zhu (1996).

$$\begin{aligned} & \min \left(\sum_{j=1}^{m} \alpha_j\theta_{lj}\right) / \sum_{j=1}^{m} \alpha_j, \\ & \text{Subject to} \\ & \sum_{i=1}^{n} x_{ij}\lambda_i \leq x_{lj}\theta_{lj}, \quad \text{for } j = 1, 2, \ldots, m, \\ & \sum_{i=1}^{n} y_{ik}\lambda_i \geq y_{lk}, \quad \text{for } k = 1, 2, \ldots, p, \\ & \theta_{lj} \text{ free}, \\ & \lambda_i \geq 0, \quad \text{for } i = 1, 2, \ldots, n. \end{aligned} \tag{11.41}$$

The optimal value of $\theta_{lj}$ can be less, equal or greater than 1, for $j = 1, 2, \ldots, m$. In addition, the equality occurs in the corresponded input constraints for the optimal solutions, that is, $\sum_{i=1}^{n} x_{ij}\lambda_i^* = x_{lj}\theta_{lj}^*$, for $j = 1, 2, \ldots, m$. Because, if $\sum_{i=1}^{n} x_{ij}\lambda_i^* < x_{lj}\theta_{lj}^*$, for some $j = 1, 2, \ldots, m$, this means $\theta_{lj}^*$ is not the optimal (minimum) value, as there will be a smaller value $\theta_{lj}$ such that $\theta_{lj} < \theta_{lj}^*$ which can result in equality. Therefore, there is no positive input slack exists in the optimal solutions of Eqs. 11.41 and 11.42, that is, $s_{lj}^{-*} = 0$, for $j = 1, 2, \ldots, m$, and the following theorem is concluded:

**Theorem 11.1** Any optimal solution in Eqs. 11.41 and 11.42 will always have all input slacks equal to zero.

In order to show that the objective in Eqs. 11.41 and 11.42 is between 0 and 1, we have the following theorem.

**Theorem 11.2** The objective in Eqs. 11.41 and 11.42 is less than or equal to 1.

*Proof* We first find the dual linear programming for Eq. 11.24. For such an aim, suppose the weights $w_j^-$ and $w_k^+$ are used to sum up the constraints, for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$. Thus, we have the following equations, respectively:

$$\begin{cases} \sum_{j=1}^{m} w_j^- \left(\sum_{i=1}^{n} x_{ij}\lambda_i\right) \leq \sum_{j=1}^{m} x_{lj} w_j^- \theta_{lj}, \\ \sum_{k=1}^{p} y_{ik} w_k^+ \left(\sum_{i=1}^{n} \lambda_i\right) \geq \sum_{k=1}^{p} y_{lk} w_k^+, \end{cases} \tag{11.42}$$

$\Leftrightarrow$

$$\begin{cases} -\sum_{j=1}^{m} \sum_{i=1}^{n} x_{ij} w_j^- \lambda_i \geq -\sum_{j=1}^{m} x_{lj} w_j^- \theta_{lj}, \\ \sum_{k=1}^{p} \sum_{i=1}^{n} y_{ik} w_k^+ \lambda_i \geq \sum_{k=1}^{p} y_{lk} w_k^+, \end{cases} \tag{11.43}$$

$\Leftrightarrow$

$$\begin{cases} -\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} w_j^- \lambda_i \geq -\sum_{j=1}^{m} x_{lj} w_j^- \theta_{lj}, \\ \sum_{i=1}^{n} \sum_{k=1}^{p} y_{ik} w_k^+ \lambda_i \geq \sum_{k=1}^{p} y_{lk} w_k^+, \end{cases} \tag{11.44}$$

$\Leftrightarrow$

$$\begin{cases} -\sum_{i=1}^{n} \lambda_i \sum_{j=1}^{m} x_{ij} w_j^- \geq -\sum_{j=1}^{m} x_{lj} w_j^- \theta_{lj}, \\ \sum_{i=1}^{n} \lambda_i \sum_{k=1}^{p} y_{ik} w_k^+ \geq \sum_{k=1}^{p} y_{lk} w_k^+, \end{cases} \tag{11.45}$$

$\Leftrightarrow$

$$\sum_{i=1}^{n} \lambda_i \left(\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^-\right) \geq \sum_{k=1}^{p} y_{lk} w_k^+ - \sum_{j=1}^{m} x_{lj} w_j^- \theta_{lj}, \tag{11.46}$$

$\Leftrightarrow$

$$\sum_{j=1}^{m} x_{lj} w_j^- \theta_{lj} \geq \sum_{k=1}^{p} y_{lk} w_k^+ - \sum_{i=1}^{n} \lambda_i \left(\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^-\right), \tag{11.47}$$

Assume that $\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \leq 0$, for $i = 1, 2, \ldots, n$, and also $\alpha_j / \sum_{j=1}^{m} \alpha_j = x_{lj} w_j^-$, for $j = 1, 2, \ldots, m$. Therefore, we have the following dual linear programming

$$\begin{aligned}
&\max \sum_{k=1}^{p} y_{lk} w_k^+, \\
&\text{Subject to} \\
&\sum_{k=1}^{p} y_{ik} w_k^+ - \sum_{j=1}^{m} x_{ij} w_j^- \leq 0, \quad \text{for } i = 1, 2, \ldots, n, \\
&x_{lj} w_j^- = \alpha_j / \sum_{j=1}^{m} \alpha_j, \quad \text{for } j = 1, 2, \ldots, m, \\
&w_k^+ \geq 0, \quad \text{for } k = 1, 2, \ldots, p.
\end{aligned} \tag{11.48}$$

Now, since

$$\sum_{j=1}^{m} x_{lj} w_j^- = \sum_{j=1}^{m} \frac{\alpha_j}{\sum_{j=1}^{m} \alpha_j} = 1. \tag{11.49}$$

Thus, the constraint $\sum_{k=1}^{p} y_{lk} w_k^+ - \sum_{j=1}^{m} x_{lj} w_j^- \leq 0$, yields that $\sum_{k=1}^{p} w_k^+ y_{lk} \leq 1$, and the proof is completed. □

Model 11.42 measures the technical efficiency of DMU$_l$ ($l = 1, 2, \ldots, n$) under weights $\alpha_j$, for $j = 1, 2, \ldots, m$, and determines a preferred empirical production frontier. If $\alpha_j = 0$, or some $j = 1, 2, \ldots, m$, we set the corresponding $\theta_{lj}$ equal to 1. When the greater value for a weight $\alpha_j$ ($j = 1, 2, \ldots, m$) is considered DMUs give higher priority to reduce the corresponding input value. The value of $\alpha_j$ can be selected differently based upon the user judgment. For example $\alpha_j$ can be defined as $1/x_{lj}$ or $x_{lj}/\sum_{j=1}^{n} x_{lj}$.

Let's consider the DMUs in Fig. 11.17. DMU E is an inefficient DMU. Figure 11.18 depicts the targets for E by applying the radial CCR and non-radial Model 11.42. The target for E by CCR is $x_{1E}^* = 4.8$ and $x_{2E}^* = 3.6$, whereas the target for E by Model 11.42 is B, where $\alpha_1 = 1$ and $\alpha_2 = 1$.

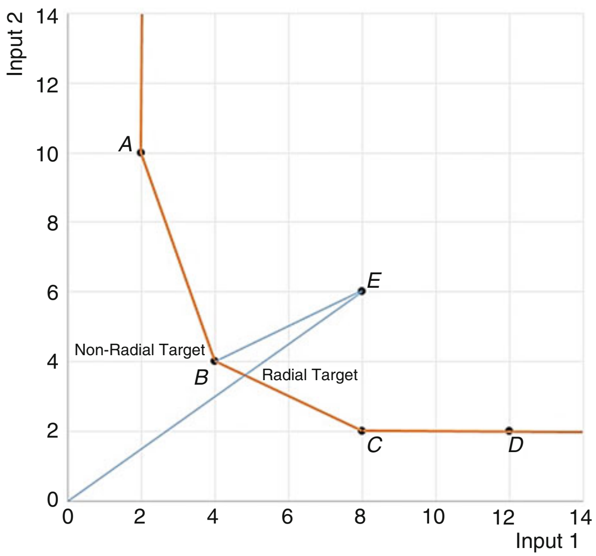



**Fig. 11.18** Radial and non-radial targets

We now introduce the four models to measure the non-radial Malmquist efficiency index which is incorporated with the preference over the individual input improvements, and does not allow the existence of non-zero input slacks. Equations 11.50, 11.51, 11.52 and 11.53 show these four models.

The non-radial input oriented Model 11.50 measures the technical efficiency score of DMU$_l$ ($l$=1,2,. . .,$n$) in time period $t$ according to the generated PPS of all DMUs in time period $t$.

$$
\begin{aligned}
&D_{l^t}^t = \min\left(\sum_{j=1}^m \alpha_j \theta_{l^t j}^t\right) / \sum_{j=1}^m \alpha_j,\\
&\text{Subject to}\\
&\sum_{i=1}^n x_{ij}^t \lambda_i \le x_{l^t j}^t \theta_{l^t j}^t, \quad \text{for } j = 1, 2, \ldots, m,\\
&\sum_{i=1}^n y_{ik}^t \lambda_i \ge y_{l^t k}^t, \quad \text{for } k = 1, 2, \ldots, p,\\
&\theta_{l^t j}^t \text{ free},\\
&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n.
\end{aligned} \tag{11.50}
$$

The non-radial input oriented Model 11.51 calculates the technical efficiency score of DMU$_l$ ($l$=1,2,. . .,$n$) in time period $t$ according to the generated PPS of all DMUs in time period $t + 1$.

$$
\begin{aligned}
&D_{l^t}^{t+1} = \min\left(\sum_{j=1}^m \alpha_j \theta_{l^t j}^{t+1}\right) / \sum_{j=1}^m \alpha_j,\\
&\text{Subject to}\\
&\sum_{i=1}^n x_{ij}^{t+1} \lambda_i \le x_{l^t j}^t \theta_{l^t j}^{t+1}, \quad \text{for } j = 1, 2, \ldots, m,\\
&\sum_{i=1}^n y_{ik}^{t+1} \lambda_i \ge y_{l^t k}^t, \quad \text{for } k = 1, 2, \ldots, p,\\
&\theta_{l^t j}^{t+1} \text{ free},\\
&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n.
\end{aligned} \tag{11.51}
$$

The non-radial input oriented Model 11.52 measures the technical efficiency score of DMU$_l$ ($l$=1,2,. . .,$n$) in time period $t + 1$ according to the generated PPS of all DMUs in time period $t$.

$$
\begin{aligned}
&D_{l^{t+1}}^t = \min\left(\sum_{j=1}^m \alpha_j \theta_{l^{t+1} j}^t\right) / \sum_{j=1}^m \alpha_j,\\
&\text{Subject to}\\
&\sum_{i=1}^n x_{ij}^t \lambda_i \le x_{l^{t+1} j}^t \theta_{l^{t+1} j}^t, \quad \text{for } j = 1, 2, \ldots, m,\\
&\sum_{i=1}^n y_{ik}^t \lambda_i \ge y_{l^{t+1} k}^t, \quad \text{for } k = 1, 2, \ldots, p,\\
&\theta_{l^{t+1} j}^t \text{ free},\\
&\lambda_i \ge 0, \quad \text{for } i = 1, 2, \ldots, n.
\end{aligned} \tag{11.52}
$$

The non-radial input oriented Model 11.53 calculates the technical efficiency score of DMU$_l$ ($l$=1,2,. . .,$n$) in time period $t + 1$ according to the generated PPS of all DMUs in time period $t + 1$.

$$
\begin{aligned}
&D^{t+1}_{l^{t+1}} = \min\left(\sum_{j=1}^{m} \alpha_j \theta^{t+1}_{l^{t+1}j}\right) / \sum_{j=1}^{m} \alpha_j,\\
&\text{Subject to}\\
&\sum_{i=1}^{n} x^{t+1}_{ij} \lambda_i \le x^{t}_{l^{t+1}j} \theta^{t+1}_{l^{t+1}j}, \text{for } j = 1, 2, \ldots, m,\\
&\sum_{i=1}^{n} y^{t+1}_{ik} \lambda_i \ge y^{t}_{l^{t+1}k}, \text{for } k = 1, 2, \ldots, p,\\
&\theta^{t+1}_{l^{t+1}j} \text{ free},\\
&\lambda_i \ge 0, \text{for } i = 1, 2, \ldots, n.
\end{aligned}
\tag{11.53}
$$

Equation 11.54 represents the non-radial input oriented Malmquist efficiency index

$$
ME_l = \frac{D^{t+1}_{l^{t+1}}}{D^{t}_{l^{t}}} \times \left( \frac{D^{t}_{l^{t}}}{D^{t+1}_{l^{t}}} \times \frac{D^{t}_{l^{t+1}}}{D^{t+1}_{l^{t+1}}} \right)^{1/2}. \tag{11.54}
$$

The first term in Eq. 11.54, that is, $D^{t+1}_{l^{t+1}} / D^{t}_{l^{t}}$, calculates the weighted non-radial input oriented efficiency change, and the second term, that is, $\left(D^{t}_{l^{t}} / D^{t+1}_{l^{t}}\right)\left(D^{t}_{l^{t+1}} / D^{t+1}_{l^{t+1}}\right)$ calculates the frontier shift in preferred empirical production function. It is also possible that the two different empirical production frontiers (created in two different time periods) have intersections and some sides (facets) shift backwards and some forwards. In other words, the sides of empirical production frontier may not all shift in one direction. In such a situation, the movement of empirical production frontier is DMU-specific, that is, the Malmquist efficiency index calculates the performance of a specific DMU in terms of the movement of its referent DMUs.

## 11.6 An Example of Three Major Chinese Industries

In this section, the radial and non-radial models are applied to calculate the efficiency change and the impact of economic development plans on efficiency changes of three Chinese major industries: (1) Textiles, (2) Chemicals and (3) metallurgy during five-year-plan in four different periods, from 1966 to 1985.

Table 11.15 illustrates the five DMUs and the different time periods. Each DMU is defined as a year in the first period from 1966 to 1970. The year after 5 years corresponding to the first period, that is, 1971–1975, 1976–1980, and 1981–1985 are

**Table 11.15** An example of four five-year plan periods

| DMUs\Period | $t = 1$ | $t = 2$ | $t = 3$ | $t = 4$ |
|---|---|---|---|---|
| DMU1 | 1966 | 1971 | 1976 | 1981 |
| DMU2 | 1967 | 1972 | 1977 | 1982 |
| DMU3 | 1968 | 1973 | 1978 | 1983 |
| DMU4 | 1969 | 1974 | 1979 | 1984 |
| DMU5 | 1970 | 1975 | 1980 | 1985 |

defined as the same DMU1-DMU5 in second, third and fourth periods. For example, DMU 1 is the year 1966 in the first period, the year 1971 in the second period, the year 1976 in the third period and the year 1981 in the fourth period. In other words, the first years in all five-year-plan periods are considered as the same DMU, but in different time periods.

Note that a series of 5-year plans were launched by the Chinese government since the year 1953. Here, $t = 1$ refers to the third five-year-plan period, and $t = 2$ refers to the fourth five-year-plan period, and so on. In other words, the period 1966–1985 refers to four five-year-plan periods during which the third five-year-plan period came on stream during 1966–1970. In each five-year plan, some economic development targets and plans were arranged. Thus, it is meaningful to review the efficiency change between two sequential five-year-plan periods, in order to measure the impact of economic development plans.

In order to characterize an industry in which is labor intensive, the textile industry is considered. The chemical industry is considered to characterize an industry which is capital intensive.

In addition, the metallurgy industry is considered to characterize an industry which both labor intensive and capital intensive. These selections allow us to determine the use of the discussed Malmquist efficiency indexes.

Tables 11.16, 11.17 and 11.18 illustrates the data for the three major Chinese industries: (1) Textiles, (2) Chemicals and (3) metallurgy from 1966 to 1985. Each

**Table 11.16** The data of textiles industry

| DMUs/periods | Year | Capital | Labor | AGIOV |
|---|---|---|---|---|
| $T_1^1$ | 1966 | 48,002 | 106,910 | 145,163 |
| $T_2^1$ | 1967 | 38,135 | 107,047 | 121,057 |
| $T_3^1$ | 1968 | 40,584 | 111,428 | 135,572 |
| $T_4^1$ | 1969 | 50,454 | 115,222 | 177,849 |
| $T_5^1$ | 1970 | 49,218 | 122,428 | 183,630 |
| $T_1^2$ | 1971 | 43,604 | 127,393 | 160,291 |
| $T_2^2$ | 1972 | 44,677 | 131,991 | 161,853 |
| $T_3^2$ | 1973 | 50,958 | 133,555 | 181,968 |
| $T_4^2$ | 1974 | 53,419 | 134,456 | 188,066 |
| $T_5^2$ | 1975 | 59,430 | 135,642 | 206,317 |
| $T_1^3$ | 1976 | 57,914 | 139,195 | 196,584 |
| $T_2^3$ | 1977 | 56,410 | 145,303 | 187,317 |
| $T_3^3$ | 1978 | 70,558 | 125,627 | 229,308 |
| $T_4^3$ | 1979 | 73,542 | 134,303 | 258,529 |
| $T_5^3$ | 1980 | 87,180 | 149,533 | 313,734 |
| $T_1^4$ | 1981 | 105,123 | 166,794 | 361,155 |
| $T_2^4$ | 1982 | 122,385 | 194,801 | 347,229 |
| $T_3^4$ | 1983 | 143,098 | 202,171 | 369,631 |
| $T_4^4$ | 1984 | 276,615 | 218,937 | 465,235 |
| $T_5^4$ | 1985 | 311,977 | 214,961 | 492,053 |

**Table 11.17** The data of chemicals industry

| DMUs/periods | Year | Capital | Labor | AGIOV |
|---|---|---|---|---|
| $C_1^1$ | 1966 | 57,785 | 68,800 | 116,884 |
| $C_2^1$ | 1967 | 47,570 | 68,959 | 100,354 |
| $C_3^1$ | 1968 | 46,565 | 71,665 | 102,307 |
| $C_4^1$ | 1969 | 64,768 | 77,788 | 160,948 |
| $C_5^1$ | 1970 | 67,488 | 85,612 | 171,774 |
| $C_1^2$ | 1971 | 77,468 | 94,896 | 195,428 |
| $C_2^2$ | 1972 | 79,225 | 100,775 | 197,665 |
| $C_3^2$ | 1973 | 84,557 | 103,336 | 209,838 |
| $C_4^2$ | 1974 | 87,820 | 105,591 | 219,256 |
| $C_5^2$ | 1975 | 92,777 | 110,636 | 232,676 |
| $C_1^3$ | 1976 | 83,593 | 114,661 | 209,057 |
| $C_2^3$ | 1977 | 87,818 | 124,392 | 214,105 |
| $C_3^3$ | 1978 | 107,463 | 125,328 | 272,935 |
| $C_4^3$ | 1979 | 117,348 | 126,285 | 278,142 |
| $C_5^3$ | 1980 | 125,644 | 144,818 | 286,499 |
| $C_1^4$ | 1981 | 143,419 | 153,201 | 309,717 |
| $C_2^4$ | 1982 | 160,309 | 159,691 | 338,989 |
| $C_3^4$ | 1983 | 173,788 | 160,238 | 354,515 |
| $C_4^4$ | 1984 | 188,013 | 177,426 | 373,221 |
| $C_5^4$ | 1985 | 222,236 | 169,310 | 410,720 |

DMU has two input factors, Capital and Labor, and a single output, Annual Gross Industrial Output Value (AGIOV). These factors were considered to be consistent with the previous researchers and are enough to run the proposed models. For each industry, the single output factor, AGIOV, and the input factor, Capital, are measured in 10,000 RMB1 by the 1980 official prices. The labor factor is the number of workers and staff in the corresponding industry.

The selected input and output factors for these three industries are the used key measures by the Chinese government in assessing the industrial performance. The data are also gathered from the Yearbook of China's 40 Years which is issued by the Chinese Statistical Bureau.

This is obvious and completely reasonable that DEA is sensitive to variable selection. Indeed, adding/removing a factor or a DMU definitely should change the results, as DEA is naturally consistent with the real-life situation. DEA estimates the production frontier based upon the observed DMUs while there are no guesses or any known relationships between factors. It is almost always impossible to find causation from the observed data, thus, it is not realistic to assume that the estimated production frontier by DEA should be unique or not sensible to variable selection. In other words, managers usually consider the factors that might be related in measuring the performance of a set of firms. The strong logic of DEA compares each pair of

**Table 11.18** The data of metallurgy industry

| DMUs/periods | Year | Capital | Labor | AGIOV |
|---|---|---|---|---|
| $M_1^1$ | 1966 | 26,504 | 24,974 | 51,060 |
| $M_2^1$ | 1967 | 27,400 | 25,603 | 50,087 |
| $M_3^1$ | 1968 | 29,940 | 27,430 | 46,259 |
| $M_4^1$ | 1969 | 38,374 | 29,174 | 59,358 |
| $M_5^1$ | 1970 | 31,720 | 37,580 | 64,959 |
| $M_1^2$ | 1971 | 36,777 | 27,152 | 70,446 |
| $M_2^2$ | 1972 | 30,634 | 30,754 | 89,592 |
| $M_3^2$ | 1973 | 39,781 | 35,151 | 112,588 |
| $M_4^2$ | 1974 | 39,988 | 32,170 | 121,013 |
| $M_5^2$ | 1975 | 49,216 | 31,933 | 132,093 |
| $M_1^3$ | 1976 | 42,909 | 33,849 | 109,393 |
| $M_2^3$ | 1977 | 47,857 | 36,506 | 119,411 |
| $M_3^3$ | 1978 | 68,538 | 36,183 | 171,228 |
| $M_4^3$ | 1979 | 78,746 | 35,363 | 185,703 |
| $M_5^3$ | 1980 | 86,504 | 37,403 | 196,998 |
| $M_1^4$ | 1981 | 89,003 | 39,412 | 199,055 |
| $M_2^4$ | 1982 | 94,149 | 40,005 | 208,897 |
| $M_3^4$ | 1983 | 103,925 | 45,650 | 226,187 |
| $M_4^4$ | 1984 | 108,083 | 41,094 | 228,109 |
| $M_5^4$ | 1985 | 125,269 | 41,146 | 235,981 |

firms based on the selected observed factors only. Even if the real production frontier is known, it is possible that none of the DMUs lie on that real production frontier, but DEA does focus on reality and considers the best practices as references for inefficient DMUs.

Of course, it is good to establish a priori of the existence of an association between the inputs and outputs. In this example, there is a strong association between the two input factors and the output factor in Tables 11.16, 11.17 and 11.18.

The input oriented CCR model is used in this example, as success in meeting physical targets is important for the Chinese government and managers were rewarded from this view.

The following instructions show the steps to measure the components of the Malmquist efficiency index for the textile industry in the first and second periods.

1. Copy data from 1966 to 1975 in Table 11.16 on an Excel sheet into cells A1: D11, as Fig. 11.19 illustrates.
2. Label E1 as 'Lambdas', F1 as 'Firm', G1 as '$D_1^1$', H1 as '$D_1^2$', I1 as '$D_2^1$', J1 as '$D_2^2$', K1 as 'Malmquist Index', A13 as 'Theta', G13 as 'Index 1', G14 as 'Index 2', A16 as 'Model 1', A17 as 'Model 2', A18 as 'Model 3', A19 as 'Model 4', and A21 as 'Selected Model' (Fig. 11.20).
3. Assign number 1 to H13 and H14.

**Fig. 11.19** Copying data in Table 11.16 in an excel sheet

| | A | B | C | D |
|---|---|---|---|---|
| 1 | Year \ Textiles | Capital | Labor | AGIOV |
| 2 | 1966 | 48002 | 106910 | 145163 |
| 3 | 1967 | 38135 | 107047 | 121057 |
| 4 | 1968 | 40584 | 111428 | 135572 |
| 5 | 1969 | 50454 | 115222 | 177849 |
| 6 | 1970 | 49218 | 122428 | 183630 |
| 7 | 1971 | 43604 | 127393 | 160291 |
| 8 | 1972 | 44677 | 131991 | 161853 |
| 9 | 1973 | 50958 | 133555 | 181968 |
| 10 | 1974 | 53419 | 134456 | 188066 |
| 11 | 1975 | 59430 | 135642 | 206317 |

F16 =INDEX(B2:B6,$H12)*$B13

| | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Year \ Textiles | Capital | Labor | AGIOV | Lambda | CCR-Scores | $D_1^1$ | $D_1^2$ | $D_2^1$ | $D_2^2$ | M |
| 2 | 1966 | 48002 | 106910 | 145163 | | DMU1 | | | | | |
| 3 | 1967 | 38135 | 107047 | 121057 | | DMU2 | | | | | |
| 4 | 1968 | 40584 | 111428 | 135572 | | DMU3 | | | | | |
| 5 | 1969 | 50454 | 115222 | 177849 | | DMU4 | | | | | |
| 6 | 1970 | 49218 | 122428 | 183630 | | DMU5 | | | | | |
| 7 | 1971 | 43604 | 127393 | 160291 | | | | | | | |
| 8 | 1972 | 44677 | 131991 | 161853 | | | | | | | |
| 9 | 1973 | 50958 | 133555 | 181968 | | Run | | | | | |
| 10 | 1974 | 53419 | 134456 | 188066 | | | | | | | |
| 11 | 1975 | 59430 | 135642 | 206317 | | | | | | | |
| 12 | | | | | | | Index 1 | 1 | | | |
| 13 | Theta | 1 | | | | | Index 2 | 1 | | | |
| 14 | | | | | | | | | | | |
| 15 | | | | | | | | | | | |
| 16 | 1 | 0 | 0 | 0 | | 48002 | 106910 | 145163 | | | |
| 17 | 2 | 0 | 0 | 0 | | 48002 | 106910 | 145163 | | | |
| 18 | 3 | 0 | 0 | 0 | | 43604 | 127393 | 160291 | | | |
| 19 | 4 | 0 | 0 | 0 | | 43604 | 127393 | 160291 | | | |
| 20 | | | | | | | | | | | |
| 21 | Selected | 0 | 0 | 0 | | 48002 | 106910 | 145163 | | | |

**Fig. 11.20** Setting Excel sheet for data in Table 11.16

4. Assign the following command into B16 and B18,

   '=Sumproduct(B2:B6,$E2:$E6)'.

5. Copy B16 and paste it into C16 and D16.
6. Copy B18 and paste it into C18 and D18.

7. Assign the following command into B17 and B19,

   '=Sumproduct(B7:B11,$E7:$E11)'.

8. Copy B17 and paste it into C17 and D17.
9. Copy B19 and paste it into C19 and D19.
10. Assign the following command into F16 and F17,

    '=Index(B2:B6,$H12)*$B13'.

11. Copy F16 and paste it into G16.
12. Copy F17 and paste it into G17.
13. Assign the following command into F18 and F19,

    '=Index(B7:B11,$H12)*$B13'.

14. Copy F18 and paste it into G18.
15. Copy F19 and paste it into G19.
16. Assign the following command into H16 and H17,

    '=Index(D2:D6,$H12)'.

17. Assign the following command into H18 and H19,

    '=Index(D7:D11,$H12)'.

18. Assign the following command into B21,

    '=Index(B16:B19,$H13)'.

19. Copy B21 and paste it to C21, D21, F21, G21 and H21.
20. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 11.21 illustrates.
21. Assign 'B13' into 'Set Objective' and choose 'Min'.
22. Assign 'E2:E11, B13' into 'By Changing Variable Cells'.
23. Click on 'Add' and assign 'B21:C21' into 'Cell Reference', then select '<=', and assign 'F21:G21' into 'Constraint'.
24. Click on 'Add' and assign 'D21' into 'Cell Reference', then select '>=' and assign 'H21' into 'Constraint'. Then click on 'OK'.
25. Tick 'Make Unconstrained Variables Non-Negative'.
26. Choose 'Simplex LP' from 'Select a Solving Method'.
27. Click on 'Solve'.
28. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.
29. Click on the first icon, 'Button (Form Control)', and then click on a place on the Excel sheet.
30. In the opened window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.

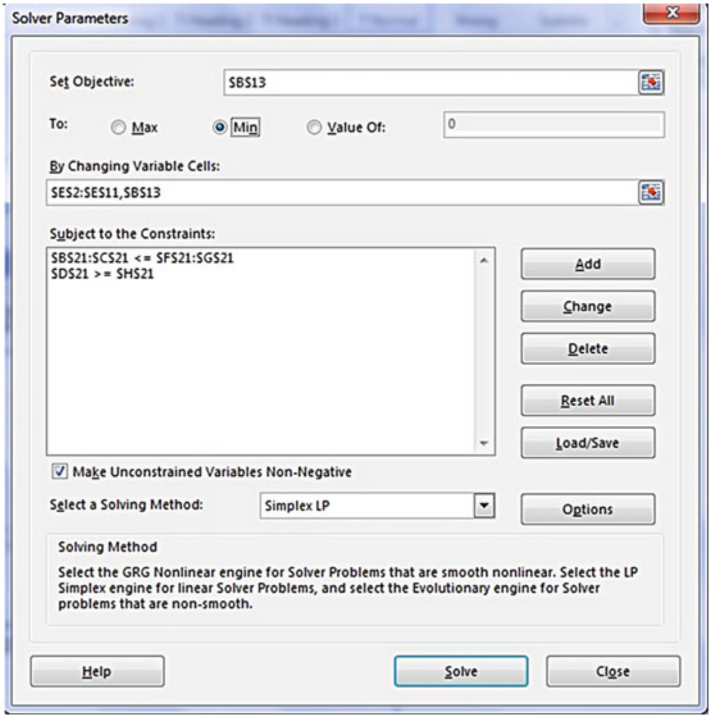



**Fig. 11.21** Setting solver for data in Table 11.16

31. Inside of the 'Microsoft Visual Basic for Applications' window, write the following commands between 'Sub Button1_Click ()' and 'End Sub', as Fig. 11.22 shows.

```
Dim i, j As Integer
For i = 1 To 5
    Range("H12") = i
    For j = 1 To 4
        Range("H13") = j
        SolverSolve Userfinish:=True
        Cells(i + 1, j + 6) = Range("B13")
    Next j
Next i
For i = 1 To 5
    Range("K" & i + 1) = ((Range("I" & i + 1) * Range("J" & i + 1)) / _
                    (Range("H" & i + 1) * Range("G" & i + 1))) ^ 0.5
Next i
```

32. Close the 'Microsoft Visual Basic for Applications' window.

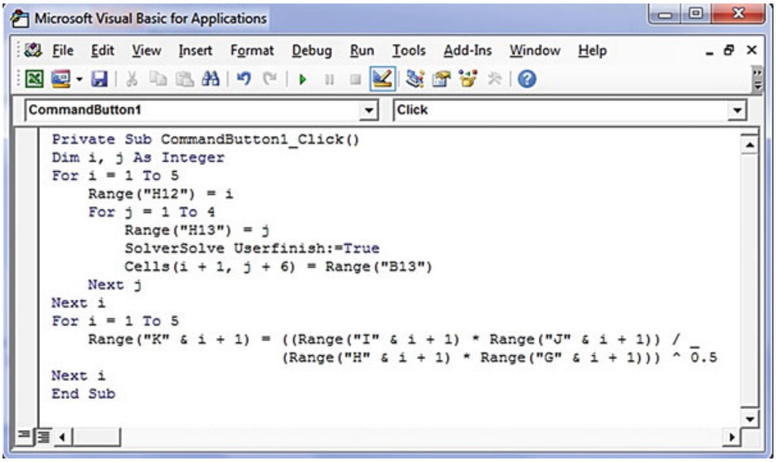
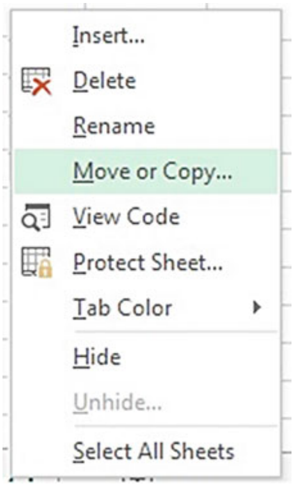


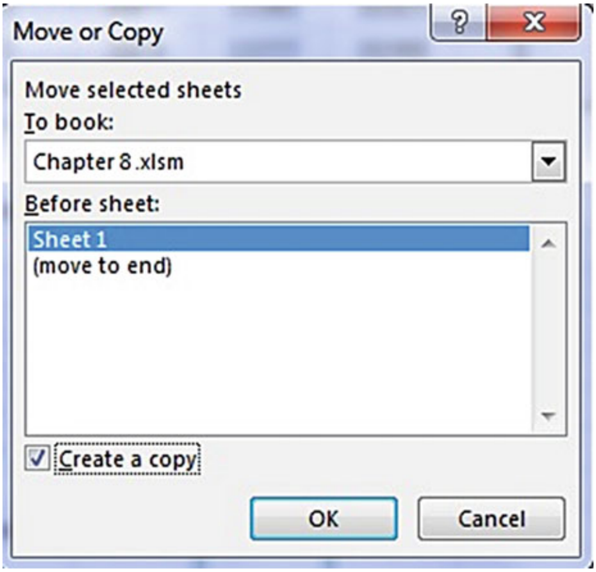

**Fig. 11.22** Setting VBA for data in Table 11.16

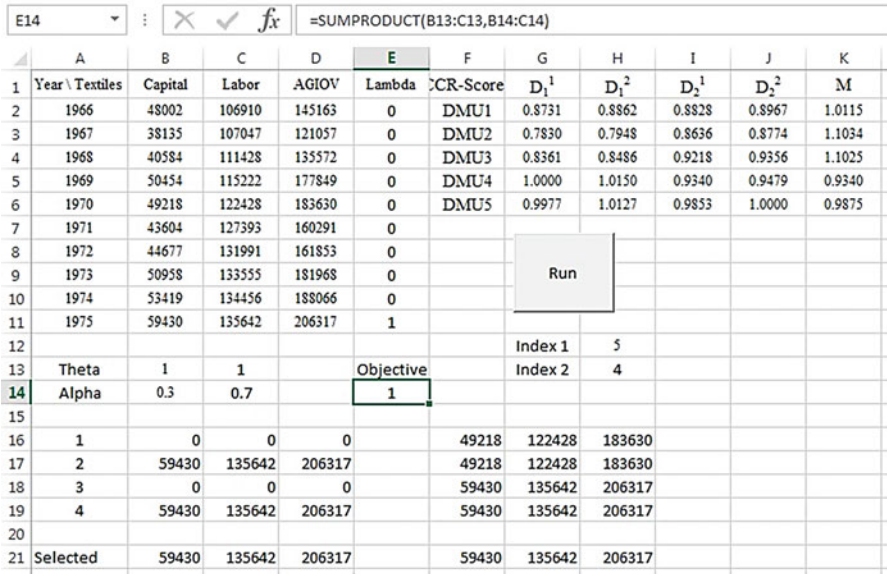
**Fig. 11.23** make a copy of a worksheet




33. Click on the small rectangle which was automatically made on the Excel sheet and created by step 29. The results are represented to cells G2:K6.
34. Right click on the worksheet tab and select 'Move or Copy . . .', as Fig. 11.23 shows.
35. Select 'Create a copy', as Fig. 11.24 depicts, and click on 'OK'.
36. Label A14 as 'Alpha', and E13 as 'Objective', as Fig. 11.25 shows.
37. Assign 0.3 and 0.7 into B14 and C14.

**Fig. 11.24** Create a copy of a worksheet




| | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Year \ Textiles | Capital | Labor | AGIOV | Lambda | CCR-Score | $D_1^1$ | $D_1^2$ | $D_2^1$ | $D_2^2$ | M |
| 2 | 1966 | 48002 | 106910 | 145163 | 0 | DMU1 | 0.8731 | 0.8862 | 0.8828 | 0.8967 | 1.0115 |
| 3 | 1967 | 38135 | 107047 | 121057 | 0 | DMU2 | 0.7830 | 0.7948 | 0.8636 | 0.8774 | 1.1034 |
| 4 | 1968 | 40584 | 111428 | 135572 | 0 | DMU3 | 0.8361 | 0.8486 | 0.9218 | 0.9356 | 1.1025 |
| 5 | 1969 | 50454 | 115222 | 177849 | 0 | DMU4 | 1.0000 | 1.0150 | 0.9340 | 0.9479 | 0.9340 |
| 6 | 1970 | 49218 | 122428 | 183630 | 0 | DMU5 | 0.9977 | 1.0127 | 0.9853 | 1.0000 | 0.9875 |
| 7 | 1971 | 43604 | 127393 | 160291 | 0 | | | | | | |
| 8 | 1972 | 44677 | 131991 | 161853 | 0 | | | | | | |
| 9 | 1973 | 50958 | 133555 | 181968 | 0 | | | | | | |
| 10 | 1974 | 53419 | 134456 | 188066 | 0 | | | | | | |
| 11 | 1975 | 59430 | 135642 | 206317 | 1 | | | | | | |
| 12 | | | | | | | Index 1 | 5 | | | |
| 13 | Theta | 1 | 1 | | Objective | | Index 2 | 4 | | | |
| 14 | Alpha | 0.3 | 0.7 | | 1 | | | | | | |
| 15 | | | | | | | | | | | |
| 16 | 1 | 0 | 0 | 0 | | 49218 | 122428 | 183630 | | | |
| 17 | 2 | 59430 | 135642 | 206317 | | 49218 | 122428 | 183630 | | | |
| 18 | 3 | 0 | 0 | 0 | | 59430 | 135642 | 206317 | | | |
| 19 | 4 | 59430 | 135642 | 206317 | | 59430 | 135642 | 206317 | | | |
| 20 | | | | | | | | | | | |
| 21 | Selected | 59430 | 135642 | 206317 | | 59430 | 135642 | 206317 | | | |

**Fig. 11.25** Setting Excel for taking the alpha values

38. Assign the following command into G16 and G17,

    '=Index(C2:C6,$H12)*C13'.

39. Assign the following command into B21,

    '=Index(C7:C11,$H12)*C13'.

40. Assign the following command into E14,

    '=Sumproduct(B13:C13,B14:C14)'.

41. Open 'Solver Parameters' window, from 'DATA' in toolbar menu, as Fig. 11.26 illustrates.
42. Assign 'E14' into 'Set Objective' and choose 'Min'.
43. Assign 'E2:E11, B13:C13' into 'By Changing Variable Cells'.
44. Click on 'Solve'.
45. Right click on the small rectangle which was automatically made on the Excel sheet and created by step 29, and click on 'Assign Macro...' (Fig. 11.27).

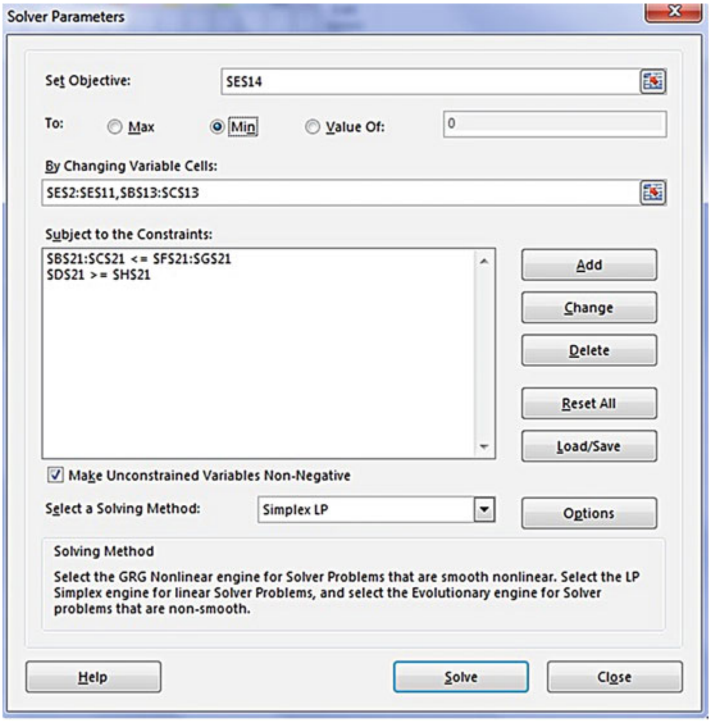



**Fig. 11.26** Setting VBA for program in Fig. 11.25

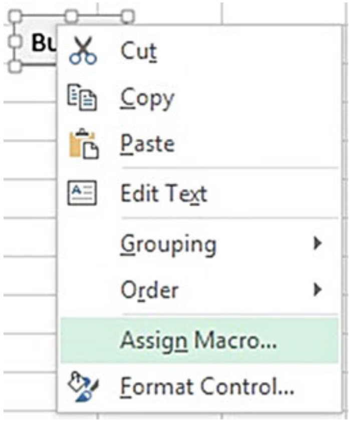



**Fig. 11.27** Assign a macro to a worksheet

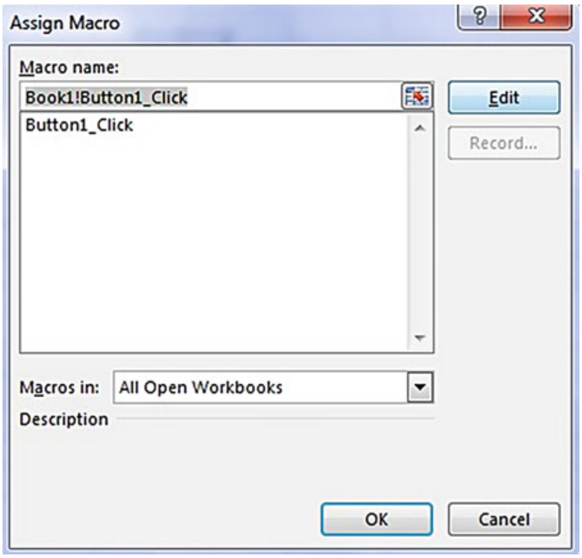



**Fig. 11.28** Edit a macro in a worksheet

46. Click on 'Edit', as shown in Fig. 11.28, and instead 'Range("B13")' in the command 'Cells(i + 1, j + 6) = Range("B13")', write Range("E14").

In other words, inside of the 'Microsoft Visual Basic for Applications' window, the following commands between 'Sub Button1_Click ()' and 'End Sub', should be written.

```
Dim i, j As Integer
For i = 1 To 5
   Range("H12") = i
   For j = 1 To 4
      Range("H13") = j
      SolverSolve Userfinish:=True
      Cells(i + 1, j + 6) = Range("E14")
   Next j
Next i
For i = 1 To 5
   Range("K" & i + 1) = ((Range("I" & i + 1) * Range("J" & i + 1)) / _
                    (Range("H" & i + 1) * Range("G" & i + 1))) ^ 0.5
Next i
```

47. Close the ‘Microsoft Visual Basic for Applications’ window.
48. Click on the small rectangle. The results are represented to cells G2:K6.

Tables 11.19, 11.20, 11.21, 11.22, 11.23, 11.24, 11.25, 11.26, 11.27, 11.28, 11.29, 11.30, 11.31, 11.32, 11.33, 11.34, 11.35 and 11.36 illustrates the results of the CCR radial model to measure the Malmquist efficiency indexes for the three Chinese industries during four different time periods from 1966 to 1985.

The first two tables, Tables 11.19 and 11.20, represent the result of the model for the textile industry from the first period to the second period. Table 11.19 illustrates

**Table 11.19** Textile radial performance from 1st period to 2nd

| DMUs | $D_1^1$ | $D_1^2$ | $D_2^1$ | $D_2^2$ | M |
|---|---|---|---|---|---|
| $T_1$ | 0.8797 | 0.8927 | 0.9853 | 1.0000 | 1.1201 |
| $T_2$ | 0.8508 | 0.8722 | 0.9710 | 0.9855 | 1.1355 |
| $T_3$ | 0.8954 | 0.9229 | 0.9571 | 0.9975 | 1.0749 |
| $T_4$ | 1.0000 | 1.0153 | 0.9436 | 0.9927 | 0.9605 |
| $T_5$ | 1.0000 | 1.0548 | 0.9854 | 1.0000 | 0.9666 |

**Table 11.20** Textile radial Malmquist index from 1st period to 2nd

| DMUs | $D_2^2/D_1^1$ | $D_1^1/D_1^2$ | $D_2^1/D_2^2$ | $\left(D_1^1 D_2^1/\left(D_1^2 D_2^2\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $T_1$ | 1.1368 | 0.9854 | 0.9853 | 0.9854 |
| $T_2$ | 1.1583 | 0.9755 | 0.9853 | 0.9804 |
| $T_3$ | 1.1141 | 0.9702 | 0.9595 | 0.9648 |
| $T_4$ | 0.9927 | 0.9850 | 0.9506 | 0.9676 |
| $T_5$ | 1.0000 | 0.9481 | 0.9854 | 0.9666 |

**Table 11.21** Textile radial performance from 2nd period to 3rd

| DMUs | $D_2^2$ | $D_2^3$ | $D_3^2$ | $D_3^3$ | M |
|---|---|---|---|---|---|
| $T_1$ | 1.0000 | 1.0215 | 0.9670 | 0.9432 | 0.9449 |
| $T_2$ | 0.9855 | 1.0067 | 0.9313 | 0.9227 | 0.9307 |
| $T_3$ | 0.9975 | 0.9923 | 1.2000 | 0.9031 | 1.0464 |
| $T_4$ | 0.9927 | 0.9783 | 1.2656 | 0.9769 | 1.1283 |
| $T_5$ | 1.0000 | 0.9647 | 1.3794 | 1.0000 | 1.1958 |

**Table 11.22** Textile radial Malmquist index from 2nd period to 3rd

| DMUs | $D_3^3/D_2^2$ | $D_2^2/D_2^3$ | $D_3^2/D_3^3$ | $\left(D_2^2D_3^2/\left(D_2^3D_3^3\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $T_1$ | 0.9432 | 0.9790 | 1.0252 | 1.0018 |
| $T_2$ | 0.9363 | 0.9790 | 1.0093 | 0.9940 |
| $T_3$ | 0.9053 | 1.0053 | 1.3288 | 1.1558 |
| $T_4$ | 0.9841 | 1.0147 | 1.2955 | 1.1465 |
| $T_5$ | 1.0000 | 1.0366 | 1.3794 | 1.1958 |

**Table 11.23** Textile radial performance from 3rd period to 4th

| DMUs | $D_3^3$ | $D_3^4$ | $D_4^3$ | $D_4^4$ | M |
|---|---|---|---|---|---|
| $T_1$ | 0.9432 | 0.9880 | 1.0320 | 1.0000 | 1.0523 |
| $T_2$ | 0.9227 | 0.9666 | 0.8496 | 0.8258 | 0.8869 |
| $T_3$ | 0.9031 | 0.9460 | 0.8714 | 0.8398 | 0.9256 |
| $T_4$ | 0.9769 | 1.0232 | 1.0128 | 0.9400 | 0.9759 |
| $T_5$ | 1.0000 | 1.0475 | 1.0910 | 1.0000 | 1.0206 |

**Table 11.24** Textile radial Malmquist index from 3rd period to 4th

| DMUs | $D_4^4/D_3^3$ | $D_3^3/D_3^4$ | $D_4^3/D_4^4$ | $\left(D_3^3D_4^3/\left(D_3^4D_4^4\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $T_1$ | 1.0602 | 0.9547 | 1.0320 | 0.9926 |
| $T_2$ | 0.8950 | 0.9547 | 1.0287 | 0.9910 |
| $T_3$ | 0.9300 | 0.9547 | 1.0376 | 0.9953 |
| $T_4$ | 0.9622 | 0.9547 | 1.0775 | 1.0142 |
| $T_5$ | 1.0000 | 0.9547 | 1.0910 | 1.0206 |

**Table 11.25** Chemicals radial performance from 1st period to 2nd

| DMUs | $D_1^1$ | $D_1^2$ | $D_2^1$ | $D_2^2$ | M |
|---|---|---|---|---|---|
| $C_1$ | 0.8211 | 0.8078 | 1.0065 | 1.0000 | 1.2319 |
| $C_2$ | 0.8288 | 0.8363 | 0.9802 | 0.9890 | 1.1827 |
| $C_3$ | 0.8632 | 0.8709 | 0.9911 | 0.9842 | 1.1391 |
| $C_4$ | 1.0000 | 0.9893 | 1.0042 | 0.9937 | 1.0043 |
| $C_5$ | 1.0000 | 1.0089 | 1.0164 | 1.0000 | 1.0037 |

**Table 11.26** Chemicals radial Malmquist index from 1st period to 2nd

| DMUs | $D_2^2/D_1^1$ | $D_1^1/D_1^2$ | $D_2^1/D_2^2$ | $\left(D_1^1D_2^1/\left(D_1^2D_2^2\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $C_1$ | 1.2179 | 1.0164 | 1.0065 | 1.0115 |
| $C_2$ | 1.1933 | 0.9911 | 0.9911 | 0.9911 |
| $C_3$ | 1.1402 | 0.9911 | 1.0070 | 0.9990 |
| $C_4$ | 0.9937 | 1.0108 | 1.0105 | 1.0107 |
| $C_5$ | 1.0000 | 0.9911 | 1.0164 | 1.0037 |

**Table 11.27** Chemicals radial performance from 2nd period to 3rd

| DMUs | $D_2^2$ | $D_2^3$ | $D_3^2$ | $D_3^3$ | M |
|---|---|---|---|---|---|
| $C_1$ | 1.0000 | 0.9933 | 0.9914 | 0.9847 | 0.9914 |
| $C_2$ | 0.9890 | 0.9824 | 0.9664 | 0.9599 | 0.9772 |
| $C_3$ | 0.9842 | 0.9771 | 1.0355 | 1.0000 | 1.0377 |
| $C_4$ | 0.9937 | 0.9830 | 1.0473 | 1.0000 | 1.0354 |
| $C_5$ | 1.0000 | 0.9874 | 0.9407 | 0.9070 | 0.9295 |

**Table 11.28** Chemicals radial Malmquist index from 2nd period to 3rd

| DMUs | $D_3^3/D_2^2$ | $D_2^2/D_2^3$ | $D_3^2/D_3^3$ | $\left(D_2^2D_3^2/\left(D_2^3D_3^3\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $C_1$ | 0.9847 | 1.0068 | 1.0068 | 1.0068 |
| $C_2$ | 0.9706 | 1.0068 | 1.0068 | 1.0068 |
| $C_3$ | 1.0160 | 1.0073 | 1.0355 | 1.0213 |
| $C_4$ | 1.0063 | 1.0109 | 1.0473 | 1.0289 |
| $C_5$ | 0.9070 | 1.0127 | 1.0372 | 1.0249 |

**Table 11.29** Chemicals radial performance from 3rd period to 4th

| DMUs | $D_3^3$ | $D_3^4$ | $D_4^3$ | $D_4^4$ | M |
|---|---|---|---|---|---|
| $C_1$ | 0.9847 | 1.1581 | 0.9179 | 1.0000 | 0.8972 |
| $C_2$ | 0.9599 | 1.1290 | 0.9638 | 1.0000 | 0.9430 |
| $C_3$ | 1.0000 | 1.1761 | 1.0045 | 1.0000 | 0.9242 |
| $C_4$ | 1.0000 | 1.0976 | 0.9551 | 0.9628 | 0.9153 |
| $C_5$ | 0.9070 | 1.0559 | 1.1014 | 1.0000 | 1.0724 |

**Table 11.30** Chemicals radial Malmquist index from 3rd period to 4th

| DMUs | $D_4^4/D_3^3$ | $D_3^3/D_3^4$ | $D_4^3/D_4^4$ | $\left(D_3^3D_4^3/\left(D_3^4D_4^4\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $C_1$ | 1.0156 | 0.8503 | 0.9179 | 0.8834 |
| $C_2$ | 1.0417 | 0.8503 | 0.9638 | 0.9053 |
| $C_3$ | 1.0000 | 0.8503 | 1.0045 | 0.9242 |
| $C_4$ | 0.9628 | 0.9111 | 0.9920 | 0.9507 |
| $C_5$ | 1.1026 | 0.8590 | 1.1014 | 0.9727 |

**Table 11.31** Metallurgy radial performance from 1st period to 2nd

| DMUs | $D_1^1$ | $D_1^2$ | $D_2^1$ | $D_2^2$ | M |
|---|---|---|---|---|---|
| $M_1$ | 0.8414 | 1.2713 | 0.6627 | 1.0000 | 0.7871 |
| $M_2$ | 0.8755 | 1.3262 | 0.8474 | 1.0000 | 0.8543 |
| $M_3$ | 1.0000 | 1.5385 | 0.7458 | 0.9025 | 0.6615 |
| $M_4$ | 0.8290 | 1.2752 | 0.6790 | 0.8014 | 0.7175 |
| $M_5$ | 1.0000 | 1.5324 | 0.5477 | 0.6898 | 0.4965 |

**Table 11.32** Metallurgy radial Malmquist index from 1st period to 2nd

| DMUs | $D_2^2/D_1^1$ | $D_1^1/D_1^2$ | $D_2^1/D_2^2$ | $\left(D_1^1D_2^1/\left(D_1^2D_2^2\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $M_1$ | 1.1885 | 0.6618 | 0.6627 | 0.6623 |
| $M_2$ | 1.1422 | 0.6601 | 0.8474 | 0.7479 |
| $M_3$ | 0.9025 | 0.6500 | 0.8264 | 0.7329 |
| $M_4$ | 0.9667 | 0.6501 | 0.8474 | 0.7422 |
| $M_5$ | 0.6898 | 0.6526 | 0.7939 | 0.7198 |

**Table 11.33** Metallurgy radial performance from 2nd period to 3rd

| DMUs | $D_2^2$ | $D_2^3$ | $D_3^2$ | $D_3^3$ | M |
|---|---|---|---|---|---|
| $M_1$ | 1.0000 | 1.2456 | 0.8715 | 1.0000 | 0.8365 |
| $M_2$ | 1.0000 | 1.2726 | 0.8563 | 0.9880 | 0.8153 |
| $M_3$ | 0.9025 | 1.1201 | 0.5920 | 0.6829 | 0.6324 |
| $M_4$ | 0.8014 | 1.0198 | 0.5253 | 0.6154 | 0.6290 |
| $M_5$ | 0.6898 | 0.8225 | 0.5187 | 0.6136 | 0.7490 |

**Table 11.34** Metallurgy radial Malmquist index from 2nd period to 3rd

| DMUs | $D_3^3/D_2^2$ | $D_2^2/D_2^3$ | $D_3^2/D_3^3$ | $\left(D_2^2D_3^2/(D_2^3D_3^3)\right)^{1/2}$ |
|---|---|---|---|---|
| $M_1$ | 1.0000 | 0.8028 | 0.8715 | 0.8365 |
| $M_2$ | 0.9880 | 0.7858 | 0.8666 | 0.8252 |
| $M_3$ | 0.7567 | 0.8057 | 0.8669 | 0.8358 |
| $M_4$ | 0.7680 | 0.7858 | 0.8536 | 0.8190 |
| $M_5$ | 0.8895 | 0.8387 | 0.8454 | 0.8420 |

**Table 11.35** Metallurgy radial performance from 3rd period to 4th

| DMUs | $D_3^3$ | $D_3^4$ | $D_4^3$ | $D_4^4$ | M |
|---|---|---|---|---|---|
| $M_1$ | 1.0000 | 1.7814 | 0.6399 | 1.0000 | 0.5993 |
| $M_2$ | 0.9880 | 1.7226 | 0.6189 | 0.9618 | 0.5914 |
| $M_3$ | 0.6829 | 1.1922 | 0.6523 | 1.0000 | 0.8950 |
| $M_4$ | 0.6154 | 1.0141 | 0.5822 | 0.8926 | 0.9125 |
| $M_5$ | 0.6136 | 0.9764 | 0.5635 | 0.8639 | 0.9014 |

**Table 11.36** Metallurgy radial Malmquist index from 3rd period to 4th

| DMUs | $D_4^4/D_3^3$ | $D_3^3/D_3^4$ | $D_4^3/D_4^4$ | $\left(D_3^3D_4^3/(D_3^4D_4^4)\right)^{1/2}$ |
|---|---|---|---|---|
| $M_1$ | 1.0000 | 0.5613 | 0.6399 | 0.5993 |
| $M_2$ | 0.9735 | 0.5735 | 0.6435 | 0.6075 |
| $M_3$ | 1.4643 | 0.5728 | 0.6523 | 0.6113 |
| $M_4$ | 1.4504 | 0.6068 | 0.6523 | 0.6291 |
| $M_5$ | 1.4080 | 0.6284 | 0.6523 | 0.6402 |

the scores of CCR by Models 11.10–11.13 as well as the Malmquist efficiency indexes for each DMU. Table 11.20 also shows the components of the Malmquist efficiency indexes.

From Table 11.20, the ratio $D_2^2/D_1^1$ represents the value of technical efficiency change which is greater than 1 for all DMUs except DMU 4. Thus, on the average, the radial distance of DMUs in the second time period to the production frontier in the second time period is closer than the radial distance of DMUs in the first time period to the production frontier in the first period time. In contrast, the production frontier change value is less than 1 for all DMUs. This indicates a negative shift and technical regress occurred. On average, we have Case 4, which is represented in Fig. 11.16, for the textile industry from the first period to the second period. DMU 4 had the worst case, that is, Case 8, indicating decline in both technical efficiency and frontier changes. The Malmquist efficiency indexes in Table 11.19 show performance progress for the first three DMUs and regress for the last two DMUs, from the first period to the second period.

From the second time period to the third time period, the ratio $D_2^2/D_1^1$ is less than 1 for all DMUs except DMU 5, as shown in Tables 11.21 and 11.22. This indicates that, on the average, the radial distance of DMUs in the second time period to the production frontier in the second time period is closer than the radial distance of DMUs in the third time period to the production frontier in the third period time. In addition, the production frontier change value is greater than 1 for all DMUs and we have Case a, that is, DMUs moved onto a side which has positive shift and progress in the production technology.

Overall, technical progress/regress happened after each five-year-plan period in the textile industry. The same illustrations can be written for the rest of the information in Tables 11.16, 11.17, 11.18, 11.19, 11.20, 11.21, 11.22, 11.23, 11.24, 11.25, 11.26, 11.27, 11.28, 11.29, 11.30, 11.31, 11.32, 11.33, 11.34, 11.35 and 11.36.

Now, assume that $\alpha_1 = 0.3$ and $\alpha_2 = 0.7$ for the textile industry, $\alpha_1 = 0.7$ and $\alpha_2 = 0.3$ for the chemical industry, and $\alpha_1 = 0.5$ and $\alpha_2 = 0.5$ for metallurgy. By identifying these weights, we suppose that (i) it is more important to decrease the amount of labor in the textile industry when we plan to improve the performance of the textile industry, because the industry is labor intensive; (ii) it is more important to decrease the amount of capital in the chemical industry when we plan to improve the performance of the chemical industry, because the industry is capital intensive.

In the metallurgy industry, no preference over the two inputs is given. In other words, the metallurgy industry is labor intensive and capital intensive, because the two inputs are equally important.

The results of applying the non-radial Models 11.50–11.53 are illustrated in Tables 11.37, 11.38, 11.39, 11.40, 11.41, 11.42, 11.43, 11.44, 11.45, 11.46, 11.47, 11.48, 11.49, 11.50, 11.51, 11.52, 11.53, and 11.54. Similarly, the average efficiency change along with the average technical efficiency change and the average production frontier movement are illustrated for each industry.

**Table 11.37** Textile non-radial performance from 1st period to 2nd

| DMUs | $D_1^1$ | $D_1^2$ | $D_2^1$ | $D_2^2$ | M |
|---|---|---|---|---|---|
| $T_1$ | 0.8731 | 0.8862 | 0.8828 | 0.8967 | 1.0115 |
| $T_2$ | 0.7830 | 0.7948 | 0.8636 | 0.8774 | 1.1034 |
| $T_3$ | 0.8361 | 0.8486 | 0.9218 | 0.9356 | 1.1025 |
| $T_4$ | 1.0000 | 1.0150 | 0.9340 | 0.9479 | 0.9340 |
| $T_5$ | 0.9977 | 1.0127 | 0.9853 | 1.0000 | 0.9875 |

**Table 11.38** Textile non-radial Malmquist index from 1st period to 2nd

| DMUs | $D_2^2/D_1^1$ | $D_1^1/D_1^2$ | $D_2^1/D_2^2$ | $\left(D_1^1D_2^1/\left(D_1^2D_2^2\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $T_1$ | 1.0270 | 0.9853 | 0.98s45 | 0.9849 |
| $T_2$ | 1.1205 | 0.9852 | 0.9843 | 0.9847 |
| $T_3$ | 1.1191 | 0.9852 | 0.9852 | 0.9852 |
| $T_4$ | 0.9479 | 0.9853 | 0.9852 | 0.9853 |
| $T_5$ | 1.0023 | 0.9852 | 0.9853 | 0.9853 |

**Table 11.39** Textile non-radial performance from 2nd period to 3rd

| DMUs | $D_2^2$ | $D_2^3$ | $D_3^2$ | $D_3^3$ | M |
|---|---|---|---|---|---|
| $T_1$ | 0.8967 | 0.7262 | 0.9433 | 0.7542 | 1.0452 |
| $T_2$ | 0.8774 | 0.7111 | 0.8802 | 0.7069 | 0.9987 |
| $T_3$ | 0.9356 | 0.7523 | 1.1209 | 0.8799 | 1.1838 |
| $T_4$ | 0.9479 | 0.7602 | 1.1897 | 0.9353 | 1.2426 |
| $T_5$ | 1.0000 | 0.7969 | 1.2765 | 1.0000 | 1.2657 |

**Table 11.40** Textile non-radial Malmquist index from 2nd period to 3rd

| DMUs | $D_3^3/D_2^2$ | $D_2^2/D_2^3$ | $D_3^2/D_3^3$ | $\left(D_2^2D_3^2/\left(D_2^3D_3^3\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $T_1$ | 0.8410 | 1.2347 | 1.2508 | 1.2427 |
| $T_2$ | 0.8057 | 1.2338 | 1.2452 | 1.2395 |
| $T_3$ | 0.9405 | 1.2437 | 1.2738 | 1.2587 |
| $T_4$ | 0.9867 | 1.2470 | 1.2720 | 1.2594 |
| $T_5$ | 1.0000 | 1.2549 | 1.2765 | 1.2657 |

**Table 11.41** Textile non-radial performance from 3rd period to 4th

| DMUs | $D_3^3$ | $D_3^4$ | $D_4^3$ | $D_4^4$ | M |
|---|---|---|---|---|---|
| $T_1$ | 0.7542 | 0.7530 | 1.0088 | 1.0000 | 1.3329 |
| $T_2$ | 0.7069 | 0.7067 | 0.8312 | 0.8240 | 1.1709 |
| $T_3$ | 0.8799 | 0.8739 | 0.8253 | 0.8166 | 0.9362 |
| $T_4$ | 0.9353 | 0.9293 | 0.8492 | 0.8338 | 0.9026 |
| $T_5$ | 1.0000 | 0.9925 | 0.8952 | 0.8777 | 0.8897 |

**Table 11.42** Textile non-radial Malmquist index from 3rd period to 4th

| DMUs | $D_4^4/D_3^3$ | $D_3^3/D_3^4$ | $D_4^3/D_4^4$ | $\left(D_3^3D_4^3/\left(D_3^4D_4^4\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $T_1$ | 1.3260 | 1.0016 | 1.0088 | 1.0052 |
| $T_2$ | 1.1656 | 1.0003 | 1.0088 | 1.0045 |
| $T_3$ | 0.9281 | 1.0069 | 1.0107 | 1.0088 |
| $T_4$ | 0.8915 | 1.0065 | 1.0184 | 1.0124 |
| $T_5$ | 0.8777 | 1.0075 | 1.0199 | 1.0137 |

**Table 11.43** Chemicals non-radial performance from 1st period to 2nd

| DMUs | $D_1^1$ | $D_1^2$ | $D_2^1$ | $D_2^2$ | M |
|---|---|---|---|---|---|
| $C_1$ | 0.8103 | 0.8069 | 1.0017 | 0.9979 | 1.2364 |
| $C_2$ | 0.7978 | 0.7964 | 0.9794 | 0.9762 | 1.2267 |
| $C_3$ | 0.8177 | 0.8169 | 0.9861 | 0.9823 | 1.2043 |
| $C_4$ | 0.9928 | 0.9888 | 0.9971 | 0.9931 | 1.0044 |
| $C_5$ | 1.0000 | 0.9966 | 1.0042 | 1.0000 | 1.0038 |

**Table 11.44** Chemicals non-radial Malmquist index from 1st period to 2nd

| DMUs | $D_2^2/D_1^1$ | $D_1^1/D_1^2$ | $D_2^1/D_2^2$ | $\left(D_1^1D_2^1/\left(D_1^2D_2^2\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $C_1$ | 1.2315 | 1.0042 | 1.0038 | 1.0040 |
| $C_2$ | 1.2236 | 1.0017 | 1.0033 | 1.0025 |
| $C_3$ | 1.2013 | 1.0010 | 1.0039 | 1.0024 |
| $C_4$ | 1.0003 | 1.0041 | 1.0041 | 1.0041 |
| $C_5$ | 1.0000 | 1.0034 | 1.0042 | 1.0038 |

**Table 11.45** Chemicals non-radial performance from 2nd period to 3rd

| DMUs | $D_2^2$ | $D_2^3$ | $D_3^2$ | $D_3^3$ | M |
|---|---|---|---|---|---|
| $C_1$ | 0.9979 | 0.9790 | 0.9581 | 0.9404 | 0.9604 |
| $C_2$ | 0.9762 | 0.9578 | 0.9260 | 0.9091 | 0.9488 |
| $C_3$ | 0.9823 | 0.9637 | 1.0196 | 1.0000 | 1.0378 |
| $C_4$ | 0.9931 | 0.9742 | 0.9758 | 0.9567 | 0.9823 |
| $C_5$ | 1.0000 | 0.9809 | 0.9187 | 0.9010 | 0.9186 |

**Table 11.46** Chemicals non-radial Malmquist index from 2nd period to 3rd

| DMUs | $D_3^3/D_2^2$ | $D_2^2/D_2^3$ | $D_3^2/D_3^3$ | $\left(D_2^2D_3^2/\left(D_2^3D_3^3\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $C_1$ | 0.9424 | 1.0193 | 1.0188 | 1.0191 |
| $C_2$ | 0.9312 | 1.0191 | 1.0187 | 1.0189 |
| $C_3$ | 1.0180 | 1.0193 | 1.0196 | 1.0194 |
| $C_4$ | 0.9634 | 1.0194 | 1.0199 | 1.0197 |
| $C_5$ | 0.9010 | 1.0195 | 1.0196 | 1.0195 |

**Table 11.47** Chemicals non-radial performance from 3rd period to 4th

| DMUs | $D_3^3$ | $D_3^4$ | $D_4^3$ | $D_4^4$ | M |
|---|---|---|---|---|---|
| $C_1$ | 0.9404 | 1.0812 | 0.8737 | 1.0000 | 0.9269 |
| $C_2$ | 0.9091 | 1.0457 | 0.8752 | 1.0000 | 0.9595 |
| $C_3$ | 1.0000 | 1.1464 | 0.8670 | 0.9880 | 0.8644 |
| $C_4$ | 0.9567 | 1.0951 | 0.8369 | 0.9544 | 0.8731 |
| $C_5$ | 0.9010 | 1.0327 | 0.8435 | 0.9546 | 0.9303 |

**Table 11.48** Chemicals non-radial Malmquist index from 3rd period to 4th

| DMUs | $D_4^4/D_3^3$ | $D_3^3/D_3^4$ | $D_4^3/D_4^4$ | $\left(D_3^3D_4^3/\left(D_3^4D_4^4\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $C_1$ | 1.0633 | 0.8698 | 0.8737 | 0.8717 |
| $C_2$ | 1.1000 | 0.8693 | 0.8752 | 0.8723 |
| $C_3$ | 0.9880 | 0.8723 | 0.8776 | 0.8749 |
| $C_4$ | 0.9976 | 0.8736 | 0.8769 | 0.8752 |
| $C_5$ | 1.0595 | 0.8725 | 0.8836 | 0.8780 |

**Table 11.49** Metallurgy non-radial performance from 1st period to 2nd

| DMUs | $D_1^1$ | $D_1^2$ | $D_2^1$ | $D_2^2$ | M |
|---|---|---|---|---|---|
| $M_1$ | 1.0000 | 0.5901 | 1.1316 | 0.6613 | 1.1262 |
| $M_2$ | 0.9529 | 0.5621 | 1.4715 | 0.8704 | 1.5465 |
| $M_3$ | 0.8134 | 0.4794 | 1.5178 | 0.8933 | 1.8647 |
| $M_4$ | 0.8990 | 0.5260 | 1.7054 | 1.0000 | 1.8990 |
| $M_5$ | 0.9542 | 0.5681 | 1.7082 | 0.9933 | 1.7691 |

**Table 11.50** Metallurgy non-radial Malmquist index from 1st period to 2nd

| DMUs | $D_2^2/D_1^1$ | $D_1^1/D_1^2$ | $D_2^1/D_2^2$ | $\left(D_1^1D_2^1/\left(D_1^2D_2^2\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $M_1$ | 0.6613 | 1.6947 | 1.7111 | 1.7029 |
| $M_2$ | 0.9135 | 1.6953 | 1.6905 | 1.6929 |
| $M_3$ | 1.0983 | 1.6966 | 1.6991 | 1.6978 |
| $M_4$ | 1.1123 | 1.7092 | 1.7054 | 1.7073 |
| $M_5$ | 1.0409 | 1.6796 | 1.7198 | 1.6996 |

**Table 11.51** Metallurgy non-radial performance from 2nd period to 3rd

| DMUs | $D_2^2$ | $D_2^3$ | $D_3^2$ | $D_3^3$ | M |
|---|---|---|---|---|---|
| $M_1$ | 0.6613 | 0.6532 | 0.8508 | 0.8482 | 1.2926 |
| $M_2$ | 0.8704 | 0.8931 | 0.8470 | 0.8405 | 0.9570 |
| $M_3$ | 0.8933 | 0.9048 | 1.0374 | 0.9803 | 1.1216 |
| $M_4$ | 1.0000 | 0.9998 | 1.0741 | 1.0000 | 1.0365 |
| $M_5$ | 0.9933 | 0.9629 | 1.0609 | 0.9843 | 1.0449 |

**Table 11.52** Metallurgy non-radial Malmquist index from 2nd period to 3rd

| DMUs | $D_3^3/D_2^2$ | $D_2^2/D_2^3$ | $D_3^2/D_3^3$ | $\left(D_2^2D_3^2/\left(D_2^3D_3^3\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $M_1$ | 1.2826 | 1.0125 | 1.0030 | 1.0078 |
| $M_2$ | 0.9656 | 0.9746 | 1.0078 | 0.9911 |
| $M_3$ | 1.0973 | 0.9873 | 1.0583 | 1.0222 |
| $M_4$ | 1.0000 | 1.0002 | 1.0741 | 1.0365 |
| $M_5$ | 0.9910 | 1.0315 | 1.0778 | 1.0544 |

**Table 11.53** Metallurgy non-radial performance from 3rd period to 4th

| DMUs | $D_3^3$ | $D_3^4$ | $D_4^3$ | $D_4^4$ | M |
|---|---|---|---|---|---|
| $M_1$ | 0.8840 | 0.9551 | 0.9848 | 1.1200 | 1.1200 |
| $M_2$ | 0.8755 | 0.9676 | 0.9960 | 1.1444 | 1.1444 |
| $M_3$ | 1.0161 | 0.9332 | 0.9619 | 0.9493 | 0.9493 |
| $M_4$ | 1.0317 | 0.9760 | 1.0000 | 0.9726 | 0.9726 |
| $M_5$ | 1.0139 | 0.9455 | 0.9629 | 0.9551 | 0.9551 |

**Table 11.54** Metallurgy non-radial Malmquist index from 3rd period to 4th

| DMUs | $D_4^4/D_3^3$ | $D_3^3/D_3^4$ | $D_4^3/D_4^4$ | $\left(D_3^3D_4^3/\left(D_3^4D_4^4\right)\right)^{1/2}$ |
|---|---|---|---|---|
| $M_1$ | 1.1610 | 0.9596 | 0.9698 | 0.9647 |
| $M_2$ | 1.1851 | 0.9600 | 0.9715 | 0.9657 |
| $M_3$ | 0.9813 | 0.9647 | 0.9702 | 0.9674 |
| $M_4$ | 1.0000 | 0.9693 | 0.9760 | 0.9726 |
| $M_5$ | 0.9782 | 0.9708 | 0.9819 | 0.9763 |

When a larger weight is defined on the labor input factor in the textile industry by the non-radial model, on average the efficiency improvement can be seen in each time period, whereas there was a decline in efficiency from the third to fourth period by the radial model. In other words, the efficiency improvement from the first period to the second period was 23% on average, from the second period to the third period was 15% on average, and from third period to fourth period was 5% on average. In addition, the non-radial model results that, on average, the technical progress and positive frontier shift happened after each five-year-plan period in the textile industry.

On the other hand, when a larger weight is defined on the capital input in the chemical industry, the results are almost the same on average. Nevertheless, while the radial model results that on average the efficiency declined in each five-year-plan period in the metallurgy industry, the non-radial results opposite. This means, on average there was efficiency progress in each five-year-plan period in the metallurgy industry. The improvement in efficiency was on average 64% from the first period to the second period, 9% from the second period to the third period, and 3% from the third to the fourth period. This is due to the fact that the radial efficiency index only considers the proportional changes of all inputs and ignores the non-zero slacks.

The newly defined non-radial Malmquist efficiency index represents that on average efficiency improvement happened in each industry from the third to the fourth five-year-plan period, while the radial Malmquist efficiency index represents the opposite outcomes. In fact, the Chinese industrial growth was especially rapid in the early 1960s, except for a dip early in the 1966–1976 Cultural Revolution period. This shows that the non-radial Malmquist efficiency index delivers outcomes in consistent with the variations during the 1978–1983 economic reform period.

## 11.7 Conclusion

In this chapter, the DEA Malmquist productivity approach is discussed. Each individual component of Malmquist productivity index is analyzed, and show that the analyses are essential to indicate the performance of a firm specifically. The Malmquist approach is extended to identify the strategic change of DMUs in a particular period, and whether or not the strategic change is favorable. The DEA Malmquist efficiency index can be employed to assess the technology and productivity shifts result from economic development plans. A non-radial Malmquist efficiency index and its prose are also illustrated to incorporate the preference over the performance progress and to measure the slacks. The methods are exemplified for an example of the three major Chinese industries whose industrial activities constitute important components of China's five-year economic development planning efforts.

## 11.8 Exercises

11.1. Develop the Malmquist Efficiency index using CCR output-oriented model.

11.2. Develop an output-oriented non-radial Malmquist efficiency index, when input factors are fixed at their current levels; that is, using CCR output-oriented.

11.3. Apply the output-oriented radial and non-radial Malmquist indexes in Exercises 11.1 and 11.2 for data in Table 11.1.

11.4. Apply the output-oriented radial and non-radial Malmquist indexes in Exercises 11.1 and 11.2 for data in Table 11.16.

11.5. Develop the Malmquist Efficiency index using SBM.

11.6. Write a VBA procedure to run the models in Exercise 11.5 for the data in Table 11.1.

11.7. Develop the Malmquist efficiency index using one set of weights models, such as Eqs. 6.1 and 6.2.

11.8. Develop the Malmquist Efficiency index using Eq. 6.31.

11.9. Write a VBA procedure to run the models in Exercise 11.9 for the data in Table 11.1.

11.10. Apply the output-oriented radial and non-radial Malmquist indexes in Exercises 11.1 and 11.2 for data in Table 11.1.

# Chapter 12
# Delta Neighborhood Extension

## 12.1 Introduction

In this chapter, the introduced mathematical model in Chap. 6 is improved to fairly rank firms in various conditions, and the chapter is finished with several outcomes of the model.

## 12.2 The Delta KAM

Suppose that there are $n$ firms, labeled $F_i$ ($i = 1, 2, \ldots, n$), and each firm has $m$ input factors with the values $x_{ij}$ ($j = 1, 2, \ldots, m$) and $p$ output factors with the values $y_{ik}$ ($k = 1, 2, \ldots, p$). Assume that the weights/prices or the approximation of the relationships between input and output factors are $W_j^-$ and $W_k^+$, for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$, respectively. Suppose that, $V_j^-$ and $V_k^+$ are defined as Eq. 12.1, for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$:

$$V_j^- = \frac{W_j^-}{\sum_{j=1}^{m} W_j^- x_{lj}} \quad \& \quad V_k^+ = \frac{W_k^+}{\sum_{k=1}^{p} W_k^+ y_{lk}}. \tag{12.1}$$

Assume that the delta vector, with bolded notation $\boldsymbol{\Delta}$, is given by $\boldsymbol{\Delta} = \left(\delta_1^-, \delta_2^-, \ldots, \delta_m^-, \delta_1^+, \delta_2^+, \ldots, \delta_p^+\right)$, to introduce a delta neighborhood of firm $F_l$, ($l = 1, 2, \ldots, n$). The components of delta vector are introduced by Eq. 12.2, for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.



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$$\delta_j^- = \delta / W_j^- \quad \text{and} \quad \delta_k^+ = \delta / W_k^+. \tag{12.2}$$

The value of delta has the same meaning for each factor, when Eq. 12.2 is considered. If the components of delta vector are defined by Eq. 12.3, where $\delta \in [0, +\infty)$, for $j = 1, 2, \ldots, m$, and $k = 1, 2, \ldots, p$, the $\boldsymbol{\Delta}$-KAM is the same as Eq. 5.25, where $m = 4$ and $p = 3$.

$$\delta_j^- = \delta \times x_{lj} \quad \text{and} \quad \delta_k^+ = \delta \times y_{lk}. \tag{12.3}$$

There are a lot of ways to introduce the components of delta vector, according to the aim of discrimination. For instance, $\delta_j^-$ and $\delta_k^+$ can be introduced as Eq. 12.4, for $j = 1, 2, \ldots, m$, and $k = 1, 2, \ldots, p$. In this case, the components of delta vector are commensurate with the corresponded input and output factors, but are not changed from one firm to another.

$$\delta_j^- = \delta \times \text{ave}_{1 \le i \le n} x_{ij} \quad \text{and} \quad \delta_k^+ = \delta \times \text{ave}_{1 \le i \le n} y_{ik}. \tag{12.4}$$

It is also possible to introduce one (or more) of the components of delta vector as 0, regarding the purpose of discrimination. For example, the components of delta can be introduced by $\delta_j^- = \delta \times x_{lj}$ and $\delta_k^+ = 0$, for $j = 1, 2, \ldots, m$, and $k = 1, 2, \ldots, p$, which let's consider the errors in input factors only.

Equation 12.5 illustrates the $\boldsymbol{\Delta}$-KAM, when the performance of $F_l$ is measured $(l = 1, 2, \ldots, n)$.

$$\begin{aligned}
&\min \frac{\sum_{j=1}^{m} V_j^- (x_{lj} + \delta_j^- - s_j^-)}{\sum_{k=1}^{p} V_k^+ (y_{lk} - \delta_k^+ + s_k^+)}, \\
&\text{Subject to} \\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = x_{lj} + \delta_j^-, \ \text{for } j = 1, 2, \ldots, m, \\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{lk} - \delta_k^+, \ \text{for } k = 1, 2, \ldots, p, \\
&\lambda_i \ge 0, \ \text{for } i = 1, 2, \ldots, n, \\
&s_j^- \ge 0, \ \text{for } j = 1, 2, \ldots, m, \\
&s_k^+ \ge 0, \ \text{for } k = 1, 2, \ldots, p.
\end{aligned} \tag{12.5}$$

The $\boldsymbol{\Delta}$-KAM (Eq. 12.5) can linearly be solved by Eq. 12.6. In order to solve the model by Microsoft Excel Solver software, similar instructions to solve Eqs. 5.31 and 5.32 can be used.

$$\begin{aligned}
&\min\left[\sum_{j=1}^{m} V_j^-(tx_{lj} + t\delta_j^- - s_j^-)\right],\\
&\text{Subject to}\\
&\left[\sum_{k=1}^{p} V_k^+(y_{lk}t - \delta_k^+ t + s_k^+)\right] = 1,\\
&\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = x_{lj}t + \delta_j^- t, \ \text{for } j = 1,2,\ldots,m,\\
&\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{lk}t - \delta_k^+ t, \ \text{for } k = 1,2,\ldots,p,\\
&\lambda_i \geq 0, \ \text{for } i = 1,2,\ldots,n,\\
&s_j^- \geq 0, \ \text{for } j = 1,2,\ldots,m,\\
&s_k^+ \geq 0, \ \text{for } k = 1,2,\ldots,p,\\
&t > 0.
\end{aligned} \tag{12.6}$$

The score of KAM represents that the efficiency score of firm $l$, that is, $\sum_{k=1}^{p} W_k^+ y_{lk} / \sum_{j=1}^{m} W_j^- x_{lj}$, is compared with the efficiency score of a point on the estimated production function, that is, $\sum_{k=1}^{p} W_k^+ (y_{lk} - \delta_k^+ + s_k^+) / \sum_{j=1}^{m} W_j^- \left(x_{lj} + \delta_j^- - s_j^-\right)$, such that, the ratio of $\sum_{k=1}^{p} W_k^+ y_{lk} / \sum_{j=1}^{m} W_j^- x_{lj}$ to $\sum_{k=1}^{p} W_k^+ (y_{lk} - \delta_k^+ + s_k^+) / \sum_{j=1}^{m} W_j^- \left(x_{lj} + \delta_j^- - s_j^-\right)$ becomes minimum. Since the efficiency of firm $l$ is a constant value, KAM finds a point on the estimated production function which has an equal or greater efficiency score in comparison with that of firm $l$, regarding the value of delta and introduced $W_j^-$ and $W_k^+$.

The target for firm $l$ ($l$ = 1, 2, ..., $n$) from Eq. 12.6, which lies on the estimated production function, and has a greater (or equal) ratio of the linear combination of output factors to the linear combination of input factors, regarding the value of delta and introduced $W_j^-$ and $W_k^+$, is given by:

$$\begin{aligned}
x_{lj}^* &= x_{lj} + \delta_j^- - s_j^{-*}/t^*, \ \text{for } j = 1,2,\ldots,m,\\
y_{lk}^* &= y_{lk} - \delta_k^+ + s_k^{+*}/t^*, \ \text{for } k = 1,2,\ldots,p,
\end{aligned} \tag{12.7}$$

The dual linear programming of Eq. 12.6 is also given by Eq. 12.8.

$$\begin{aligned}
&\max \tau,\\
&\left(\sum_{k=1}^{p} V_k^+(y_{lk} - \delta_k^+)\right)\tau + \sum_{j=1}^{m}(x_{lj} + \delta_j^-)w_j^- - \sum_{\mathrm{k}=1}^{\mathrm{p}}(y_{lk} - \delta_k^+)w_k^+ = \sum_{j=1}^{m} V_j^-(x_{lj} + \delta_j^-),\\
&\sum_{k=1}^{p} y_{ik}w_k^+ - \sum_{j=1}^{m} x_{ij}w_j^- \leq 0, \ \text{for } i = 1,2,\ldots,n,\\
&w_j^- \geq V_j^- \ \text{for } j = 1,2,\ldots,m,\\
&w_k^+ \geq \tau V_k^+ \ \text{for } k = 1,2,\ldots,p.
\end{aligned} \tag{12.8}$$

Suppose that the component of delta vector is introduced by Eq. 12.2. Please note we usually use the notation $\delta$-KAM instead of $\boldsymbol{\Delta}$-KAM in this book. When $\delta = 0$, the 0-KAM measures the technical efficiency of firms, and divides the firms into two categories similar to DEA models. If the score of 0-KAM is less than 1 for a firm, that firm is technically inefficient, and if the score of 0-KAM is 1, the firm is technically efficient. However, the scores of 0-KAM (similar to DEA models) should neither be used to rank firms, nor the proposed targets can be used to benchmark firms.

The $\delta$-KAM is SBM (Eq. 5.26), where $\delta = 0$, $W_j^- = 1/x_{lj}$ and $W_k^+ = 1/y_{lk}$. When $\delta > 0$, and $W_j^- = 1/x_{lj}$ and $W_k^+ = 1/y_{lk}$, we express '*the $\delta$-KAM with SBM approach*'. Note that, the SBM approach does not satisfy Theorem 4.1 in order to measure efficiency, but can be used to introduce the technically efficient firms. When $\delta > 0$, and $W_j^- = 1/\min\{x_{lj} : x_{lj} \neq 0\}$ and $W_k^+ = 1/\min\{y_{lk} : y_{lk} \neq 0\}$, we state '*the $\delta$-KAM with minimum approach*'.

Suppose that $W_j^-$ and $W_k^+$ are given as the available costs of input and output factors, where $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$. We can decrease the discrimination power of $\delta$-KAM, as Eqs. 12.9, 12.10 represent, to illustrate the lack of CF and RF (or PF) measurements, respectively, (see also Exercises 6.10–6.15).

$$\begin{aligned}
&\min \textstyle\sum_{j=1}^{m} V_j^-(x_{lj} + \delta_j^- - s_j^-),\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = x_{lj} + \delta_j^-, \quad \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{lk}, \quad \text{for } k = 1, 2, \ldots, p,\\
&\lambda_i \geq 0, \quad \text{for } i = 1, 2, \ldots, n,\\
&s_j^- \geq 0, \quad \text{for } j = 1, 2, \ldots, m,\\
&s_k^+ \geq 0, \quad \text{for } k = 1, 2, \ldots, p.
\end{aligned} \tag{12.9}$$

$$\begin{aligned}
&\max \textstyle\sum_{k=1}^{p} V_k^+(y_{lk} - \delta_k^+ + s_k^+),\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = x_{lj}, \quad \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{lk} - \delta_k^+, \quad \text{for } k = 1, 2, \ldots, p,\\
&\lambda_i \geq 0, \quad \text{for } i = 1, 2, \ldots, n,\\
&s_j^- \geq 0, \quad \text{for } j = 1, 2, \ldots, m,\\
&s_k^+ \geq 0, \quad \text{for } k = 1, 2, \ldots, p.
\end{aligned} \tag{12.10}$$

In other words, none of Eqs. 12.9 and 12.10 provide a fair measure to discrimination the efficiency of firms. A suitable model should at least satisfy the introduced Types 1–6. For instance, Eq. 12.9 does not satisfy Type 4, and Eq. 12.10 does not satisfy Type 3.

As explained in the previous chapters, a firm, $F_l$, partially dominates another firm, $F_{l'}$, if and only if, the value of Eq. 12.11 for $F_l$ is greater than that of $F_{l'}$, where $l$ and $l'$ belongs to $\{1, 2, \ldots, n\}$.

$$\frac{\sum_{k=1}^{p} W_k^+ y_k}{\sum_{j=1}^{m} W_j^- x_j} = \frac{W_k^+ y_1 + W_k^+ y_2 + \ldots + W_k^+ y_p}{W_j^- x_1 + W_j^- x_2 + \ldots + W_j^- x_m}. \tag{12.11}$$

The six introduced types which increase the value of Eq. 12.11 are expressed by:

**Type 1**: Decreasing the value of denominator in Eq. 12.11 when the value of numerator is fixed or increased.

**Type 2**: Increasing the value of numerator in Eq. 12.11 when the value of denominator is fixed or decreased.

**Type 3**: If the rate of increasing the value of numerator in Eq. 12.11 is greater than the rate of increasing the value of denominator.

**Type 4**: If the rate of decreasing the value of numerator in Eq. 12.11 is greater than the rate of decreasing the value of denominator.

**Type 5**: The value of denominator in Eq. 12.11 is decreased (increased) by: (1) decreasing (increasing) the value of one or more of input factors, or (2) increasing (decreasing) a small value of one or more of input factors and decreasing (increasing) a large value of one or more of other input factors.

**Type 6**: The value of numerator in Eq. 12.11 is increased (decreased) by: (1) increasing (decreasing) the value of one or more of output factors, or (2) decreasing (increasing) a small value of one or more of output factors and increasing (decreasing) a large value of one or more of other output factors.

In short, Eq. 12.5 shows that KAM compares the efficiency of a firm with the efficiency of the points on the estimated production function, and finds the best target for the firm, regarding the value of delta and introduced weights. The discrimination power of KAM is greater than CF, RF and PF models, and KAM provides a fair measure to assess the inefficiencies of the firms. In the next sections, KAM is improved and the optimum of delta is also measured.

## 12.3 KAM and Uncontrollable Factors

In the airport example in Chap. 3, the runway is an input factor which has the standard area, according to the documents of International Civil Aviation Organization (ICAO). The area of runway should not be less than the standard value, and depends on the aircrafts which want to land and take off from that runway. Because of the safety of passengers, decreasing the length/area of a runway is not suggested. This kind of factor is called *non-controllable* or *uncontrollable* factor (Banker and Morey 1986; Charnes et al. 1987; Cooper et al. 2007). In other words, an uncontrollable factor may not be controlled by managers, although, it may affect the performance of firms.

For instance, suppose that $j^{\text{th}}$ input factor ($j = 1, 2, \ldots, m$) is an uncontrollable factor and the efficiency of $F_l$ is measured ($l = 1, 2, \ldots, n$). If this uncontrollable factor should not be decreased and increased, the corresponded linear combination of this $j^{\text{th}}$ input factor of firms in Eq. 12.2, (that is, $\sum_{i=1}^{n} x_{ij}\lambda_i$) should be equal to the corresponded value of $j^{\text{th}}$ factor of firm $l$ (that is, $x_{lj}$), that is, $\sum_{i=1}^{n} x_{ij}\lambda_i = x_{lj}$, (where $l = 1, 2, \ldots, n$). Nonetheless, it is valuable to examine a delta neighborhood of an uncontrollable factor, in order to measure the effect of such restriction on other factors. For such an aim, the constraint $s_j^- \le \delta_j^-$ (or $s_j^- \le t\delta_j^-$) can be added to the constraints in Eq. 12.5 (Eq. 12.6).

The term $\delta_j^-$ is a value which is added to $j^{\text{th}}$ input factor of $F_l$, to introduce a neighborhood of this factor, thus the linear combination of this uncontrollable factor of firms is at least equal to $x_{lj}$, that is, $\sum_{i=1}^{n} x_{ij}\lambda_i = x_{lj} + \delta_j^- - s_j^- \ge x_{lj}$. Therefore, the optimal value of $j^{\text{th}}$ input factor is not decreased, but it may slightly be increased according to value of $\delta_j^-$.

For the case that the optimal value of $j^{\text{th}}$ input factor should not also be increased, the value of delta can be considered very small, such that, the value of $\delta_j^-$ is quite negligible. For instance, if $j^{\text{th}}$ input factor is measured with three decimal digits, the negligible error can be less than 0.005. Similar discussion can be illustrated for an uncontrollable output factor as well.

Now, suppose that $J_u$ is a subset of input factor indexes, $\{1, 2, \ldots, m\}$, corresponded to the uncontrollable input factors, and $K_u$ is a subset of output factor indexes, $\{1, 2, \ldots, p\}$, corresponded to the uncontrollable output factors. The $\delta$-KAM is given by Eq. 12.12.

$$
\begin{aligned}
&\min \left[\sum_{j=1}^{m} V_j^-\left(tx_{lj} + t\delta_j^- - s_j^-\right)\right],\\
&\text{Subject to}\\
&\left[\sum_{k=1}^{p} V_k^+\left(y_{lk}t - \delta_k^+ t + s_k^+\right)\right] = 1,\\
&\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = x_{lj}t + \delta_j^- t, \ \text{ for } j = 1, 2, \ldots, m,\\
&\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{lk}t - \delta_k^+ t, \ \text{ for } k = 1, 2, \ldots, p,\\
&s_j^- \le \delta_j^- t \ \text{ for } j \in J_u,\\
&s_k^+ \ge \delta_k^+ t \ \text{ for } k \in K_u,\\
&s_j^- \ge 0 \ \text{ for } j = 1, 2, \ldots, m,\\
&s_k^+ \ge 0 \ \text{ for } k = 1, 2, \ldots, p,\\
&t > 0.
\end{aligned}
\tag{12.12}
$$

For an example of uncontrollable factor, suppose that fourth input factor (runway) is uncontrollable in the airport example in Chap. 5. Assume that the same conditions to solve Eq. 5.31 in Chap. 5 are considered, that is, $W_j^- = 1/x_{lj}$, for $j = 1, 2, 3, 4$, $W_k^+ = 1/y_{lk}$, for $k = 1, 2, 3$, where $\delta$ is 0, 0.0001, 0.01 and 0.1, and the component of delta vectors are introduced by Eq. 12.3. Since the weights are introduced by the inverse of data similar to SBM, the approach is the SBM approach and the results are only used to express the methodology.

When $\delta = 0$, in Eq. 12.12, the corresponded constraints for fourth input factor of firm $F_l$, (that is, $s_4^- \le \delta x_{l4}t$, and $\sum_{i=1}^{8} x_{i4}\lambda_i + s_4^- = (1+\delta)x_{l4}t$) are equal to zero, that

**Table 12.1** The $\delta$-KAM scores where $s_4^- \leq t\delta x_{14}$

| Delta | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 0.0001 | 0.99983 | 0.99997 | 0.99924 | 0.98588 | 0.99258 | 0.99993 | 0.99990 | 0.99999 |
| 0.01 | 0.98307 | 0.99747 | 0.92921 | 0.63590 | 0.54534 | 0.99274 | 0.98991 | 0.99857 |
| 0.1 | 0.85672 | 0.97644 | 0.55163 | 0.50990 | 0.36364 | 0.93414 | 0.91129 | 0.98565 |

is, $s_4^- = \delta x_{l4}t = 0$, and $\sum_{i=1}^{8} x_{i4}\lambda_i = x_{l4}t$. As Table 12.1 illustrates, the 0-KAM by SBM approach represents that all airports A-H are technically efficient. In other words, this simple restriction on fourth factor of airports yields that airports C and E become technically efficient.

As the value of delta is increased, (for instance, when $\delta$ is 0.0001, 0.01 and 0.1), $\delta$-KAM discriminates the airports according to the introduced value of delta and the SBM approach.

If the uncontrollable factor cannot be decreased and increased, the constraint $s_4^- \leq \delta x_{l4}t$ can be replaced by $s_4^- = \delta x_{l4}t$. The results of $\delta$-KAM for this assumption are illustrated in Table 12.1, which can be compared with the results in Table 6.10 as well. In this case, the value of fourth factor for airport number $l$is not changed. Indeed, the targets for firm $l$ ($l = 1, 2, \ldots, n$) can be measured from Eq. 12.7. Since $s_4^- = \delta_j^- t = \delta x_{l4}t$, for instance, when $\delta = 0.0001$, Table 12.2 represents that the suggested target for fourth input factor of airport number $l$ is not changed, ($l = 1, 2, \ldots, 8$).

The suggested targets in Table 12.2 (or the targets by Eq. 12.5) lie on the frontier of feasible area, and have better performance in comparison with the real data in Table 5.1, regarding the assumptions of discrimination. In other words, 0.0001-KAM not only discriminates the airports according to the introduced errors, but it also benchmarks all technically efficient airports, and suggests how they can regulate their factors according to Types 1–6, in order to improve their performances, (regarding the introduced assumptions). For instance, 0.0001-KAM says that A should decrease the area of airport and increase the numbers of flights and passengers, even if the area of apron and terminal are increased, or the amount of cargo decreased, according to the value of delta and the SBM approach.

Now, suppose that $s_4^- \leq \delta x_{l4}t$ is only added to Eq. 5.31, the corresponded results to Tables 12.1 and 6.10 are displayed in Table 12.3. The constraint, $s_4^- \leq \delta x_{l4}t$, does not let the value of fourth factor decrease, but it may be increased in order to find a better situation on the production frontier, according to the SBM approach.

Table 12.4 represents the targets of KAM when $\delta = 0.0001$. According to the table, A should seriously improve the numbers of flights and passengers, even if the input factors are increased and the value of cargo is decreased. In other words, the inefficiency of A is due to the small numbers of flights and passengers, according to the data of other airports. All of these assessments are approximations and do not mean that it is impossible to increase the number of flights of airport A without increasing the values of its input factors, but at the same time, express the way of increasing the efficiency of A.

**Table 12.2** The $10^{-4}$-KAM targets where $s_4^- \leq t\delta x_{14}$

| Airport | Area | Apron | Terminal | Runway | Flights | Passengers | Cargo |
|---|---|---|---|---|---|---|---|
| A | 1187.28 | 325,581.90 | 48,948.45 | 353,610.00 | 44,770.19 | 5,080,228.95 | 66,765.60 |
| B | 553.30 | 230,417.98 | 40,786.56 | 348,120.00 | 51,816.17 | 5,187,091.37 | 20,175.81 |
| C | 418.46 | 45,103.30 | 12,980.00 | 269,955.00 | 23,914.76 | 1,258,627.24 | 3681.02 |
| D | 778.22 | 123,710.40 | 23,155.00 | 395,730.00 | 35,883.90 | 2,441589.64 | 17,250.27 |
| E | 297.08 | 33,000.00 | 8800.00 | 192,330.00 | 16,583.00 | 865,955.98 | 2825.11 |
| F | 497.53 | 69300.00 | 23,630.14 | 389,115.00 | 40,918.80 | 2,218,202.93 | 7181.15 |
| G | 486.49 | 51,931.00 | 10,230.00 | 268,995.00 | 19,494.13 | 1,047,313.36 | 5212.20 |
| H | 1357.68 | 483,620.85 | 73,294.86 | 421,305.00 | 116,237.70 | 10,746,024.81 | 46,368.88 |

**Table 12.3** The $\delta$-KAM scores where $s_4^- \le t\delta x_{14}$

| Delta | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 0.0001 | 0.99980 | 0.99997 | 0.99924 | 0.98468 | 0.99258 | 0.99993 | 0.99990 | 0.99997 |
| 0.01 | 0.98056 | 0.99747 | 0.92921 | 0.63590 | 0.54534 | 0.99274 | 0.98991 | 0.99739 |
| 0.1 | 0.84096 | 0.97644 | 0.55163 | 0.50990 | 0.36364 | 0.93414 | 0.91129 | 0.97558 |

The same illustration can be discussed for other airports as well. In addition, instead of SBM approach, the average measurement approach can also be used, that is, $W_j^- = 1/\text{ave}\{x_{ij} : i = 1, 2, \ldots, 8\}$, for $j$=1, 2, 3, 4, $W_k^- = 1/\text{ave}\{y_{ik} : i = 1, 2, \ldots, 8\}$, for $k$ = 1, 2, 3, and so on for any other interested approaches introduced by expert judgment.

## 12.4 KAM and the Production Tape

In Chaps. 1 and 2, several approaches are proposed to introduce practical points from a set of homogenous firms. The largest feasible area, which is linearly generated, is called the CRS-PPS or $T_C$ which is introduced by Eq. 7.4. Now, from the outcomes of the $\delta$-KAM, the feasible area can be extended corresponded to the value of delta. In other words, when the practical points are generated by the wholly dominant, the convexity and the radiate approaches, a delta neighborhood of the feasible area can be practical as well. The three introduced approaches may not be suitable to apply for the points in the delta neighborhood of the feasible area, but an exact data can rarely be measured in real life applications, and even if data are exact, this technique is useful to recuperate the estimated production function from linear approaches. Indeed, it is hard to prove that a production function does not have any curves, as can be seen in Fig. 12.1.

Assume that the blue curve in Fig. 12.1 represents the exact production function for firms A-F. The CRS technology suggests the frontier of $T_C$, which is generated by applying the radiate and the convexity approaches for the observed data at the first step, and at the second step by applying the wholly dominant approach, as proved by Theorems 2.4–2.7.

After introducing the frontier of $T_C$, a delta neighborhood can linearly be added to $T_C$, by shifting the frontier toward the directions which improve the factors, regarding the value of delta. This extension of $T_C$ is also the smallest set to recuperate $T_C$, regarding the value of delta.

Suppose that a $T_C$ is given. There is always a δ>0 to introduce an extension of $T_C$, as Eq. 12.13 represents.

The delta in Eq. 12.13 is a real value, and can also be introduced as a vector with $m + p$ components, such as, $\boldsymbol{\Delta} = \left(\delta_1^-, \delta_2^-, \ldots, \delta_m^-, \delta_1^+, \delta_2^+, \ldots, \delta_p^+\right) \in \mathbb{R}_+^{m+p}$, to introduce $T_C^{+\boldsymbol{\Delta}}$. For instance, the components of delta vector in $\delta$-KAM by the SBM

**Table 12.4** The $10^{-4}$-KAM targets where $s_4^- \leq t\delta x_{14}$

| Airport | Area | Apron | Terminal | Runway | Flights | Passengers | Cargo |
|---|---|---|---|---|---|---|---|
| A | 1200.00 | 304,211.34 | 45,604.56 | 353,615.42 | 30,723.69 | 4,032,130.88 | 74,176.58 |
| B | 503.05 | 213,745.69 | 38,780.01 | 348,120.00 | 46,879.94 | 4,783,523.97 | 19,051.13 |
| C | 799.62 | 41,007.10 | 11,801.18 | 269,955.00 | 15,616.31 | 1,040,185.66 | 1589.09 |
| D | 1031.88 | 112,475.25 | 21,052.11 | 395,769.57 | 39,867.01 | 1,769,654.15 | 5046.57 |
| E | 997.61 | 30,003.00 | 8000.80 | 192,330.00 | 4949.97 | 430,744.19 | 1573.84 |
| F | 478.02 | 63,006.30 | 23,000.63 | 389,115.00 | 41,087.83 | 2,165,624.63 | 5415.77 |
| G | 481.01 | 47,214.72 | 9300.93 | 268,995.00 | 19,010.48 | 971,389.00 | 3827.39 |
| H | 1346.13 | 503,280.12 | 76,370.76 | 421,340.40 | 129,140.08 | 11,708,951.44 | 39,571.86 |

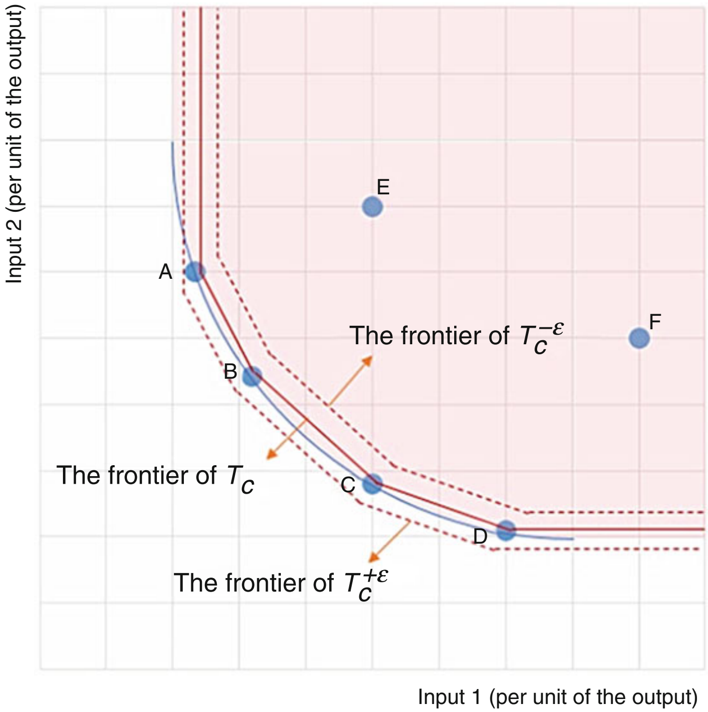


**Fig. 12.1** The production tape

approach when firm $F_l$ is evaluated are $\delta_{lj}^- = \delta \times x_{lj}$, for $j = 1, 2, \ldots, m$, and $\delta_{lk}^+ = \delta \times y_{ik}$, for $k = 1, 2, \ldots, p$.

$$T_C^{+\delta} = \Big\{(X', Y') \in \mathbb{R}^{m+p} : \ \Lambda \cdot X^j - \delta \le x_j', \ \Lambda \cdot Y^k + \delta \ge y_k', \ \lambda_i \ge 0, \\ \text{for } i = 1, 2, \ldots, n, j = 1, 2, \ldots, m, k = 1, 2, \ldots, p\Big\}. \tag{12.13}$$

Similarly, the generated feasible area by CRS technology can be contracted regarding the value of delta, as Eq. 12.14 represents.

$$T_C^{-\delta} = \Big\{(X', Y') \in \mathbb{R}^{m+p} : \Lambda \cdot X^j + \delta \le x_j', \ \Lambda \cdot Y^k - \delta \ge y_k', \ \lambda_i \ge 0, \\ \text{for } i = 1, 2, \ldots, n, j = 1, 2, \ldots, m, k = 1, 2, \ldots, p\Big\}. \tag{12.14}$$

Then, the *production tape* can be defined by $T_C^{+\delta} - T_C^{-\delta} = TT_C^{\delta}$. When the targets of $\delta$-KAM are measured by Eq. 12.7, the targets are on the frontier of $T_C^0 = T_C$, then the targets can be transferred to the frontier of $T_C^{+\delta}$ by Eq. 12.15.

$$\begin{aligned} x_{lj}^* &= x_{lj} - s_j^{-*}/t^*, \ \text{ for } j = 1, 2, \ldots, m, \\ y_{lk}^* &= y_{lk} + s_k^{+*}/t^*, \ \text{ for } k = 1, 2, \ldots, p. \end{aligned} \tag{12.15}$$

It is possible that the values of input factors for the suggested targets in Eq. 12.15 become negative. Thus, the constraints $tx_{lj} - s_j^- \geq 0$, for $j = 1, 2, \ldots, m$, are added to Eq. 12.6 to guarantee positive values for the suggested targets of input factors. We may also look for the minimum value of $\sum_{j=1}^{4} V_j^- \left(x_{lj} - s_j^-\right)$ over $\sum_{k=1}^{3} V_k^+ \left(y_{lk} + s_k^+\right)$ subject to the constraints of Eq. 12.5, and the constraints $x_{lj} - s_j^- \geq 0$, for $j = 1, 2, \ldots, m$, for a given $\delta > 0$.

The same illustration can be discussed for the targets which are transferred to the frontier of $T_C^{-\delta}$, as Eq. 12.16 represents.

$$\begin{aligned} x_{lj}^* &= x_{lj} + 2\delta_j^- - s_j^{-*}/t^* \ \text{ for } j = 1, 2, \ldots, m, \\ y_{lk}^* &= y_{lk} - 2\delta_k^+ + s_k^{+*}/t^* \ \text{ for } k = 1, 2, \ldots, p. \end{aligned} \tag{12.16}$$

Since the values of output factors for the suggested targets in Eq. 12.16 can be negative, the constraints $ty_{lk} - 2t\delta_k^+ + s_k^+ \geq 0$, for $k = 1, 2, \ldots, p$, should be added to Eq. 12.6 to guarantee positive values for the suggested targets of output factors. As a result, in order to introduce the production tape, Eq. 12.17 should be solved for a given $\delta > 0$.

$$\begin{aligned} &\min \left[\sum_{j=1}^{m} V_j^- \left(x_{lj}t + \delta_j^- t - s_j^-\right)\right], \\ &\left[\sum_{k=1}^{p} V_k^+ \left(y_{lk}t - \delta_k^+ t + s_k^+\right)\right] = 1, \\ &\sum_{i=1}^{n} x_{ij}\lambda_i + s_j^- = x_{lj}t + \delta_j^- t, \ \text{ for } j = 1, 2, \ldots, m, \\ &\sum_{i=1}^{n} y_{ik}\lambda_i - s_k^+ = y_{lk}t - \delta_k^+ t, \ \text{ for } k = 1, 2, \ldots, p, \\ &x_{lj}t - s_{lj}^- \geq 0, \ \text{ for } j = 1, 2, \ldots, m, \\ &y_{lk}t - 2\delta_{lk}^+ t + s_{lk}^+ \geq 0, \ \text{ for } k = 1, 2, \ldots, p, \\ &\lambda_i \geq 0, \ \text{ for } i = 1, 2, \ldots, n, \\ &s_j^- \geq 0, \ \text{ for } j = 1, 2, \ldots, m, \\ &s_k^+ \geq 0, \ \text{ for } k = 1, 2, \ldots, p, \\ &t > 0. \end{aligned} \tag{12.17}$$

In order to introduce a production tape, the delta value should be small. We can also measure the optimum value of $\delta$ such that at least two corresponded components to the input (output) factors of the $T_C^{+\delta}$'s frontier ($T_C^{-\delta}$'s frontier) become positive. (How?)

## 12.5 An Improvement of KAM

Firms may have a request that the values of a factor should be in an introduced range or interval, for instance, due to standardization or the intention of firms. Sometimes a manager plans (or is asked) to decrease (increase) a certain amount of an input (output) factor in a period of time. A manager may only able to improve or control a special amount of a factor. All these situations can affect the efficiency (productivity) measurement and assessing the performance of a set of homogenous firms. Indeed, the provided relative scores to discriminate, to rank and to benchmark firms should satisfy such restrictions and desired requests as well. Thus, KAM is improved to handle such purposes.

Suppose that the $\boldsymbol{\Delta}$-KAM by Eq. 12.5 is given. Thus, the $\boldsymbol{\Delta}$-KAM targets for firm number $l$ are $x_{lj} + \delta_{lj}^- - s_{lj}^-$ and $y_{lk} - \delta_{lk}^+ + s_{lk}^+$, for $j = 1, 2, \ldots, m$, and $k = 1, 2, \ldots, p$. Assume that the target of $j^{\text{th}}$ input factor should belong to $[a_j, b_j]$ and the target of $k^{\text{th}}$ output factor should belong to $[c_k, d_k]$, where $a_j$, $b_j$, $c_k$ and $d_k$ are non-negative real values, $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$. Equation 12.18 illustrates the constraints which should be added to the constraints of Eq. 12.5 to satisfy the purpose of discrimination.

$$
\begin{aligned}
& a_j \le x_{lj} + \delta_{lj}^- - s_{lj}^-, \ \text{for } j = 1, 2, \ldots, m, \\
& x_{lj} + \delta_{lj}^- - s_{lj}^- \le b_j, \ \text{for } j = 1, 2, \ldots, m, \\
& c_k \le y_{lk} - \delta_{lk}^+ + s_{lk}^+, \ \text{for } k = 1, 2, \ldots, p, \\
& y_{lk} - \delta_{lk}^+ + s_{lk}^+ \le d_k, \ \text{for } k = 1, 2, \ldots, p.
\end{aligned}
\tag{12.18}
$$

The values of $a_j$, $b_j$, $c_k$ and $d_k$ should reasonably be selected in order to have a feasible area, regarding the components of delta vector. These variables should also be commensurate with the units of corresponded factors. In order to decrease the number of introduced variables by expert judgment and make sure there is always a feasible area as well as simplifying the model and improving the applicability of the model, let's assume that the components of delta vector are exchangeable and at least one of the firms satisfies the introduced ranges, $[a_j, b_j]$ and $[c_k, d_k]$, for $j = 1, 2\ldots$, $m$ and $k = 1, 2, \ldots, p$. Thus, the optimal values of the delta components can also be measured and Eq. 12.4 can be extended as Eq. 12.19.

$$
\begin{aligned}
&KA^*_{\Delta_l} = \min \frac{\sum_{j=1}^{m} V_j^-(x_{lj} + \delta_{lj}^- - s_{lj}^-)}{\sum_{k=1}^{p} V_k^+(y_{lk} - \delta_{lk}^+ + s_{lk}^+)},\\
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_{li} + s_{lj}^- = x_{lj} + \delta_{lj}^-, \ \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_{li} - s_{lk}^+ = y_{lk} - \delta_{lk}^+, \ \text{for } k = 1, 2, \ldots, p,\\
&a_j \le x_{lj} + \delta_{lj}^- - s_{lj}^-, \ \text{for } j = 1, 2, \ldots, m,\\
&x_{lj} + \delta_{lj}^- - s_{lj}^- \le b_j, \ \text{for } j = 1, 2, \ldots, m,\\
&c_k \le y_{lk} - \delta_{lk}^+ + s_{lk}^+, \ \text{for } k = 1, 2, \ldots, p,\\
&y_{lk} - \delta_{lk}^+ + s_{lk}^+ \le d_k, \ \text{for } k = 1, 2, \ldots, p,\\
&\lambda_{li} \ge 0, \ \text{for } i = 1, 2, \ldots, n,\\
&s_{lj}^- \ge 0, \ \text{for } j = 1, 2, \ldots, m,\\
&\delta_{lj}^- \ge 0, \ \text{for } j = 1, 2, \ldots, m,\\
&s_{lk}^+ \ge 0, \ \text{for } k = 1, 2, \ldots, p,\\
&\delta_{lk}^+ \ge 0 \ \text{for } k = 1, 2, \ldots, p.
\end{aligned}
\tag{12.19}
$$

If there are no restrictions for the target values, the variables $a_j$, $b_j$, $c_k$ and $d_k$ can be introduced as $a_j = 0$ and $b_j = M_j$, for $j = 1, 2, \ldots, m$, and $c_k = 0$ and $d_k = N_k$, for $k = 1, 2, \ldots, p$. Here the meaning of $M_j$ and $N_k$ are the suitable large numbers according to data or the used software. The variable can also be different from one firm to another as long as the homogeneity of firms is satisfied.

The optimal values for the components of delta vector may be different from one firm to another; however, the provided scores by Eq. 12.19 for $F_i$'s ($i = 1, 2\ldots, n$) are relatively meaningful, regarding to a vector of delta that its corresponded components are the maximum values of the components of the measured optimal delta vectors of $F_i$'s ($i = 1, 2\ldots, n$).

The components of delta vector can also be introduced by Eq. 12.2. In this case, the $\delta^*$-KAM is given by Eq. 12.20, where $\delta^* = \max\left\{\delta_l^* : l = 1, 2, \ldots, n\right\}$.

$$KA^*_{\delta^*_l} = \min \frac{\sum_{j=1}^{m} W_j^-(x_{lj} + \delta_l/W_j^- - s_{lj}^-)/\sum_{j=1}^{m} W_j^- x_{lj}}{\sum_{k=1}^{p} W_k^+(y_{lk} - \delta_l/W_k^+ + s_{lk}^+)/\sum_{k=1}^{p} W_k^+ y_{lk}},$$

$$\begin{aligned}
&\text{Subject to}\\
&\textstyle\sum_{i=1}^{n} x_{ij}\lambda_{li} + s_{lj}^- = x_{lj} + \delta_l/W_j^-, \ \text{for } j = 1, 2, \ldots, m,\\
&\textstyle\sum_{i=1}^{n} y_{ik}\lambda_{li} - s_{lk}^+ = y_{lk} - \delta_l/W_k^+, \ \text{for } k = 1, 2, \ldots, p,\\
&x_{lj} + \delta_l/W_j^- - s_{lj}^- \le b_j, \ \text{for } j = 1, 2, \ldots, m,\\
&a_j \le x_{lj} + \delta_l/W_j^- - s_{lj}^-, \ \text{for } j = 1, 2, \ldots, m,\\
&c_k \le y_{lk} - \delta_l/W_k^+ + s_{lk}^+, \ \text{for } k = 1, 2, \ldots, p,\\
&\lambda_{li} \ge 0, \ \text{for } i = 1, 2, \ldots, n,\\
&s_{lj}^- \ge 0, \ \text{for } j = 1, 2, \ldots, m,\\
&s_{lk}^+ \ge 0, \ \text{for } k = 1, 2, \ldots, p,\\
&\delta_l \ge 0.
\end{aligned} \tag{12.20}$$

The score of $\delta$-KAM is non-increasing, that is, $KA^*_{\delta_1} \ge KA^*_{\delta_2}$ if $\delta_1 \le \delta_2$. Moreover, $KA^*_{\delta^*_l} = KA^*_{\delta}$, for $\delta \ge \delta^*_l$. The $\delta^*$-KAM scores are relatively meaningful and depend on the introduced $W_j^-$, for $j$ = 1, 2…, $m$, and $W_k^+$, for $k$ = 1, 2…, $p$.

## 12.5.1 Solving KAM for a Request

Suppose that managers in the airport example can only regulate at about 10% of the factors' values, where $W_1^- = 10$, $W_2^- = 1$, $W_3^- = 100$, $W_4^- = 10$, $W_1^+ = 100$, $W_2^+ = 1$, and $W_3^+ = 10$. In other words, assume that $a_j = x_{lj} - 10\ \%\ x_{lj}$ and $b_j = x_{lj} + 10\ \%\ x_{lj}$, for $j$ = 1, 2, 3, 4, and $c_k = y_{lk} - 10\ \%\ y_{lk}$ and $d_k = y_{lk} + 10\ \%\ y_{lk}$, for $k$ = 1, 2, 3.

From such assumption, the feasible area for the airports may vary from one airport to another. Nonetheless, KAM compares the efficiency scores of the points in the estimated feasible area with the efficiency score of the evaluated airport, and finds the best target for that airport, regarding the assumptions of the example.

On the other hand, Eq. 12.20 can linearly be solved by Eq. 12.21.

$$
\begin{aligned}
& KA^*_{\delta^*_l} = \min \sum_{j=1}^m W_j^-(x_{lj}t_l + \delta_l/W_j^- - s_{lj}^-) / \sum_{j=1}^m W_j^- x_{lj}, \\
& \text{Subject to} \\
& \sum_{i=1}^n x_{ij}\lambda_{li} + s_{lj}^- = x_{lj}t_l + \delta_l/W_j^-, \ \text{for } j = 1, 2, \ldots, m, \\
& \sum_{i=1}^n y_{ik}\lambda_{li} - s_{lk}^+ = y_{lk}t_l - \delta_l/W_k^+, \ \text{for } k = 1, 2, \ldots, p, \\
& a_j t_l \le x_{lj}t_l + \delta_l/W_j^- - s_{lj}^-, \ \text{for } j = 1, 2, \ldots, m, \\
& x_{lj}t_l + \delta_l/W_j^- - s_{lj}^- \le b_j t_l, \ \text{for } j = 1, 2, \ldots, m, \\
& c_k t_l \le y_{lk}t_l - \delta_l/W_k^+ + s_{lk}^+, \ \text{for } k = 1, 2, \ldots, p, \\
& y_{lk}t_l - \delta_l/W_k^+ + s_{lk}^+ \le d_k t_l, \ \text{for } k = 1, 2, \ldots, p, \\
& \lambda_{li} \ge 0, \ \text{for } i = 1, 2, \ldots, n, \\
& s_{lj}^- \ge 0, \ \text{for } j = 1, 2, \ldots, m, \\
& s_{lk}^+ \ge 0, \ \text{for } k = 1, 2, \ldots, p, \\
& \delta_l \ge 0, \\
& t_l \ge 0.
\end{aligned}
\tag{12.21}
$$

The following instructions illustrate how to solve Eq. 12.21 for the airports in Table 5.1, by the Microsoft Excel Solver 2013 software.

1. Copy the 9 columns of Table 5.1 on an Excel sheet into cells A1:I9, as Fig. 5.1 depicts.
2. Label B11 as 'Index1', E11 as 'Range', H11 as 'Delta', B13 as 'Ws', B15 as 'Constraint', E15 as '*t*', H15 as 'Objective', B17 as 'Left side', B18 as 'Slacks', B19 as 'Right side', B20 as 'Deltas', B22 as 'Targets', B24 as 'Lower bounds', B25 as 'Upper bounds', J1 as 'Lambdas', K1 as 'Solver Code', L1 as 'KAM score', and M1 as 'Delta'.
3. Assign number 1 to C11, and 0.1 to F11.
4. Assign number 10 to C13, 1 to D13, 100 to E13, 10 to F13, 100 to G13, 1 to H13, and 10 to I13.
5. Assign the following command to C15

   '=Sumproduct(G13:I13,G19:I19 + G18:I18)/Sumproduct(G13:I13,Index(G2:I9,C11,0))'.

6. Assign the following command to I15

   '=Sumproduct(C13:F13,C19:F19-C18:F18)/Sumproduct(C13:F13,Index (C2:F9,C11,0))'.

7. Assign the command '=Sumproduct(C2:C9,$J2:$J9) + C18' to C17. Then, copy C17 and paste it to D17, E17 and F17.
8. Assign the command '=Sumproduct(G2:G9,$J2:$J9)-G18' to G17. Then, copy G17 and paste it to H17 and I17.
9. Assign '=$F15*Index(C2:C9,$C11) + C20' to C19. Then, copy C19 and paste it to D19, E19 and F19.

10. Assign '=$F15*Index(G2:G9,$C11)-G20' to G19. Then, copy G19 and paste it to H19 and I19.
11. Assign '=$I11/C13' to C20. Then copy C20 and paste it to D20-I20.
12. Assign '=C19-C18' to C22. Then copy C22 and paste it to D22-F22.
13. Assign '=G19 + G18' to G22. Then copy G22 and paste it to H22 and I22.
14. Assign '=(1-$F11)*Index(C2:C9,$C11)*$F15' to C24. Then copy C24 and paste it to D24-I24.
15. Assign '=(1 + $F11)*Index(C2:C9,$C11)*$F15' to C25. Then copy C25 and paste it to D25-I25.
16. Open 'Solver Parameters' window from 'DATA' in Excel toolbar.
17. Assign 'I15' into 'Set Objective' and choose 'Min'.
18. Assign 'J2:J9, C18:I18, F15, I11', into 'By Changing Variable Cells'.
19. Click on 'Add' and assign 'C15' into 'Cell Reference', then select '=', and assign '1' into 'Constraint'.
20. Click on 'Add' and assign 'C17:I17' into 'Cell Reference', then select '=', and assign 'C19:I19' into 'Constraint'.
21. Click on 'Add' and assign 'C22:I22' into 'Cell Reference', then select '>=', and assign 'C24:I24' into 'Constraint'.
22. Click on 'Add' and assign 'C22:I22' into 'Cell Reference', then select '<=', and assign 'C25:I25' into 'Constraint'. Then click on 'OK'.
23. Tick 'Make Unconstrained Variables Non-Negative'.
24. Choose 'Simplex LP' from 'Select a Solving Method' and then 'Solve'.
25. From 'Developer' in the toolbar menu, click on the 'Insert' icon to open the 'Form Control' window.
26. Click on the first icon, 'Button (Form Control)', and then click on a place in the Excel sheet.
27. In the open window with the title 'Assign Macro', click on 'New'. So, the 'Microsoft Visual Basic for Applications' window is opened.
28. From the toolbar menu, click on 'Tools> References...>' and make sure 'Solver' is ticked, and then 'OK'.
29. Inside of the 'Microsoft Visual Basic for Applications' window, write the following commands between 'Sub Button1_Click ()' and 'End Sub'.

```
Dim i As Integer
For i = 1 To 8
   Range("C11") = i
   Range("K" & i + 1) = SolverSolve(Userfinish:=True)
   Range("L" & i + 1) = Range("I15")
   Range("M" & i + 1) = Range("I11")/ Range("F15")
Next i
Range("M10") = WorksheetFunction.Max(Range("M2:M9"))
```

30. Close the 'Microsoft Visual Basic for Applications' window.
31. Click on the small rectangle which was automatically made in the Excel sheet in Step 27.

**Table 12.5** Returns values by Microsoft Excel Solver 2013

| Solver | Description |
|---|---|
| 0 | Constraints and optimality conditions are satisfied and Solver has found a solution. |
| 1 | Constraints are satisfied, but Solver has converged to the current solution. |
| 2 | Constraints are satisfied, but Solver cannot improve the current solution. |
| 3 | Solver is stopped chosen because of the maximum repetition's limit was reached. |
| 4 | The set cell values do not converge. |
| 5 | Solver could not find a feasible solution. |
| 6 | Solver is stopped at user's request. |
| 7 | The required linearity conditions for Solver are not satisfied. |
| 8 | The problem is too large and Solver cannot handle it. |
| 9 | Solver encountered an error value in a target or a constraint cell. |
| 10 | Solver is stopped chosen because of the maximum times' limit has been reached. |
| 11 | There is not enough memory available to solve the problem. |

**Table 12.6** The outcomes of Eq. 12.21 for the introduced assumptions

| N | Airport | Solver Code | KAM Score | Delta values |
|---|---|---|---|---|
| 1 | A | 0 | 0.9103466 | 74184.00000 |
| 2 | B | 0 | 0.8839073 | 74257.83920 |
| 3 | C | 0 | 0.9294655 | 60481.71351 |
| 4 | D | 0 | 0.9323123 | 149631.38643 |
| 5 | E | 0 | 0.8416030 | 605.56693 |
| 6 | F | 0 | 0.9417391 | 165675.79624 |
| 7 | G | 0 | 0.8920495 | 93000.00000 |
| 8 | H | 0 | 1.0000000 | 0.00000 |

32. The corresponded Solver code, KAM score, and Delta for each airport are represented into cells K2:M9, respectively, and the maximum value of delta is measured in M10.

The Solver code illustrates whether the Microsoft Excel Solver 2013 software was able to find an optimal solution for Eq. 12.21, as Table 12.5 displays. When the Solver code is 0, it means that the Solver finds an optimal solution which optimally satisfies all the constraints and conditions.

Table 12.6 also expresses the scores of KAM according to the measured delta and the assumptions of this example, as well as the Solver codes which are 0 for all airports and display that Solver is found the optimal solution for the objective when all the constraints are optimally satisfied.

The corresponded targets to the results in Table 12.6 are also illustrated in Table 12.7.

When the prices for the factors are $W_1^- = 10$, $W_2^- = 1$, $W_3^- = 100$, $W_4^- = 10$, $W_1^+ = 100$, $W_2^+ = 1$, and $W_3^+ = 10$, and managers can only regulate at about 10% of the factors' values, KAM robustly measures the inefficiency of each airport according to Table 12.6 and suggests how the efficiency of each airport can be improved, according to the results in Table 12.7.

**Table 12.7** The targets of airports from Eq. 12.21

| Airport | Area | Apron | Terminal | Runway | Flights | Passengers | Cargo |
|---|---|---|---|---|---|---|---|
| A | 1118.45 | 292,337.26 | 43,874.67 | 330,923.65 | 33,777.70 | 4,148,385.19 | 66,765.60 |
| B | 553.30 | 201,919.60 | 36,177.97 | 313,308.00 | 51,562.50 | 4,708,862.16 | 17,145.00 |
| C | 720.00 | 43,476.53 | 11,253.40 | 247,903.15 | 16,222.70 | 979,485.29 | 1745.70 |
| D | 936.90 | 111,542.31 | 20,509.11 | 356,157.00 | 38,374.69 | 1,856,032.66 | 5410.90 |
| E | 901.80 | 30,605.57 | 7713.14 | 174,652.28 | 5375.70 | 470,771.40 | 1702.56 |
| F | 454.19 | 69300.00 | 22,190.56 | 350,203.50 | 39,431.24 | 2,220,238.62 | 5848.43 |
| G | 462.64 | 51,931.00 | 10,230.00 | 242,095.50 | 20,037.49 | 1,068,444.30 | 3824.83 |
| H | 1346.00 | 503,274.00 | 76,370.00 | 421,305.00 | 129,153.00 | 11,709,741.00 | 39,556.00 |

**Table 12.8** The outcomes of Eq. 12.21 by SBM approach

| N | Airport | Solver Code | KAM Score | Delta values |
|---|---|---|---|---|
| 1 | A | 0 | 0.9386553 | 0.10000 |
| 2 | B | 0 | 0.9523211 | 0.10000 |
| 3 | C | 0 | 0.9101377 | 0.00000 |
| 4 | D | 0 | 0.8880952 | 0.09307 |
| 5 | E | 0 | 0.8624157 | 0.01806 |
| 6 | F | 0 | 0.9536431 | 0.10000 |
| 7 | G | 0 | 0.9113718 | 0.10000 |
| 8 | H | 0 | 0.9891776 | 0.10000 |

Instead of the introduced prices for each factor, for instance, the SBM approach can be applied. Table 12.8 illustrates the results of KAM when SBM approach is applied.

### 12.5.2 A Specified Request for KAM

In the previous example, every airport has different restrictions. Now, suppose that the managers of airports want to regulate the factors of airports according to the following conditions: (1) the area of airport should not be more than 1200 Hectares, (2) the number of passengers should be greater than 10,000,000 people, and (3) the amount of cargo should at least be 30,000 metric tons. Assume that $W_1^- = 10$, $W_2^- = 1$, $W_3^- = 100$, $W_4^- = 10$, $W_1^+ = 100$, $W_2^+ = 1$, and $W_3^+ = 10$. Is there any feasible area for such purposes? What are the inefficiencies of the airport according to these conditions?

In order to answer these questions, Eq. 12.22 should be solved.

$$
\begin{aligned}
& KA^*_{\delta_l^*} = \min\left[\sum_{j=1}^{4} W_j^-(x_{lj}t_l + \delta_l/W_{lj}^- - s_{lj}^-)\right] / \sum_{j=1}^{4} W_j^- x_{lj}, \\
& \text{Subject to} \\
& \left[\sum_{k=1}^{3} W_k^+(y_{lk}t_l - \delta_l/W_k^+ + s_{lk}^+)\right] / \sum_{k=1}^{3} W_k^+ y_{lk} = 1, \\
& \sum_{i=1}^{8} x_{ij}\lambda_{li} + s_{lj}^- = x_{lj}t_l + \delta_l/W_j^-, \ \text{for } j = 1,2,3,4, \\
& \sum_{i=1}^{8} y_{ik}\lambda_{li} - s_{lk}^+ = y_{lk}t_l + \delta_l/W_k^+, \ \text{for } k = 1,2,3, \\
& x_{l1}t_l + \delta_l/W_1^- - s_{l1}^- \le 10000t_l, \\
& 10000000t_l \le y_{l2}t_l - \delta_l/W_2^+ + s_{l2}^+, \\
& 50000t_l \le y_{l3}t_l - \delta_l/W_3^+ + s_{l3}^+, \\
& \lambda_{li} \ge 0, \ \text{for } i = 1,2,\ldots,8, \\
& s_{lj}^- \ge 0, \ \text{for } j = 1,2,3,4, \\
& s_{lk}^+ \ge 0, \ \text{for } k = 1,2,3, \\
& \delta_l > 0, \\
& t_l > 0.
\end{aligned}
\tag{12.22}
$$

**Table 12.9** The results of Eq. 12.22

| N | Airport | Solver Code | KAM Score | Delta values |
|---|---|---|---|---|
| 1 | A | 0 | 0.4608393 | 1,961,920.50863 |
| 2 | B | 0 | 0.6301526 | 2,644,120.50799 |
| 3 | C | 0 | 0.3292076 | 5,341,920.50548 |
| 4 | D | 0 | 0.4619512 | 4,416,920.50634 |
| 5 | E | 0 | 0.1667772 | 5,721,920.50512 |
| 6 | F | 0 | 0.4997589 | 4,221,920.50652 |
| 7 | G | 0 | 0.3917765 | 5,591,920.50525 |
| 8 | H | 0 | 1.0000000 | 1,885,764.84456 |

The results are represented in Table 12.9. According to the Solver codes, the feasible area is available and the KAM score for each airport is optimally measured. The measured target for all airports is also displayed in Table 12.10.

As can be seen, H still has the best performance to satisfy the purpose of this example. However, the relative scores of other airports drop sharply down. For instance, KAM measures that the ratio of the efficiency of A to the efficiency of the measured target is 0.4608393 to 1. The value 0.4608393 shows the inefficiency of A in comparison with the measured target in Table 12.9. Thus, A should decrease 12.21% of the area of airport, increase 41.29% of the area of apron, increase 43.02% of the area of terminal and so on, to satisfy the conditions of this example, as Table 12.11 illustrates. The same illustration can be expressed for the other airports, as well.

## 12.6 KAM Scores and Decomposition of Inefficiency

Suppose that there are $n$ firms, labeled $F_i$ ($i = 1, 2, \ldots, n$), and each firm has $m$ input factors with the values $x_{ij}$ ($j = 1, 2, \ldots, m$) and $p$ output factors with the values $y_{ik}$ ($k = 1, 2, \ldots, p$). Assume that the weights/prices or the approximation of the relationships between input and output factors are $W_j^-$ and $W_k^+$, for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$, respectively. The $\delta$-KAM is given by Eq. 12.23 to measure the efficiency score of $F_l$'s ($l = 1, 2, \ldots, n$) and optimize the value of delta.

$$
\begin{aligned}
& KA^*_{\delta_l^*} = \min \textstyle\sum_{j=1}^{m} W_j^-\left(x_{lj}t_l + \delta_l/W_j^- - s_{lj}^-\right)/\sum_{j=1}^{m} W_j^- x_{lj}, \\
& \text{Subject to} \\
& \textstyle\sum_{k=1}^{p} W_k^+\left(y_{lk}t_l - \delta_l/W_k^+ + s_{lk}^+\right)/\sum_{k=1}^{p} W_k^+ y_{lk} = 1, \\
& \textstyle\sum_{i=1}^{n} x_{ij}\lambda_{li} + s_{lj}^- = x_{lj}t_l + \delta_l/W_j^-, \text{ for } j = 1, 2, \ldots, m, \\
& \textstyle\sum_{i=1}^{n} y_{ik}\lambda_{li} - s_{lk}^+ = y_{lk}t_l - \delta_l/W_k^+, \text{ for } k = 1, 2, \ldots, p, \\
& \lambda_{li} \geq 0, \text{ for } i = 1, 2, \ldots, n, \\
& s_{lj}^- \geq 0, \text{ for } j = 1, 2, \ldots, m, \\
& s_{lk}^+ \geq 0, \text{ for } k = 1, 2, \ldots, p, \\
& \delta_l \geq 0, \\
& t_l \geq 0.
\end{aligned}
\tag{12.23}
$$

**Table 12.10** The targets of airports from Eq. 12.22

| Airport | Area | Apron | Terminal | Runway | Flights | Passengers | Cargo |
|---|---|---|---|---|---|---|---|
| A-H | 1149.47 | 429,790.89 | 65,219.21 | 359,790.20 | 110,295.35 | 10,000,000.00 | 33,780.42 |

**Table 12.11** The rate of regulating the factors by Eq. 12.22

| Airport | Area (%) | Apron (%) | Terminal (%) | Runway (%) | Flights (%) | Passengers (%) | Cargo (%) |
|---|---|---|---|---|---|---|---|
| A | −4.21 | 41.29 | 43.02 | 1.75 | 259.19 | 148.09 | −54.46 |
| B | 128.52 | 101.09 | 68.19 | 3.35 | 135.30 | 109.07 | 77.33 |
| C | 43.68 | 948.19 | 452.71 | 33.28 | 606.66 | 861.57 | 2028.57 |
| D | 10.42 | 282.16 | 209.83 | −9.08 | 176.63 | 473.22 | 586.73 |
| E | 14.72 | 1332.64 | 715.24 | 87.07 | 2156.91 | 2236.59 | 2046.15 |
| F | 140.47 | 582.21 | 183.56 | −7.54 | 168.44 | 361.77 | 523.95 |
| G | 138.98 | 810.38 | 601.28 | 33.75 | 480.20 | 929.53 | 782.92 |
| H | −14.60 | −14.60 | −14.60 | −14.60 | −14.60 | −14.60 | −14.60 |

The scores of $\delta^*$-KAM, where $\delta^* = \max\left\{\delta_l^* : l = 1, 2, \ldots, n\right\}$, are the relative efficiency scores of $F_i$'s ($i = 1, 2, \ldots, n$) and can absolutely be used to rank a set of homogenous firms.

As illustrated in Sect. 6.4.3, KAM represents the minimum value of Eq. 5.23, and compares the efficiency of a firm with the efficiency of the best possible location in the estimated feasible area, that is, KAM measures all the inefficiency of the firm, and its scores are relatively meaningful.

Even if the constraint $\sum_{i=1}^{n} \lambda_{li} = t$ (or $\sum_{i=1}^{n} \lambda_{li} \geq t$ or $\sum_{i=1}^{n} \lambda_{li} \leq t$) is added to the constraints of Eq. 12.23, the scores of KAM are the same. This is due to this fact that, all the observed firms are in the estimated feasible area by Eq. 12.23, so KAM compares all the firms to each other and regulates the factors to find the best location of the feasible area. Therefore, even if the observed firms are only considered (that is, a discrete set with $n$ points) the scores of KAM are the same as when the CRS technology is applied by Eq. 12.23. This phenomenon lets us completely decompose the inefficiency of firms with the scores of KAM.

If the delta value is equal to zero, that is, $\delta = 0$, the Technical Efficiency (TE) scores of the firms are measured. If the constraint $\sum_{i=1}^{n} \lambda_{li} = t$ is added to Eq. 12.23, when $\delta = 0$, the Variable Returns to Scale-Technical Efficiency (VRS-TE) scores are measured. The Non-Technical Efficiency (Non-TE) scores (allocative efficiency scores) are measured by the efficiency scores of the firms over the TE scores, and the Non-Variable Returns to Scale-Technical Efficiency (Non-VRS-TE) scores (scale efficiency scores) of the firms are measured by the ratio of TE scores to VRS-TE scores. All of these scores belong to interval [0,1]. Indeed, the score of $\delta^*$-KAM is not greater than the score of 0-KAM (TE score), and the score of TE is not greater than the score of VRS-TE.

For example, suppose that the airports in Table 5.1 is given, where $W_1^- = 10$, $W_2^- = 1$, $W_3^- = 100$, $W_4^- = 10$, $W_1^+ = 100$, $W_2^+ = 1$, and $W_3^+ = 10$. Table 12.12 illustrates the efficiency, TE, Non-TE, VRS-TE, and Non-VRS-TE scores of the airports, according to the introduced $W_j^-$'s and $W_k^+$'s.

None of the Non-TE, TE, VRS-TE, and Non-VRS-TE scores can be used to rank the airports, and they only represent the reasons of inefficiency. Only the

**Table 12.12** The decomposition of efficiency scores of the airports

| N | Airport | Efficiency | Non-TE | TE | VRS-TE | Non-VRS-TE |
|---|---|---|---|---|---|---|
| 1 | A | 0.4608393 | 0.4608393 | 1.0000000 | 1.0000000 | 1.0000000 |
| 2 | B | 0.6301526 | 0.6301526 | 1.0000000 | 1.0000000 | 1.0000000 |
| 3 | C | 0.3292076 | 0.6754831 | 0.4873662 | 1.0000000 | 0.4873662 |
| 4 | D | 0.4619512 | 0.4619512 | 1.0000000 | 1.0000000 | 1.0000000 |
| 5 | E | 0.1667772 | 1.0000000 | 0.1667772 | 1.0000000 | 0.1667772 |
| 6 | F | 0.4997589 | 0.4997589 | 1.0000000 | 1.0000000 | 1.0000000 |
| 7 | G | 0.3917765 | 0.3917765 | 1.0000000 | 1.0000000 | 1.0000000 |
| 8 | H | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |

efficiency scores in the third column of Table 12.8 should be used to rank the airports for $\delta \geq \delta^*$, (see also Table 5.2).

From Table 12.12, the inefficiency of C and E are due to the scale inefficiency (Non-VRS-TE), whereas the inefficiency of A, B, D, F and G are because of the allocative inefficiency (Non-TE) (see Khezrimotlagh et al. 2012b and Mohsenpour et al. 2013).

If extra restrictions are added to the model, they may change the efficiency scores of firms, and the measured outcomes. In this situation, the inefficiency by the extra restrictions should also be measured. For instance, when it is supposed that managers in the airport example can only regulate at about 10% value of each factor, KAM can only compare an airport with the points in the area that the corresponded components have 10% lesser or greater values than that of the airport.

In order to elucidate this statement, suppose that the divisions in the petroleum example in Chap. 2 are selected, where $W_1^- = W_2^- = 1$, and the output is a constant single value for all divisions, such as 1. Assume that only 10% values of the diesel fuel and the gasoline amounts can be regulated, that is, $a_j = x_{lj} - 10\ \%\ x_{lj}$ and $b_j = x_{lj} + 10\ \%\ x_{lj}$, for $j = 1, 2$. The $\delta_l$-KAM, when the components of delta are introduced with Eq. 12.3, is given by Eq. 12.24 ($l = 1, 2, \ldots, 18$).

$$
\begin{aligned}
& KA^*_{\delta^*_l} = \min \left[\sum_{j=1}^{2} \left(x_{lj} + \delta_l x_{lj} - s^-_{lj}\right)\right] / \sum_{j=1}^{2} x_{lj}, \\
& \text{Subject to} \\
& \sum_{i=1}^{18} \lambda_{li} x_{ij} + s^-_{lj} = x_{lj} + \delta_l x_{lj}, \ \text{for } j = 1, 2, \\
& \sum_{i=1}^{18} \lambda_{li} = 1, \\
& -10\% x_{lj} \leq \delta_l x_{lj} - s^-_{lj}, \ \text{for } j = 1, 2, \\
& \delta_l x_{lj} - s^-_{lj} \leq 10\% x_{lj}, \ \text{for } j = 1, 2, \\
& \lambda_{li} \geq 0, \ \text{for } i = 1, 2, \ldots, 18, \\
& s^-_{lj} \geq 0, \ \text{for } j = 1, 2, \\
& \delta_l > 0.
\end{aligned}
\tag{12.24}
$$

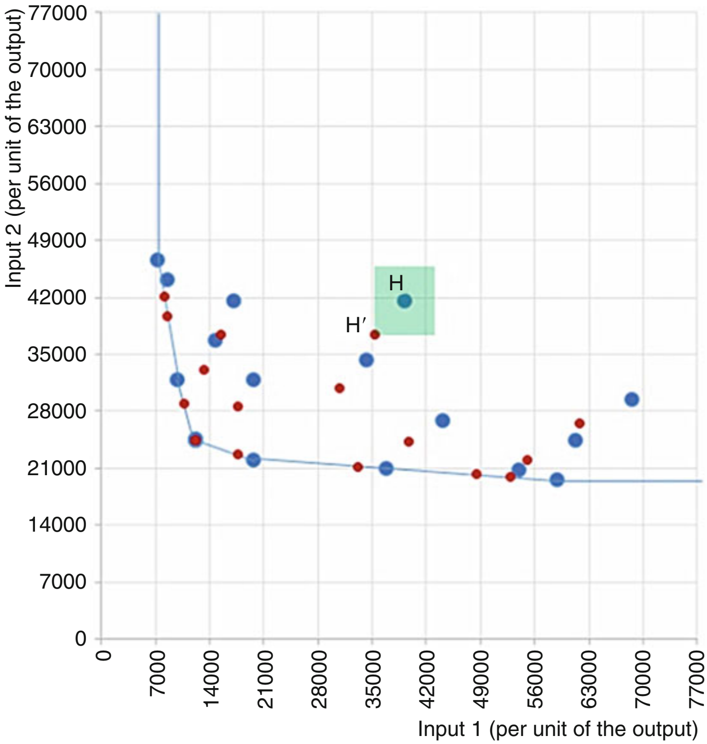


**Fig. 12.2** The benchmarking by Eq. 12.24 with 10% regulation

Figure 12.2 represents the location of divisions with large blue circles, and the location of their targets with small red circles. For instance, according to the restrictions in Eq. 12.24, when H is evaluated, the feasible area is a square which is shaded with green color in Fig. 12.2. Therefore, $H$ is compared with $H^{'}$, and the KAM's score for H represents the ratio of the efficiency of $H$ to the efficiency of $H^{'}$. In order to decompose the inefficiency of H, the inefficiency of $H^{'}$ should also be measured.

Figure 12.3 also represents the targets for divisions when each division can regulate 30% value of the diesel fuel and the gasoline amounts. For instance, the feasible area for division H in Fig. 12.3 shows that H is compared with C, and since $H^{'}$ wholly dominates both H and C, the suggested target for H is $H^{'}$. As can be seen, H is not compared to other divisions (except C), which means the efficiency of H is not completely measured due to the extra restrictions.

As the range of regulation is increased the divisions are benchmarked toward division A, which has the best performance among the divisions where $w_1^- = w_2^- = 1$, (see Fig. 4.16).

In other words, when all divisions belong to the measured feasible area, this means that the extra restrictions do not affect the efficiency measurement by KAM.

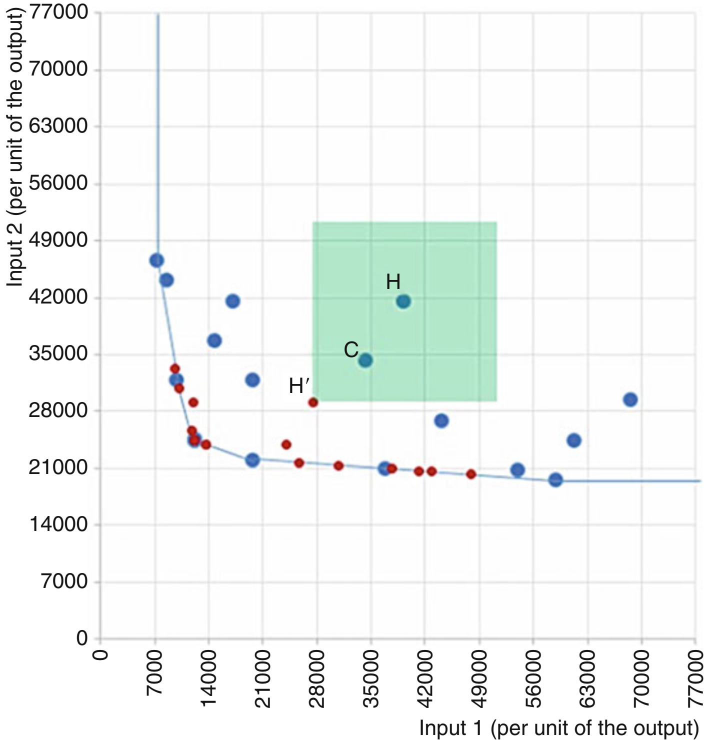


**Fig. 12.3** The benchmarking by Eq. 12.24 with 30% regulation

However, if an extra restriction changes the feasible area such that some observed firms are not included in that feasible area, the efficiency is not completely measured. As a result, the effects of extra restrictions should be calculated to decompose the inefficiency as well.

Note that, as the range of regulation in Figs. 12.2 and 12.3 increases, the intersection of the shaded square with the PPS provides the feasible area for H.

From the above illustration, it is clear that an extra restriction may change the way of interpreting the outcomes of KAM and the decomposition of measured efficiencies. But, what remains the same in any situation is that, KAM robustly compares the efficiency of a firm, $F_l$, with the efficiency of all points in the corresponded feasible area, and the target of KAM for $F_l$ is the point which has a better efficiency score in comparison with the efficiency score of $F_l$, according to the restrictions and the introduced conditions.

For more illustration, let's consider the objective of KAM, as Eq. 12.25 displays. The first fraction in the objective of KAM represents the efficiency of $F_l$ ($l = 1, 2, \ldots, n$) and the second fraction represents the efficiency of a point in the feasible area that the corresponded components of that point can have equal, lesser or greater values than that of $F_l$. KAM minimizes this ratio, that is, KAM maximizes the second fraction according to Types 1–6. Note that the first fraction has a constant value, and does not influence the optimization.

$$\min \left( \frac{\sum_{k=1}^{p} W_k^+ y_{lk}}{\sum_{j=1}^{m} W_j^- x_{lj}} \right) / \left( \frac{\sum_{k=1}^{p} W_k^+ \left( y_{lk} - \delta_k^+ + s_k^+ \right)}{\sum_{j=1}^{m} W_j^- \left( x_{lj} + \delta_j^- - s_j^- \right)} \right) \tag{12.25}$$

Therefore, the optimal target has the best efficiency score in comparison with $F_l$ in the estimated feasible area, according to the constraints and the introduced conditions.

If there are no more constraints than the constraints in the Eq. 12.23, the optimal scores of KAM are independent on the returns to scale technologies, and adding the constraint $\sum_{i=1}^{n} \lambda_{li} = t$ (or $\sum_{i=1}^{n} \lambda_{li} \geq t$ or $\sum_{i=1}^{n} \lambda_{li} \leq t$) does not change the optimal scores, because the measured feasible area in this situation includes all the observed firms. In addition, none of the returns to scale technologies change the efficiency score of a firm which is calculated with a linear combination of output factors over a linear combination of input factors.

On the other hand, the KAM benchmarks by Eq. 12.23 depend on the introduced returns to scale technology. In other words, if the constraint $\sum_{i=1}^{n} \lambda_{li} = t$ is added to Eq. 12.23, the efficiency scores of firms are not changed, but, KAM suggests the targets which are on the VRS-PPS.

As a result, while there is more than one input (output) factor and the relationships between the factors or units of measurement/prices/worth/weights of the factors are unknown, it is impossible to provide a fair relative score for each firm without some information from expert judgment. If the approach is introduced or prices are estimated for each factor, the $\delta$-KAM is the most powerful tool to linearly discriminate, rank and benchmark firms as well as decomposing the efficiency scores.

## 12.7 Outliers and Numbers of Firms via Factors

Since the discrimination between the firms is based on the best observed performers, there was a concern in the literature of operations research that such discrimination is sensitive to the possible presence of outliers. If there is no significant error in the measurement, there is not a valid concern about the extreme data when the firms are supposed as homogenous. In addition, since KAM fairly ranks the firms based on the introduced approach and the value of delta, the firms can easily be classified according to their measured efficiency scores, so the possible outliers can easily be announced statistically and the production function can be estimated by the rest of firms.

There was also a concern in the literature of operations research about increasing the number of factors when the number of firms is a constant value, or is not large enough in comparison with the number of factors. This concern is raised due to the misinterpretation of the technical efficiency as efficiency, and misusing the technical efficiency to rank firms. As frequently explained in this book, the concept of doing the job right (technical efficiency) is not logically enough for discriminating between firms or ranking them. The concept of doing the job well (efficiency) is required to rank a set of homogenous firms with multiple input factors and multiple output factors. From this fact, there is no concern about the number of factors via the number of firms when KAM is applied.

## 12.8 Conclusion

The technical efficiency scores (that is, the scores to indicate whether or not a DMU does the job right) represent the optimal output factors which can be obtained from a set of input factors, but they should not be used to rank firms; they have unfortunately been inaccurately used in the literature of operations research for the last four decades. The provided scores by KAM can relatively be meaningful according to the delta parameter, and can be used to measure the efficiency scores of firms which lets us rank and benchmark the firms logically. In this chapter, the optimal value of delta is measured, and KAM is improved to measure the efficiency of firms under specific restrictions for targets. There are no concerns about the presence of outliers or the number of firms via the number of factors, when KAM is applied. The discrimination between homogenous firms requires expert judgment either to introduce a set of weights for factors or to specify a measurement approximation to estimate the efficiency scores which are relatively meaningful. From the outcomes of KAM, the inefficiency can be decomposed to non-technical inefficiency (allocative inefficiency) and technical inefficiency, and the technical inefficiency can also be decomposed to VRS-technical inefficiency and non-VRS technical inefficiency (scale inefficiency).

## 12.9 Exercises

12.1. Write Eq. 12.5 in VRS technology and find its dual linear programming.

12.2. Solve Eq. 12.12 when $W_1^- = 10$, $W_2^- = 1$, $W_3^- = 100$, $W_4^- = 10$, $W_1^+ = 100$, $W_2^+ = 1$, and $W_3^+ = 10$, and the fourth input factor is non-controllable.

12.3. Find the dual linear programming for

12.3.1. Eq. 12.12.

12.3.2. Eq. 12.21.

12.4. Prove that if the values of $W_j^-$ and $W_k^+$ (prices) are multiplied to the corresponded data before applying KAM (Eqs. 12.5, 12.23), the results of KAM are not changed, (see Tone 2002 and Khezrimotlagh et al. 2012a).

12.5. Solve the example in Sect. 12.5.1 when SBM approach is applied and discuss the results.

12.6. Solve Eq. 12.22 when the VRS technology is applied, and compare the results with the data in Tables 12.8, 12.9, and 12.10.

12.7. Apply Eq. 12.23 for gemstone example, where $W^- = W^+ = 1$.

12.7.1. Compare the results with the results in Table 1.4.
12.7.2. Decompose the efficiency score of each candidate.
12.7.3. Apply Eq. 12.20, where each candidate can only regulate 10% of input and output factors, and describe the results.
12.7.4. Apply Eq. 12.20, where each candidate can only regulate 30% of the factors, and compare the results with previous results.
12.7.5. Assume that $W^- = 2$ and $W^+ = 3$ and answer the above questions.
12.7.6. Assume the SBM approach, that is, $W_l^- = 1/x_l$ and $W_l^+ = 1/\mathrm{y}_l$ and answer the above questions.
12.7.7. State the conclusion of this task.

12.8. Apply Eq. 12.23 for petroleum example in Sect. 4.4, where $W_1^+ = W_2^+ = 1$.

12.8.1. Compare the results with the results in Table 12.11.
12.8.2. Decompose the efficiency score of each division.
12.8.3. Apply Eq. 12.23, where each candidate can only regulate 5% of output factors, and describe the results.
12.8.4. Apply Eq. 12.23, where each candidate can only regulate 10% of output factors, and compare the results with previous results.
12.8.5. Assume the average approach, that is, $W_{l1}^+ = 1/\underset{l}{\mathrm{ave}}\{y_{l1}\}$ and $W_{l2}^+ = 1/\underset{l}{\mathrm{ave}}\{y_{l2}\}$, and answer the above questions.
12.8.6. Assume the maximum approach, that is, $W_{l1}^+ = 1/\max_l\{y_{l1}\}$ and $W_{l2}^+ = 1/\max_l\{y_{l2}\}$, and answer the above questions.
12.8.7. State the conclusion of this task.

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